Polynomial ideals of a ring based on neutrosophic sets

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Abstract

In this paper, we introduce the notion of the neutrosophic polynomial ideal A_x of a polynomial ring $R[x]$ induced by a neutrosophic ideal A of a ring R and obtain an isomorphism theorem of a ring of neutrosophic cosets of A_x. It is shown that a neutrosophic ideal A of a ring is a neutrosophic prime if and only if A_x is a neutrosophic prime ideal of $R[x]$.

Keywords: neutrosophic ideal; neutrosophic prime ideal; neutrosophic polynomial ideal; f-invariant.

1 Introduction

The concept of fuzzy sets was proposed by Zadeh.[9] The theory of fuzzy sets has several applications in real-life situations and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in.[1,3,4] The idea of intuitionistic fuzzy sets suggested by Atanassov[2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision-making.[5–7] The notion of neutrosophic sets was introduced by Smarandache[8] in 1999, which is a more general platform that extends the notions of classic sets, (intuitionistic) fuzzy sets, and interval-valued (intuitionistic) fuzzy sets. Neutrosophic set theory is applied to various parts, which is referred to on the site http://fs.unm.edu/neutrosophy.htm. In this paper, we introduce the notion of the neutrosophic polynomial ideal A_x of a polynomial ring $R[x]$ induced by a neutrosophic ideal A of a ring R and obtain an isomorphism theorem of a ring of neutrosophic cosets of A_x. It is shown that a neutrosophic ideal A of a ring is a neutrosophic prime if and only if A_x is a neutrosophic prime ideal of $R[x]$.

2 Preliminaries

Let R be a nonempty set. The neutrosophic set[8] A on R is defined to be a structure

$$A := \{\langle x, \mu(x), \gamma(x), \psi(x)\rangle \mid x \in R\}, \tag{1}$$

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where $\mu : R \to [0,1]$ is a truth membership function, $\gamma : R \to [0,1]$ is an indeterminate membership function, and $\psi : R \to [0,1]$ is a false membership function. The neutrosophic set in (1) is simply denoted by $A = (\mu_A, \gamma_A, \psi_A)$.

In this section, we review some definitions which will be used in the later section. Throughout this paper, unless stated otherwise, all rings are commutative rings with identity.

Definition 2.1. Let R be a ring. A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ of R is said to be a neutrosophic ideal of R if

$$(\forall x, y \in R) \begin{pmatrix} \mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(x-y) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(x-y) \leq \max\{\psi_A(x), \psi_A(y)\} \end{pmatrix},$$

$$(\forall x, y \in R) \begin{pmatrix} \mu_A(xy) \geq \max\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(xy) \geq \max\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(xy) \leq \min\{\psi_A(x), \psi_A(y)\} \end{pmatrix}.$$

Definition 2.2. Let R and S be any sets, and let $f : R \to S$ be a function. A neutrosophic set A of R is called an f-invariant if $f(x) = f(y) \Rightarrow \mu_A(x) = \mu_A(y), \gamma_A(x) = \gamma_A(y)$, and $\psi_A(x) = \psi_A(y)$, where $x, y \in \mathbb{R}$. If A is any f-invariant neutrosophic set of R, then $f^{-1}(f(A)) = A$.

Lemma 2.3. *Let R and S be any sets and $f : R \to S$ be any function. If A and B are neutrosophic sets of R and S, respectively, and f-invariant, then $A \cup B$ and $A \cap B$ are f-invariant.*

Lemma 2.4. *Let R and S be any sets and $f : R \to S$ be any function. Let A and B be f-invariant neutrosophic sets of R. If $A \subseteq B$, then $f(A) \subseteq f(B)$.*

Let R be a commutative ring with identity and let $R[x]$ be the ring of polynomials where x is indeterminate.

Definition 2.5. Let $f : R \to R'$ be a homomorphism of rings. A map $f_x : R[x] \to R'[x]$ defined by $f_x(\sum_{i=0}^{n} a_i x_i) = \sum_{i=0}^{n} f(a_i)x_i$ is obviously a ring homomorphism, and we call f_x an induced homomorphism by f.

3 Neutrosophic polynomial ideals

In this section, we introduce the notion of neutrosophic polynomial ideals of a ring and study their properties. The set of all real numbers is denoted by $\mathbb{R}$.

Lemma 3.1. *Let $a_i, b_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. Then* $\min_i(\min\{a_i, b_i\}) = \min\{\min_i(a_i), \min_i(b_i)\}$ *and* $\max_i(\max\{a_i, b_i\}) = \max\{\max_i(a_i), \max_i(b_i)\}$.

Lemma 3.2. *Let $a_i, b_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. Then* $\min_i(\max\{a_i, b_i\}) \geq \max\{\min_i(a_i), \min_i(b_i)\}$ *and* $\max_i(\min\{a_i, b_i\}) \leq \min\{\max_i(a_i), \max_i(b_i)\}$.

Lemma 3.3. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set of a ring R. Then for all $a_i, b_i \in \mathbb{R}$, we have*

$$\mu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \geq \max\{\min_i\{(\mu_A(a_i)), \min_i\{\mu_A(b_i)\}\},$$

$$\gamma_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \geq \max\{\min_i\{(\gamma_A(a_i)), \min_i\{\gamma_A(b_i)\}\},$$

$$\psi_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \leq \min\{\max_i(\psi_A(a_i)), \max_i\{\psi_A(b_i)\}\}.$$

Proof. Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic set of a ring R, for any $a_i, b_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$,

$$\begin{aligned} \mu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) &\geq \min_i\{\mu_A(a_ib_{n+1-i})\} \\ &\geq \min_i\{\max\{\mu_A(a_i), \mu_A(b_{n+1-i})\}\} \\ &\geq \max\{\min_i\{\mu_A(a_i), \mu_A(b_i)\}\}, \end{aligned}$$

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$$\begin{aligned}
\gamma_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) &\geq \min_i\{\gamma_A(a_ib_{n+1-i})\} \\
&\geq \min_i\{\max\{\gamma_A(a_i), \gamma_A(b_{n+1-i})\}\} \\
&\geq \max\{\min_i\{\gamma_A(a_i), \gamma_A(b_i)\}\},
\end{aligned}$$

$$\begin{aligned}
\psi_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) &\leq \max_i\{\psi_A(a_ib_{n+1-i})\} \\
&\leq \max_i\{\min\{\psi_A(a_i), \psi_A(b_{n+1-i})\}\} \\
&\leq \min\{\max_i\{\psi_A(a_i), \psi_A(b_i)\}\}.
\end{aligned}$$

□

Theorem 3.4. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic ideal of a ring R and $f(x) = \sum_{i=0}^{m} a_ix_i \in R[x]$. Define a neutrosophic set $A_x = (\mu_{A_x}, \gamma_{A_x}, \psi_{A_x})$ on $R[x]$ by $\mu_{A_x}(f(x)) = \min_i\{\mu_A(a_i)\}$, $\gamma_{A_x}(f(x)) = \min_i\{\gamma_A(a_i)\}$, and $\psi_{A_x}(f(x)) = \max_i\{\psi_A(a_i)\}$. Then A_x is a neutrosophic ideal of $R[x]$.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_ix_i, g(x) = \sum_{i=0}^{m} b_ix_i \in R[x]$. Then by Lemma 3.1, we have

$$\begin{aligned}
\mu_{A_x}(f(x) - g(x)) &= \min_i\{\mu_A(c_i)\},\ where\ c_i = a_i - b_i \\
&= \min_i\{\mu_A(a_i - b_i)\} \\
&\geq \min_i\{\min\{\mu_A(a_i), \mu_A(b_i)\}\} \\
&= \min\{\min_i\{\mu_A(a_i)\}, \min_i\{\mu_A(b_i)\}\} \\
&= \min\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\},
\end{aligned}$$

$$\begin{aligned}
\gamma_{A_x}(f(x) - g(x)) &= \min_i\{\gamma_A(c_i)\},\ where\ c_i = a_i - b_i \\
&= \min_i\{\gamma_A(a_i - b_i)\} \\
&\geq \min_i\{\min\{\gamma_A(a_i), \gamma_A(b_i)\}\} \\
&= \min\{\min_i\{\gamma_A(a_i)\}, \min_i\{\gamma_A(b_i)\}\} \\
&= \min\{\gamma_{A_x}(f(x)), \gamma_{A_x}(g(x))\},
\end{aligned}$$

$$\begin{aligned}
\psi_{A_x}(f(x) - g(x)) &= \max_i\{\psi_A(c_i)\},\ where\ c_i = a_i - b_i \\
&= \max_i\{\psi_A(a_i - b_i)\} \\
&\leq \max_i\{\max\{\psi_A(a_i), \psi_A(b_i)\}\} \\
&= \max\{\max_i\{\psi_A(a_i)\}, \max_i\{\psi_A(b_i)\}\} \\
&= \max\{\psi_{A_x}(f(x)), \psi_{A_x}(g(x))\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_{A_x}(f(x)g(x)) &= \min_i\{\mu_A(d_i)\},\ where\ d_i = \sum_{i=0}^{n+m} aib_{n+m-i} \\
&= \min_i\{\max\{\mu_A(a_i)\}, \{\mu_A(b_{n+m-i})\}\} \\
&\geq \min_i\{\max\{\mu_A(a_i), \mu_A(b_i)\}\} \\
&\geq \max\{\min_i\{\mu_A(a_i)\}, \min_i\{\mu_A(b_i)\}\} \\
&= \max\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\},
\end{aligned}$$

$$\begin{aligned}
\gamma_{A_x}(f(x)g(x)) &= \min_i\{\gamma_A(d_i)\},\ where\ d_i = \sum_{i=0}^{n+m} aib_{n+m-i} \\
&= \min_i\{\max\{\gamma_A(a_i)\}, \{\gamma_A(b_{n+m-i})\}\} \\
&\geq \min_i\{\max\{\gamma_A(a_i), \gamma_A(b_i)\}\} \\
&\geq \max\{\min_i\{\gamma_A(a_i)\}, \min_i\{\gamma_A(b_i)\}\} \\
&= \max\{\gamma_{A_x}(f(x)), \gamma_{A_x}(g(x))\},
\end{aligned}$$

$$\begin{aligned}
\psi_{A_x}(f(x)g(x)) &= \max_i\{\psi_A(d_i)\},\ where\ d_i = \sum_{i=0}^{n+m} aib_{n+m-i} \\
&= \max_i\{\min\{\psi_A(a_i)\}, \{\psi_A(b_{n+m-i})\}\} \\
&\leq \max_i\{\min\{\psi_A(a_i), \psi_A(b_i)\}\} \\
&\leq \min\{\max_i\{\psi_A(a_i)\}, \max_i\{\psi_A(b_i)\}\} \\
&= \min\{\psi_{A_x}(f(x)), \psi_{A_x}(g(x))\}.
\end{aligned}$$

Hence, A_x is a neutrosophic ideal of $R[x]$. □

Definition 3.5. The neutrosophic ideal A_x discussed in Theorem 3.4 is called the neutrosophic polynomial ideal of $R[x]$ induced by a neutrosophic ideal A of a ring R.

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Proposition 3.6. *Let $f : R \to R'$ be a homomorphism of rings and let $f_x : R[x] \to R'[x]$ be an induced homomorphism of f. If A is a neutrosophic ideal of the ring R and A_x is its neutrosophic prime ideal of $R[x]$, then A is f-invariant if and only if A_x is f_x-invariant.*

Proof. Assume that A is f-invariant. Let $f_x(r(x)) = f_x(s(x))$, where $r(x) = \sum_{i=0}^{m} a_i x_i, s(x) = \sum_{i=0}^{m} b_i x_i \in R[x]$. Then $\sum_{i=0}^{m} f(a_i)_{x_i} = \sum_{i=0}^{m} f(b_i)_{x_i}$, so $f(a_i) = f(b_i), \forall i = 1, 2, \ldots, m$. Thus,

$$\mu_{A_x}(r(x)) = \min_i\{\mu_A(a_i)\} = \min_i\{\mu_A(b_i)\} = \mu_{A_x}(s(x)),$$
$$\gamma_{A_x}(r(x)) = \min_i\{\gamma_A(a_i)\} = \min_i\{\gamma_A(b_i)\} = \gamma_{A_x}(s(x)),$$
$$\psi_{A_x}(r(x)) = \max_i\{\psi_A(a_i)\} = \max_i\{\psi_A(b_i)\} = \psi_{A_x}(s(x)).$$

Hence, A_x is fx-invariant.

Conversely, assume that A_x is f_x-invariant. If $f(a) = f(b)$, then $f_x(a) = f_x(b)$. Since A_x is f_x-invariant, we have $\mu_{A_x}(a) = \mu_{A_x}(b)$, $\gamma_{A_x}(a) = \gamma_{A_x}(b)$, and $\psi_{A_x}(a) = \psi_{A_x}(b)$, which implies that $\mu_A(a) = \mu_A(b)$, $\gamma_A(a) = \gamma_A(b)$, and $\psi_A(a) = \psi_A(b)$. Hence, A is f-invariant. □

Proposition 3.7. *Let A be a neutrosophic ideal of a ring R. Then the set $S = \{f(x) \in R[x] : \mu_{A_x}(f(x)) = \mu_{A_x}(0), \gamma_{A_x}(f(x)) = \gamma_{A_x}(0), \psi_{A_x}(f(x)) = \psi_{A_x}(0)\}$ is a subring of $R[x]$.*

Proof. Let $f(x), g(x)$ be any two elements of S. Then

$$\mu_{A_x}(f(x) - g(x)) \geq \min\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\} = \mu_{A_x}(0),$$
$$\gamma_{A_x}(f(x) - g(x)) \geq \min\{\gamma_{A_x}(f(x)), \gamma_{A_x}(g(x))\} = \gamma_{A_x}(0),$$
$$\psi_{A_x}(f(x)g(x)) \leq \max\{\psi_{A_x}(f(x)), \psi_{A_x}(g(x))\} = \psi_{A_x}(0).$$

Also,

$$\mu_{A_x}(f(x) - g(x)) \geq \min\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\} = \mu_{A_x}(0),$$
$$\gamma_{A_x}(f(x) - g(x)) \geq \min\{\gamma_{A_x}(f(x)), \gamma_{A_x}(g(x))\} = \gamma_{A_x}(0),$$
$$\psi_{A_x}(f(x)g(x)) \leq \max\{\psi_{A_x}(f(x)), \psi_{A_x}(g(x))\} = \psi_{A_x}(0).$$

On the other hand, $\mu_{A_x}(f(x)) \leq \mu_{A_x}(0)$, $\gamma_{A_x}(f(x)) \leq \gamma_{A_x}(0)$, and $\psi_{A_x}(f(x)) \geq \psi_{A_x}(0)$ for all $f(x) \in R[x]$. So, $f(x) - g(x), f(x)g(x) \in S$. Hence, S is a subring of $R[x]$. □

Remark 3.8. Let A be a neutrosophic set of a ring R. We denote a level cut set A_* by $A_* = \{x \in \mathbb{R} : \mu_A(x) = \mu_A(0), \gamma_A(x) = \gamma_A(0), \psi_A(x) = \psi_A(0)\}$. It is proved that if A is a neutrosophic ideal of a ring R, then A_* is an ideal of R. Note that if A is a neutrosophic ideal of a ring R, then $\mu_A(0) \geq \mu_A(x), \gamma_A(0) \geq \gamma_A(x)$, and $\psi_A(0) \leq \psi_A(x)$ for all $x \in \mathbb{R}$. We denote $A_*[x] = \{f(x) \in R[x] : f(x) = \sum_{i=0}^{n} a_i x_i, a_i \in A_*, \forall i = 1, 2, \ldots, n\}$.

Theorem 3.9. *Let A be a neutrosophic ideal of a ring R. Then $(A_x)_x = A_*[x]$.*

Proof. Now,

$$\begin{aligned}
(A_x)_x &= \{f(x) \in R[x] : f(x) = \sum_{i=0}^{n} a_i x_i, \mu_{A_x}(f(x)) = \mu_{A_x}(0), \gamma_{A_x}(f(x)) = \gamma_{A_x}(0), \\
&\quad \psi_{A_x}(f(x)) = \psi_{A_x}(0)\} \\
&= \{f(x) \in R[x] : f(x) = \sum_{i=0}^{n} a_i x_i, \min_i\{\mu_A(a_i)\} = \mu_A(0), \min_i\{\gamma_A(a_i)\} = \gamma_A(0), \\
&\quad \max_i\{\psi_A(a_i)\} = \psi_A(0)\} \\
&= \{f(x) \in R[x] : f(x) = \sum_{i=0}^{n} a_i x_i, \mu_A(a_i) = \mu_A(0), \gamma_A(a_i) = \gamma_A(0), \\
&\quad \psi_A(a_i) = \psi_A(0),\ \forall i = 1, 2, \ldots, n\} \\
&= \{f(x) \in R[x] : f(x) = \sum_{i=0}^{n} a_i x_i, a_i \in A_*, \forall i = 1, 2, \ldots, n\} \\
&= A_*[x].
\end{aligned}$$

□

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Theorem 3.10. *Let A and B be neutrosophic ideals of a ring R. Then*

(1) $(A\cap B)_x = A_x \cap B_x$,

(2) $(A\cup B)_x \supseteq A_x \cup B_x$,

(3) $A_x + B_x \subseteq (A+B)_x$,

(4) $A_xB_x \subseteq (AB)_x$.

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x_i$ be any element of $R[x]$.
(1) $(A\cap B)_x(f(x)) = (\mu_{(A\cap B)_x}(f(x)), \gamma_{(A\cap B)_x}(f(x)), \mu_{(A\cap B)_x}(f(x)))$, where

$$\begin{aligned}
\mu_{(A\cap B)_x}(f(x)) &= \min_i\{\mu_{A\cap B}(a_i)\} \\
&= \min_i\{\min\{\mu_A(a_i), \mu_B(a_i)\}\} \\
&= \min\{\min_i\{\mu_A(a_i), \mu_B(a_i)\}\} \\
&= \min\{\min_i\{\mu_A(a_i)\}, \min_i\{\mu_B(a_i)\}\} \\
&= \min\{\mu_{A_x}(f(x)), \mu_{B_x}(f(x))\} \\
&= \mu_{A_x\cap B_x}(f(x)),
\end{aligned}$$

$$\begin{aligned}
\gamma_{(A\cap B)_x}(f(x)) &= \min_i\{\gamma_{A\cap B}(a_i)\} \\
&= \min_i\{\min\{\gamma_A(a_i), \gamma_B(a_i)\}\} \\
&= \min\{\min_i\{\gamma_A(a_i), \gamma_B(a_i)\}\} \\
&= \min\{\min_i\{\gamma_A(a_i)\}, \min_i\{\gamma_B(a_i)\}\} \\
&= \min\{\gamma_{A_x}(f(x)), \gamma_{B_x}(f(x))\} \\
&= \gamma_{A_x\cap B_x}(f(x)),
\end{aligned}$$

$$\begin{aligned}
\psi_{(A\cap B)_x}(f(x)) &= \max_i\{\psi_{A\cap B}(a_i)\} \\
&= \max_i\{\max\{\psi_A(a_i), \psi_B(a_i)\}\} \\
&= \max\{\max_i\{\psi_A(a_i), \psi_B(a_i)\}\} \\
&= \max\{\max_i\{\psi_A(a_i)\}, \max_i\{\psi_B(a_i)\}\} \\
&= \max\{\psi_{A_x}(f(x)), \psi_{B_x}(f(x))\} \\
&= \psi_{A_x\cap B_x}(f(x)).
\end{aligned}$$

Hence, $(A\cap B)_x = A_x \cap B_x$.
(2) $\mu_{(A\cup B)_x}(f(x)) = (\mu_{(A\cup B)_x}(f(x)), \gamma_{(A\cup B)_x}(f(x)), \psi_{(A\cup B)_x}(f(x)))$, where

$$\begin{aligned}
\mu_{(A\cup B)_x}(f(x)) &= \min_i\{\mu_{A\cup B}(a_i)\} \\
&= \min_i\{\max\{\mu_A(a_i), \mu_B(a_i)\}\} \\
&\geq \max\{\min_i\{\mu_A(a_i), \mu_B(a_i)\}\} \\
&= \max\{\min_i\{\mu_A(a_i)\}, \min_i\{\mu_B(a_i)\}\} \\
&= \max\{\mu_{A_x}(f(x)), \mu_{B_x}(f(x))\} \\
&= \mu_{A_x\cup B_x}(f(x)),
\end{aligned}$$

$$\begin{aligned}
\gamma_{(A\cup B)_x}(f(x)) &= \min_i\{\gamma_{A\cup B}(a_i)\} \\
&= \min_i\{\max\{\gamma_A(a_i), \gamma_B(a_i)\}\} \\
&\geq \max\{\min_i\{\gamma_A(a_i), \gamma_B(a_i)\}\} \\
&= \max\{\min_i\{\gamma_A(a_i)\}, \min_i\{\gamma_B(a_i)\}\} \\
&= \max\{\gamma_{A_x}(f(x)), \gamma_{B_x}(f(x))\} \\
&= \gamma_{A_x\cup B_x}(f(x)),
\end{aligned}$$

$$\begin{aligned}
\psi_{(A\cup B)_x}(f(x)) &= \max_i\{\psi_{A\cup B}(a_i)\} \\
&= \max_i\{\min\{\psi_A(a_i), \psi_B(a_i)\}\} \\
&\leq \min\{\max_i\{\psi_A(a_i), \psi_B(a_i)\}\} \\
&= \min\{\max_i\{\psi_A(a_i)\}, \max_i\{\psi_B(a_i)\}\} \\
&= \min\{\psi_{A_x}(f(x)), \psi_{B_x}(f(x))\} \\
&= \psi_{A_x\cup B_x}(f(x)).
\end{aligned}$$

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Hence, $(A \cup B)_x \supseteq A_x \cup B_x$.
(3) Now, $(A_x + B_x)(f(x)) = (\mu_{A_x+B_x}(f(x)), \gamma_{A_x+B_x}(f(x)), \psi_{A_x+B_x}(f(x)))$, where

$$\begin{aligned}
\mu_{A_x+B_x}(f(x)) &= \max_{f(x)=g(x)+h(x)}\{\min\{\mu_{A_x}(g(x)), \mu_{B_x}(h(x))\}\},\\
&\quad \text{where } g(x) = \sum_{i=0}^{p} b_i x_i,\ h(x) = \sum_{i=0}^{p} c_i x_i\\
&= \max_{f(x)=g(x)+h(x)}\{\min\{\min_i\{\mu_A(b_i)\}, \min_i\{\mu_B(c_i)\}\}\}\\
&= \max_{a_i=b_i+c_i}\{\min\{\min_i\{\mu_A(b_i)\}, \min_i\{\mu_B(c_i)\}\}\}\\
&\leq \min_i\{\max_{a_i=b_i+c_i}\{\min\{\mu_A(b_i), \mu_B(c_i)\}\}\}\\
&= \min_i\{\mu_{A+B}(a_i)\}\\
&= \mu_{(A+B)_x}(f(x)),
\end{aligned}$$

$$\begin{aligned}
\gamma_{A_x+B_x}(f(x)) &= \max_{f(x)=g(x)+h(x)}\{\min\{\gamma_{A_x}(g(x)), \gamma_{B_x}(h(x))\}\},\\
&\quad \text{where } g(x) = \sum_{i=0}^{p} b_i x_i,\ h(x) = \sum_{i=0}^{p} c_i x_i\\
&= \max_{f(x)=g(x)+h(x)}\{\min\{\min_i\{\gamma_A(b_i)\}, \min_i\{\gamma_B(c_i)\}\}\}\\
&= \max_{a_i=b_i+c_i}\{\min\{\min_i\{\gamma_A(b_i)\}, \min_i\{\gamma_B(c_i)\}\}\}\\
&\leq \min_i\{\max_{a_i=b_i+c_i}\{\min\{\gamma_A(b_i), \gamma_B(c_i)\}\}\}\\
&= \min_i\{\gamma_{A+B}(a_i)\}\\
&= \gamma_{(A+B)_x}(f(x)),
\end{aligned}$$

$$\begin{aligned}
\psi_{A_x+B_x}(f(x)) &= \min_{f(x)=g(x)+h(x)}\{\max\{\psi_{A_x}(g(x)), \psi_{B_x}(h(x))\}\},\\
&\quad \text{where } g(x) = \sum_{i=0}^{p} b_i x_i,\ h(x) = \sum_{i=0}^{p} c_i x_i\\
&= \min_{f(x)=g(x)+h(x)}\{\max\{\max_i\{\psi_A(b_i)\}, \max_i\{\psi_B(c_i)\}\}\}\\
&= \min_{a_i=b_i+c_i}\{\max\{\max_i\{\psi_A(b_i)\}, \max_i\{\psi_B(c_i)\}\}\}\\
&\geq \max_i\{\min_{a_i=b_i+c_i}\{\max\{\psi_A(b_i), \psi_B(c_i)\}\}\}\\
&= \max_i\{\psi_{A+B}(a_i)\}\\
&= \psi_{(A+B)_x}(f(x)).
\end{aligned}$$

(4) Now, $(A_xB_x)(f(x)) = (\mu_{A_xB_x}(f(x)), \gamma_{A_xB_x}(f(x)), \psi_{A_xB_x}(f(x)))$, where

$$\begin{aligned}
\mu_{A_xB_x}(f(x)) &= \sup_{f(x)=g(x)h(x)}\{\min\{\mu_{A_x}(g(x)), \mu_{B_x}(h(x))\}\},\\
&\quad \text{where } g(x) = \sum_{i=0}^{n} b_i x_i,\ h(x) = \sum_{i=0}^{m} c_i x_i,\ n+m=p\\
&= \sup_{a_i=\sum_{i=0}^{n+m=p}(b_i c_{n+m-i})}\{\min\{\min_i\{\mu_A(b_i)\}, \min_i\{\mu_B(c_{n+m-i})\}\}\}\\
&= \sup_{a_i=\sum_{i=0}^{n+m=p}(b_i c_{n+m-i})}\{\min_i\{\min\{\mu_A(b_i)\}, \min_i\{\mu_B(c_{n+m-i})\}\}\}\\
&\leq \min_i\{\sup_{a_i=\sum_{i=0}^{n+m=p}(b_i c_{n+m-i})}\{\min\{\mu_A(b_i)\}, \min_i\{\mu_B(c_{n+m-i})\}\}\}\\
&= \min_i\{\mu_{AB}(a_i)\}\\
&= \mu_{(AB)_x}(f(x)),
\end{aligned}$$

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$$\begin{aligned}
\gamma_{A_xB_x}(f(x)) &= \sup_{f(x)=g(x)h(x)}\{\min\{\gamma_{A_x}(g(x)),\gamma_{B_x}(h(x))\}\},\\
&\quad \text{where } g(x)=\sum_{i=0}^{n} b_ix_i,\ h(x)=\sum_{i=0}^{m} c_ix_i,\ n+m=p\\
&= \sup_{a_i=\sum_{i=0}^{n+m=p}(b_ic_{n+m-i})}\{\min\{\min_i\{\gamma_A(b_i)\},\min_i\{\gamma_B(c_{n+m-i})\}\}\}\\
&= \sup_{a_i=\sum_{i=0}^{n+m=p}(b_ic_{n+m-i})}\{\min_i\{\min\{\gamma_A(b_i)\},\min_i\{\gamma_B(c_{n+m-i})\}\}\}\\
&\leq \min_i\{\sup_{a_i=\sum_{i=0}^{n+m=p}(b_ic_{n+m-i})}\{\min\{\gamma_A(b_i)\},\min_i\{\gamma_B(c_{n+m-i})\}\}\}\\
&= \min_i\{\gamma_{AB}(a_i)\}\\
&= \gamma_{(AB)_x}(f(x)),\\
\psi_{A_xB_x}(f(x)) &= \inf_{f(x)=g(x)h(x)}\{\max\{\psi_{A_x}(g(x)),\psi_{B_x}(h(x))\}\},\\
&\quad \text{where } g(x)=\sum_{i=0}^{n} b_ix_i,\ h(x)=\sum_{i=0}^{m} c_ix_i,\ n+m=p\\
&= \inf_{a_i=\sum_{i=0}^{n+m=p}(b_ic_{n+m-i})}\{\max\{\max_i\{\psi_A(b_i)\},\max_i\{\psi_B(c_{n+m-i})\}\}\}\\
&= \inf_{a_i=\sum_{i=0}^{n+m=p}(b_ic_{n+m-i})}\{\max_i\{\max\{\psi_A(b_i)\},\max_i\{\psi_B(c_{n+m-i})\}\}\}\\
&\geq \max_i\{\inf_{a_i=\sum_{i=0}^{n+m=p}(b_ic_{n+m-i})}\{\max\{\psi_A(b_i)\},\max_i\{\psi_B(c_{n+m-i})\}\}\}\\
&= \max_i\{\psi_{AB}(a_i)\}\\
&= \psi_{(AB)_x}(f(x)).
\end{aligned}$$

Hence, $A_xB_x \subseteq (AB)_x$. □

Theorem 3.11. *Let $f : R \to R'$ be a homomorphism from R onto R'. If A and B are neutrosophic ideals of R', then*

(1) $f^{-1}(A\cap B) = f^{-1}(A)\cap f^{-1}(B)$,

(2) $f^{-1}(A\cup B) = f^{-1}(A)\cup f^{-1}(B)$.

Proof. Let $x \in \mathbb{R}$.
(1) Now, $f^{-1}(A\cap B)(x) = (\mu_{f^{-1}(A\cap B)}(x), \gamma_{f^{-1}(A\cap B)}(x), \psi_{f^{-1}(A\cap B)}(x))$, where

$$\begin{aligned}
\mu_{f^{-1}(A\cap B)}(x) &= \mu_{(A\cap B)}(f(x))\\
&= \min\{\mu_A(f(x)),\mu_B(f(x))\}\\
&= \min\{\mu_{f^{-1}(A)}(x),\mu_{f^{-1}(B)}(x)\}\\
&= \mu_{f^{-1}(A)\cap f^{-1}(B)}(x),
\end{aligned}$$

$$\begin{aligned}
\gamma_{f^{-1}(A\cap B)}(x) &= \gamma_{(A\cap B)}(f(x))\\
&= \min\{\gamma_A(f(x)), fB(f(x))\}\\
&= \min\{\gamma_{f^{-1}(A)}(x),\gamma_{f^{-1}(B)}(x)\}\\
&= \gamma_{f^{-1}(A)\cap f^{-1}(B)}(x),
\end{aligned}$$

$$\begin{aligned}
\psi_{f^{-1}(A\cap B)}(x) &= \psi_{(A\cap B)}(f(x))\\
&= \max\{\psi_A(f(x)),\psi_B(f(x))\}\\
&= \max\{\psi_{f^{-1}(A)}(x),\psi_{f^{-1}(B)}(x)\}\\
&= \psi_{f^{-1}(A)\cup f^{-1}(B)}(x).
\end{aligned}$$

Hence, $f^{-1}(A\cap B) = f^{-1}(A)\cap f^{-1}(B)$.
(2) Now, $f^{-1}(A\cup B)(x) = (\mu_{f^{-1}(A\cup B)}(x), \gamma_{f^{-1}(A\cup B)}(x), \psi_{f^{-1}(A\cup B)}(x))$, where

$$\begin{aligned}
\mu_{f^{-1}(A\cup B)}(x) &= \mu_{A\cup B}(f(x))\\
&= \max\{\mu_A(f(x)), fB(f(x))\}\\
&= \max\{\mu_{f^{-1}(A)}(x),\mu_{f^{-1}(B)}(x)\}\\
&= \mu_{f^{-1}(A)\cup f^{-1}(B)}(x),
\end{aligned}$$

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$$\begin{aligned}
\gamma_{f^{-1}(A\cup B)}(x) &= \gamma_{A\cup B}(f(x)) \\
&= \max\{\gamma_A(f(x)), fB(f(x))\} \\
&= \max\{\gamma_{f^{-1}(A)}(x), \gamma_{f^{-1}(B)}(x)\} \\
&= \gamma_{f^{-1}(A)\cup f^{-1}(B)}(x), \\
\psi_{f^{-1}(A\cup B)}(x) &= \psi_{A\cup B}(f(x)) \\
&= \min\{\psi_A(f(x)), \psi_B(f(x))\} \\
&= \min\{\psi_{f^{-1}(A)}(x), \psi_{f^{-1}(B)}(x)\} \\
&= \psi_{f^{-1}(A)\cap f^{-1}(B)}(x).
\end{aligned}$$

Hence, $f^{-1}(A\cup B) = f^{-1}(A)\cup f^{-1}(B)$. □

Corollary 3.12. *Let $f : R \to R'$ be a homomorphism from R onto R'. Let f_x be an induced homomorphism of f. If A and B are neutrosophic ideals of R', then*

(1) $f_x^{-1}((A\cap B)_x) = f_x^{-1}(A_x)\cap f_x^{-1}(B_x)$.

(2) $f_x^{-1}((A\cup B)_x) \supseteq f_x^{-1}(A_x)\cup f_x^{-1}(B_x)$.

Proof. (1) It follows from Theorem 3.10 (1) and Theorem 3.11 (2) that $f_x^{-1}((A\cap B)_x) = f_x^{-1}(A_x\cap B_x) = f_x^{-1}(A_x)\cap f_x^{-1}(B_x)$.
(2) By Theorem 3.10 (2), we have $A_x\cup B_x \subseteq (A\cup B)_x$, so $f_x^{-1}(A_x\cup B_x)\subseteq f_x^{-1}((A\cup B)_x)$. By applying Theorem 3.10 (2) and Theorem 3.11 (2), we obtain $f_x^{-1}(A_x)\cup f_x^{-1}(B_x) = f_x^{-1}(A_x\cup B_x)\subseteq f_x^{-1}((A\cup B)_x)$, which proves (2). □

Theorem 3.13. *Let $f : R\to R'$ be a homomorphism from R onto R' and let f_x be an induced homomorphism of f. If A is a neutrosophic ideal of R', then $(f^{-1}(A))_x = f_x^{-1}(A_x)$.*

Proof. Let $r(x) = \sum_{i=0}^{n} a_i x_i$ be any element of $R[x]$. Then
$(f^{-1}(A))_x(r(x)) = (\mu_{(f^{-1}(A))_x}(r(x)), \gamma_{(f^{-1}(A))_x}(r(x)), \psi_{(f^{-1}(A))_x}(r(x)))$, where

$$\begin{aligned}
\mu_{(f^{-1}(A))_x}(r(x)) &= \min_i\{\mu_{f^{-1}(A)}(a_i)\} \\
&= \min_i\{\mu_A(f(a_i))\} \\
&= \mu_{A_x}(f_x(r(x))) \\
&= \mu_{f_x^{-1}(A_x)}(r(x)), \\
\gamma_{(f^{-1}(A))_x}(r(x)) &= \min_i\{\gamma_{f^{-1}(A)}(a_i)\} \\
&= \min_i\{\gamma_A(f(a_i))\} \\
&= \gamma_{A_x}(f_x(r(x))) \\
&= \gamma_{f_x^{-1}(A_x)}(r(x)), \\
\psi_{(f^{-1}(A))_x}(r(x)) &= \max_i\{\psi_{f^{-1}(A)}(a_i)\} \\
&= \max_i\{\psi_A(f(a_i))\} \\
&= \psi_{A_x}(f_x(r(x))) \\
&= \psi_{f_x^{-1}(A_x)}(r(x)).
\end{aligned}$$

Hence, $(f^{-1}(A))_x = f_x^{-1}(A_x)$. □

Theorem 3.14. *Let $f : R\to R'$ be a homomorphism from R onto R' and let f_x be an induced homomorphism of f. If A is an f-invariant neutrosophic ideals of R', then $(f(A))_x = f_x(A_x)$.*

Proof. For any polynomial $s(x) = \sum_{i=0}^{m} b_i x_i \in R[x]$, we let $h_j(x) = \sum_{i=0}^{m} b_{ji}x_i \in R[x]$. Then $A_x(h_j(x)) = (\mu_{A_x}(h_j(x)), \gamma_{A_x}(h_j(x)), \psi_{A_x}(h_j(x)))$, where $\mu_{A_x}(h_j(x)) = \min_i\{\mu_A(a_{ji})\}$, $\gamma_{A_x}(h_j(x)) = \min_i\{\gamma_A(a_{ji})\}$, and $\psi_{A_x}(h_j(x)) = \min_i\{\psi_A(a_{ji})\}$. Assume that $f_x(h_j(x)) = s(x)$ and $f_x(h_k(x)) = s(x)$. Then $\sum_{i=0}^{m} f(a_{ji})x_i = \sum_{i=0}^{m} b_i x_i$ and $\sum_{i=0}^{m} f(a_{ki})x_i = \sum_{i=0}^{m} b_i x_i$. It follows that $f(a_{ji}) = b_i = f(a_{ki})$, $\forall i = 1, 2, \ldots, m$. Hence,

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$\mu_{A_x}(h_j(x)) = \min_i\{\mu_A(a_{ji})\} = \min_i\{\mu_A(a_{ki})\} = \mu_{A_x}(h_k(x))$. Similarly, we can show that $\gamma_{A_x}(h_j(x)) = \min_i\{\gamma_A(a_{ji})\} = \min_i\{\gamma_A(a_{ki})\} = \gamma_{A_x}(h_k(x))$ and $\psi_{A_x}(h_j(x)) = \psi_{A_x}(h_k(x))$. Now, $(f_x(A_x))(s(x)) = (\mu_{f_x(A_x)}(s(x)), \psi_{f_x(A_x)}(s(x)))$, where

$$\begin{aligned}
\mu_{f_x(A_x)}(s(x)) &= \sup\{\mu_{A_x}(h_j(x)) : h_j(x) = \sum_{i=0}^{m} a_{j_i}x_i \text{ such that } f_x(h_j(x)) = s(x)\} \\
&= \sup\{\min_i\{\mu_A(a_{j_i})\}, j = 1, 2, \ldots\} \\
&= \mu_{A_x}(h_j(x)),
\end{aligned}$$

$$\begin{aligned}
\gamma_{f_x(A_x)}(s(x)) &= \sup\{\gamma_{A_x}(h_j(x)) : h_j(x) = \sum_{i=0}^{m} a_{j_i}x_i \text{ such that } f_x(h_j(x)) = s(x)\} \\
&= \sup\{\min_i\{\gamma_A(a_{j_i})\}, j = 1, 2, \ldots\} \\
&= \gamma_{A_x}(h_j(x)),
\end{aligned}$$

$$\begin{aligned}
\psi_{f_x(A_x)}(s(x)) &= \inf\{\psi_{A_x}(h_j(x)) : h_j(x) = \sum_{i=0}^{m} a_{j_i}x_i \text{ such that } f_x(h_j(x)) = s(x)\} \\
&= \inf\{\max_i\{\psi_A(a_{j_i})\}, j = 1, 2, \ldots\} \\
&= \psi_{A_x}(h_j(x)).
\end{aligned}$$

Now, let $i = 1, 2, \ldots, m$. Since A is f-invariant, we have $(f(A))(b_i) = (\mu_{f(A)}(b_i), \gamma_{f(A)}(b_i), \psi_{f(A)}(b_i))$, where

$$\mu_{f(A)}(b_i) = \sup\{\mu_A(a_{ji}), a_{ji} \in \mathbb{R}, f(a_{ji}) = b_i\} = \mu_A(a_{0i}) = \mu_A(a_1 i) = \ldots = \mu_A(a_{ji}),$$

$$\gamma_{f(A)}(b_i) = \sup\{\gamma_A(a_{ji}), a_{ji} \in \mathbb{R}, f(a_{ji}) = b_i\} = \gamma_A(a_{0i}) = \gamma_A(a_1 i) = \ldots = \gamma_A(a_{ji}),$$

$$\psi_{f(A)}(b_i) = \inf\{\psi_A(a_{ji}), a_{ji} \in \mathbb{R}, f(a_{ji}) = b_i\} = \psi_A(a_{0i}) = \psi_A(a_1 i) = \ldots = \psi_A(a_{ji}).$$

It follows from Theorem 3.4 that

$$\begin{aligned}
\mu_{(f(A))_x}(s(x)) &= \min_i\{\mu_{f(A)}(b_i)\} \\
&= \min_i\{\mu_{f(A)}(b_0), \mu_{f(A)}(b_1), \ldots\} \\
&= \min_i\{\mu_A(a_{j0}), \mu_A(a_{j1}), \ldots\} \\
&= \mu_{A_x}\{\sum_{i=0}^{m} a_{ji}\} \\
&= \mu_{f_x(A_x)}(s(x)),
\end{aligned}$$

$$\begin{aligned}
\gamma_{(f(A))_x}(s(x)) &= \min_i\{\gamma_{f(A)}(b_i)\} \\
&= \min_i\{\gamma_{f(A)}(b_0), \gamma_{f(A)}(b_1), \ldots\} \\
&= \min_i\{\gamma_A(a_{j0}), \gamma_A(a_{j1}), \ldots\} \\
&= \gamma_{A_x}\{\sum_{i=0}^{m} a_{ji}\} \\
&= \gamma_{f_x(A_x)}(s(x)),
\end{aligned}$$

$$\begin{aligned}
\psi_{(f(A))_x}(s(x)) &= \max_i\{\psi_{f(A)}(b_i)\} \\
&= \max_i\{\psi_{f(A)}(b_0), \psi_{f(A)}(b_1), \ldots\} \\
&= \max_i\{\psi_A(a_{j_0}), \psi_A(a_{j_1}), \ldots\} \\
&= \psi_{A_x}\{\sum_{i=0}^{m} a_{j_i}\} \\
&= \psi_{f_x(A_x)}(s(x)).
\end{aligned}$$

Hence, $(f(A))_x = f_x(A_x)$. □

Definition 3.15. Let A be a neutrosophic ideal of a ring R and let A_x be a neutrosophic polynomial ideal of $R[x]$. For any $f(x) \in R[x]$, define a neutrosophic set $f(x) + A_x$ on $R[x]$ by
$f(x) + A_x = (\mu_{f(x)+A_x}, \gamma_{f(x)+A_x}, \psi_{f(x)+A_x})$, where

$$\mu_{f(x)+A_x}(g(x)) = \mu_{A_x}(f(x) - g(x)),$$

$$\gamma_{f(x)+A_x}(g(x)) = \gamma_{A_x}(f(x) - g(x)),$$

$$\psi_{f(x)+A_x}(g(x)) = \psi_{A_x}(f(x) - g(x))$$

for all $g(x) \in R[x]$. Then $f(x) + A_x$ is called a neutrosophic coset of $R[x]$ determined by $f(x)$ and A_x.

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Theorem 3.16. *Let A be a neutrosophic ideal of a ring R and let A_x be a neutrosophic polynomial ideal of $R[x]$. Then $R[x] = A_x$, the set of all neutrosophic cosets of A_x form a ring under the composition defined by*

$$(f(x) + A_x) + (g(x) + A_x) = (f(x) + g(x)) + A_x$$

and

$$(f(x) + A_x)(g(x) + A_x) = (f(x)g(x)) + A_x$$

for all $f(x), g(x) \in R[x]$.

Proof. Straightforward. □

Lemma 3.17. *Let A be a neutrosophic ideal of a ring R and let A_x be a neutrosophic polynomial ideal of $R[x]$. Then $f(x) + A_x = g(x) + A_x$ if and only if $A_x(f(x) - g(x)) = A_x(0)$ for all $f(x), g(x) \in R[x]$.*

Proof. Firstly, assume that $f(x) + A_x = g(x) + A_x$. Then $(f(x) + A_x)(f(x)) = (g(x) + A_x)(f(x))$ implies that $(\mu_{A_x}(f(x) - f(x)), \psi_{A_x}(f(x) - f(x))) = (\mu_{A_x}(g(x) - f(x)), \psi_{A_x}(g(x) - f(x)))$, that is, $(\mu_{A_x}(0), \psi_{A_x}(0)) = (\mu_{A_x}(g(x) - f(x)), \psi_{A_x}(g(x) - f(x))))$. Thus $\mu_{A_x}(g(x) - f(x)) = \mu_{A_x}(0)$ and $\psi_{A_x}(g(x) - f(x))) = \psi_{A_x}(0))$. Hence, $A_x(g(x) - f(x)) = A_x(0)$.

Conversely, assume that $A_x(g(x) - f(x)) = A_x(0)$ for all $f(x), g(x) \in R[x]$. Let $h(x) \in R[x]$. Then

$$\begin{aligned}
\mu_{f(x)+A_x}(h(x)) &= \mu_{A_x}(h(x) - f(x)) \\
&= \mu_{A_x}(h(x) - g(x) + g(x) - f(x)) \\
&\geq \min\{\mu_{\mu_{A_x}}(h(x) - g(x)), \mu_{A_x}(g(x) - f(x))\} \\
&= \min\{\mu_{A_x}(h(x) - g(x)), \mu_{A_x}(0)\} \\
&= \mu_{A_x}(h(x) - g(x)) \\
&= \mu_{g(x)+A_x}(h(x)),
\end{aligned}$$

$$\begin{aligned}
\gamma_{f(x)+A_x}(h(x)) &= \gamma_{A_x}(h(x) - f(x)) \\
&= \gamma_{A_x}(h(x) - g(x) + g(x) - f(x)) \\
&\geq \min\{\gamma_{\gamma_{A_x}}(h(x) - g(x)), \gamma_{A_x}(g(x) - f(x))\} \\
&= \min\{\gamma_{A_x}(h(x) - g(x)), \gamma_{A_x}(0)\} \\
&= \gamma_{A_x}(h(x) - g(x)) \\
&= \gamma_{g(x)+A_x}(h(x)),
\end{aligned}$$

$$\begin{aligned}
\psi_{f(x)+A_x}(h(x)) &= \psi_{A_x}(h(x) - f(x)) \\
&= \psi_{A_x}(h(x) - g(x) + g(x) - f(x)) \\
&\leq \max\{\psi_{\mu_{A_x}}(h(x) - g(x)), \psi_{A_x}(g(x) - f(x))\} \\
&= \max\{\psi_{A_x}(h(x) - g(x)), \psi_{A_x}(0)\} \\
&= \psi_{A_x}(h(x) - g(x)) \\
&= \psi_{g(x)+A_x}(h(x)).
\end{aligned}$$

Thus $g(x) + A_x \subseteq f(x) + A_x$. In the same way, we can show that $f(x) + A_x \subseteq g(x) + A_x$. Hence, $f(x) + A_x = g(x) + A_x$. □

4 Prime neutrosophic polynomial ideals

In this section, we study some properties of prime neutrosophic polynomial ideals.

Definition 4.1. A neutrosophic ideal P of a ring R, not necessary constant, is said to be a neutrosophic prime ideal if for any neutrosophic ideals A and B of R, $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.

Proposition 4.2. *Let A be a neutrosophic prime ideal of a ring R. Then A_* is a prime ideal of R.*

Theorem 4.3. *Let A be a neutrosophic ideal of a ring R. Then A is a neutrosophic prime ideal of R if and only if A_x is a neutrosophic prime ideal of $R[x]$.*

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Proof. Let A be a neutrosophic prime ideal of R. Then A_* is a prime ideal of R. By Theorem 3.4, we have A_x is a neutrosophic ideal of $R[x]$. To show that A_x is a neutrosophic prime ideal of $R[x]$, we have to show that, by Theorem 3.9, $(A_x) = A_*[x]$ is a prime ideal of $R[x]$. Assume that $A_*[x]$ is not a prime ideal of $R[x]$. Then there exist polynomials $f(x) = \sum_{i=0}^{n} a_i x_i, g(x) = \sum_{i=0}^{m} b_i x_i \in R[x]$ such that $f(x)g(x) \in A_*[x]$, but $f(x), g(x) \notin A_*[x]$. Let i be the first smallest non-negative integer such that $\mu_A(a_i) \neq \mu_A(0)$, $\gamma_A(a_i) \neq \gamma_A(0)$, and $\psi_A(a_i) \neq \psi_A(0)$ and let j be the first smallest non-negative integer such that $\mu_A(b_i) \neq \mu_A(0)$, $\gamma_A(b_i) \neq \gamma_A(0)$, and $\psi_A(b_i) \neq \psi_A(0)$. Since $f(x)g(x) \in A_*[x]$, we have $\sum_{p,q=0,p+q=i+j}^{i+j} a_p b_q \in A_*$, because a_p (where $p = 0, 1, \ldots, i-1$) and b_p (where $p = 0, 1, \ldots, j-1$) are all in A_*. Thus $a_i b_j \in A_*$. Since A_* is a prime ideal of R, either $\mu_A(a_i) = \mu_A(0)$, $\gamma_A(a_i) = \gamma_A(0)$, and $\psi_A(a_i) = \psi_A(0)$ or $\mu_A(b_i) = \mu_A(0)$, $\gamma_A(b_i) = \gamma_A(0)$, and $\psi_A(b_i) = \psi_A(0)$, a contradiction. Hence, A_x is a neutrosophic prime ideal of $R[x]$.

Conversely, assume that A_x is a neutrosophic prime ideal of $R[x]$. We claim that A_* is a prime ideal of R. Let $a, b \in \mathbb{R}$ be such that $ab \in A_*$. Then $(ax)(bx) = abx^2 \in A_*[x] = (A_x)$. Since (A_x) is a prime ideal of $R[x]$, either $(ax) \in (A_x)$ or $(bx) \in (A_x)$, which shows that either $a \in A_*$ or $b \in A_*$. Hence, A is a neutrosophic prime ideal of R. □

Theorem 4.4. *Let $f : R \to R'$ be an epimorphism from R onto R' and let B be a neutrosophic ideal of R'. If B is a neutrosophic prime ideal of R', then $f^{-1}(B)$ is a neutrosophic prime ideal of R.*

Proof. Firstly, assume that B is a neutrosophic prime ideal of R'. Then B_* is a prime ideal of R'. Clearly, $f^{-1}(B)$ is a neutrosophic ideal of R. We claim that $(f^{-1}(B))$ is a prime ideal of R. Let $a, b \in R$ be such that $ab \in (f^{-1}(B))$. Then $\mu_{f^{-1}(B)}(ab) = \mu_{f^{-1}(B)}(0)$, $\gamma_{f^{-1}(B)}(ab) = \gamma_{f^{-1}(B)}(0)$, and $\psi_{f^{-1}(B)}(ab) = \psi_{f^{-1}(B)}(0)$, that is, $\mu_B(f(ab)) = \mu_B(0')$, $\gamma_B(f(ab)) = \gamma_B(0')$, and $\psi_B(f(ab)) = \psi_B(0')$). Thus $f(a)f(b) = f(ab) \in B_*$. Since B_* is a prime ideal of R', either $f(a) \in B_*$ or $f(b) \in B_*$. This means that either $\mu_B(f(a)) = \mu_B(0')$, $\gamma_B(f(a)) = \gamma_B(0')$, and $\psi_B(f(a)) = \psi_B(0')$ or $\mu_B(f(b)) = \mu_B(0')$, $\gamma_B(f(b)) = \gamma_B(0')$, and $\psi_B(f(b)) = \psi_B(0')$, that is, either $\mu_{f^{-1}(B)}(a) = \mu_{f^{-1}(B)}(0)$, $\gamma_{f^{-1}(B)}(a) = \gamma_{f^{-1}(B)}(0)$, and $\psi_{f^{-1}(B)}(a) = \psi_{f^{-1}(B)}(0)$ or $\mu_{f^{-1}(B)}(b) = \mu_{f^{-1}(B)}(0)$, $\gamma_{f^{-1}(B)}(b) = \gamma_{f^{-1}(B)}(0)$, and $\psi_{f^{-1}(B)}(b) = \psi_{f^{-1}(B)}(0)$. Thus either $a \in (f^{-1}(B))_*$ or $b \in (f^{-1}(B))_*$, that is, $(f^{-1}(B))$ is a prime ideal of R. Hence, $f^{-1}(B)$ is a neutrosophic ideal of R. □

Theorem 4.5. *Let $f : R \to R'$ be an epimorphism from R onto R' and let A be an f-invariant neutrosophic ideal of R. Then A is a neutrosophic prime ideal of R if and only if $f(A_*)$ is a prime ideal of R'.*

Proof. Firstly, assume that A is a neutrosophic prime ideal of R. Then A_* is a prime ideal of R. Let $x, y \in \mathbb{R}'$ be such that $xy \in f(A_*)$. Since f is onto, there exists $c \in A_*$ such that $f(c) = xy$ and there exist $a, b \in \mathbb{R}$ such that $f(a) = x$ and $f(b) = y$. Thus $f(ab) = f(a)f(b) = xy = f(c)$. As A is f-invariant, therefore, $\mu_A(ab) = \mu_A(c) = \mu_A(0)$, $\gamma_A(ab) = \gamma_A(c) = \gamma_A(0)$, and $\psi_A(ab) = \psi_A(c) = \psi_A(0)$. Thus $ab \in A_*$. Since A_* is a prime ideal of R, either $a \in A_*$ or $b \in A_*$, which shows that either $x = f(a) \in f(A_*)$ or $y = f(b) \in f(A_*)$. Hence, $f(A_*)$ is a prime ideal of R'.

Conversely, assume that $f(A_*)$ is a prime ideal of R' and let $a, b \in \mathbb{R}$ be such that $ab \in A_*$. Thus $f(a)f(b) = f(ab) \in f(A_*)$. Since $f(A_*)$ is a prime ideal of R' , either $f(a) \in f(A_*)$ or $f(b) \in f(A_*)$, which implies that either there exists $a' \in A_*$ such that $f(a) = f(a')$ or there exists $b' \in A_*$ such that $f(b) = f(b')$. Since A is f-invariant, either $\mu_A(a) = \mu_A(a') = \mu_A(0)$, $\gamma_A(a) = \gamma_A(a') = \gamma_A(0)$, and $\psi_A(a) = \psi_A(a') = \psi_A(0)$ or $\mu_A(b) = \mu_A(b') = \mu_A(0)$, $\gamma_A(b) = \gamma_A(b') = \gamma_A(0)$, and $\psi_A(b) = \psi_A(b') = \psi_A(0)$, that is, either $a \in A_*$ or $b \in A_*$. Hence, A_* is a prime ideal of R and hence, A is a neutrosophic prime ideal of R. □

Corollary 4.6. *Let $f : R \to R'$ be an epimorphism from R onto R' and let A be an f-invariant neutrosophic ideal of R. Then A is a neutrosophic prime ideal of R if and only if $f(A)$ is a neutrosophic prime ideal of R'.*

Corollary 4.7. *Let $f : R \to R'$ be an epimorphism from R onto R' and let f_x be an induced homomorphism of f. Then a neutrosophic ideal B of R' is a neutrosophic prime ideal of R' if and only if $f_x^{-1}(B_x)$ is a neutrosophic prime ideal of $R[x]$.*

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Corollary 4.8. *Let $f : R \to R'$ be an epimorphism from R onto R' and let f_x be an induced homomorphism of f. Then a neutrosophic ideal A of R is a neutrosophic prime ideal of R if and only if $f_x(A_x)$ is a neutrosophic prime ideal of $R'[x]$.*

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