Characterization of fuzzy algebraic structure based on diophantine Q-neutrosophic subbisemiring of bisemiring

V. Sreelatha devi[1], **M. Palanikumar**[1], **Aiyared Iampan**[2*]
[1]Department of Mathematics, Saveetha School of Engineering, Saveetha University, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India
[2]Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand
Emails: sreelatu@gmail.com; palanimaths86@gmail.com; aiyared.ia@up.ac.th

* Corresponding author:aiyared.ia@up.ac.th

Abstract

We propose the concept of diophantine Q-neutrosophic subbisemiring(DioQNSBS), level sets of DioQNSBS of a bisemiring. The idea of DioQNSBS is an extension of fuzzy subbisemiring over bisemiring. Exploring the concept for DioQNSBS over bisemiring. Let H be the diophantine Q-neutrosophic subset in $\mathbb{D}$, prove $H = \langle(\Gamma_H^{\mathcal{T}}, \Gamma_H^{\mathcal{I}}, \Gamma_H^{\mathcal{F}}), (\Lambda_H, \Xi_H, \Phi_H)\rangle$ is a DioQNSBS of $\mathbb{D}$ if and only if all non empty level set $H^{(t,s)}$ is a subbisemiring of $\mathbb{D}$ for $t, s \in [0, 1]$. Let H be the DioQNSBS of a bisemiring $\mathbb{D}$ and M be the strongest diophantine Q-neutrosophic relation (SDioQNSR)of $\mathbb{D}$, we notice H is a DioQNSBS of $\mathbb{D}$ if and only if M is a DioQNSBS of $\mathbb{D} \times \mathbb{D}$. Let $H_1, H_2, ..., H_n$ be the family of DioQNSBSs of $\mathbb{D}_1, \mathbb{D}_2, ..., \mathbb{D}_n$ respectively, prove $H_1 \times H_2 \times ... \times H_n$ is a DioQNSBS of $\mathbb{D}_1 \times \mathbb{D}_2 \times ... \times \mathbb{D}_n$. The homomorphic image of DioQNSBS is a DioQNSBS. The homomorphic preimage of DioQNSBS is a DioQNSBS. Illustrations are presented to demonstrate results.

Keywords: Homomorphism; neutrosophic subbisemiring;fuzzy subbisemiring;Diophantine neutrosophic bisemiring;Q diophantine neutrosophic subbisemiring.

1 Introduction

The exploration of semirings was initiated by Dedekind's involvement with the theories of commutative rings. German mathematician Dedekind initiated the research of semirings in relation to the ideals of commutative rings.As part of the extension of rings, Vandiver introduces the idea of semirings.[22] Semirings were later studied by American mathematician Vandiver, who identified them as a fundamental algebraic structure in 1934.Iseki[5,6] established a semiring ideal theory that is not always commutative under any operation.Semigroups, semirings, and hypersemigroups are a few examples of ordered algebraic structures that have been researched by a number of authors.[23]Definition states that a fuzzy set is a function that can be defined by a membership value. Degrees are measured in real unit intervals. Later, however, it was shown that this definition is insufficient when taking into account both the degree of membership and the degree of non-membership.In the absence of zero, Iseki[7]adopted this conceptual framework for semirings and used semirings to establish numerous significant results.The various ideals based on semirings have been described by a number of authors and academics.[4]Though semirings theory had been developing since 1950, these developments were already taking place.

https://doi.org/10.54216/IJNS.220207

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

The neutrosophic set is an extension of the fuzzy set and intuitionistic fuzzy set in which the truth, indeterminacy, and falsity memberships are separately stated.To manage the given uncertainty, Atanassov[2] designed a set known as an intuitionistic fuzzy set.Since this set includes several application-related issues, Smarandache[21] developed neutrosophy to address the issues of vague and inconsistent information.Riaz et al. discussed[19] linear Diophantine fuzzy set (LDFS) with the addition of reference parameters.Due to the usage of reference parameters, the LDFS is more adaptable and effective than other methods. By altering the physical meaning of reference parameters, LDFS also classifies the data in MADM difficulties[15] .

In 1993, J. Ahsan, K. Saifullah, and F. Khan[1] introduced the notion of fuzzy semirings.A semiring $(\mathbb{S}, \dagger, \odot)$ is a non-empty set in which $(\mathbb{S}, \dagger,)$ and $(\mathbb{S}, \odot)$ are semigroups such that "$\odot$" is distributive over "$\dagger$".[4] A bisemiring $(\mathbb{S}, \dagger, \circledcirc, \otimes)$ is an algebraic structure in which $(\mathbb{S}, \dagger, \circledcirc)$ and $(\mathbb{S}, \circledcirc, \otimes)$ are semirings in which $(\mathbb{S}, \dagger), (\mathbb{S}, \circledcirc)$ and $(\mathbb{S}, \otimes)$ are semigroups such that (i) $a \circledcirc (b + c) = (a \circledcirc b) + (a \circledcirc c)$, (ii) $(b + c) \circledcirc a = (b \circledcirc a) + (c \circledcirc a)$ (iii) $a \times (b \circledcirc c) = (a \otimes b) \circledcirc (a \otimes c)$ and (iv) $(b \circledcirc c) \otimes a = (b \otimes a) \circledcirc (c \otimes a), \ \forall \ a, b, c \in \mathbb{S}$.[20] A non-empty subset $\mathbb{H}$ of a bisemiring $(\mathbb{S}, \dagger, \circledcirc, \otimes)$ is a subbisemiring if and only if $a + b \in \mathbb{H}, \ a \circledcirc b \in \mathbb{H}$ and $a \otimes b \in \mathbb{H}$ for all $a, b \in \mathbb{H}$.[3] Palanikumar et al. focused on numerous applications of subbisemiring theory's ideal structure[12]-.[10] A semiring is a mathematical structure that combines properties of both rings and monoids. It serves as a fundamental concept in abstract algebra and has applications in various fields such as computer science, formal language theory, optimization, and more. The concept of semirings has evolved over time through contributions from various mathematicians and researchers. The term "semiring" was coined to describe these algebraic structures.[8]The use of "semiring" in this context can be traced back to the 1960s and 1970s with the works of mathematicians and computer scientists like Arto Salomaa and Grzegorz Rozenberg. They explored algebraic structures related to formal languages and automata theory, using semirings to describe properties of languages and their operations.Fuzzy semirings are algebraic structures that generalize the notions of semirings to accommodate uncertainty or fuzziness in arithmetic operations[9] . A fuzzy semiring consists of a se along with two binary operations usually denoted by $\oplus$ and $\otimes$ that satisfy certain properties. These properties are analogous to those of classical semirings, but with the incorporation of fuzzy or uncertain values. The $\oplus$ operation is usually treated as a maximum operation whereas $\otimes$ operation is typically treated as a minimum operation.Fuzzy semirings extend the concepts of semirings by allowing the arithmetic operations to work with fuzzy or uncertain values rather than crisp (exact) values. This makes them suitable for modeling situations where there is inherent ambiguity or imprecision.Fuzzy semirings are used to extend automata theory and formal language theory to handle fuzzy languages and behaviors.[13] This is particularly useful in modeling systems where inputs, outputs, or transitions are uncertain.It can be used to aggregate information from multiple sources that provide fuzzy values. They also play a role in decision-making processes involving uncertainty and conflicting information.It also find its applications in various fields, including fuzzy logic, approximate reasoning, image processing, machine learning, and information retrieval.[23]They can represent uncertainty and imprecision in computations and reasoning, making them valuable tools for handling real-world scenarios.Semirings have been applied in a wide range of mathematical and practical contexts, including optimization, graph theory, coding theory, and more. Semirings have diverse applications across various fields due to their flexible algebraic structure that combines both additive and multiplicative properties.[17]

This paper aims to examine many aspects and provide findings.The article is composed of the guiding ideas listed as follows.Section 1 has the introduction. Section 2 contains details on the ways to prepare for DioQNSBS. Section 3 presents theorems based on DioQNSBS. The idea of homomorphism is discussed in Section 4.

2 Preliminaries

The concept of semirings and bisemirings will be covered in this section in order to make the overview as comprehensive as feasible and makes the subsequent presentations easier.

Definition 2.1. Let $\mathscr{M}$ and $\mathbb{Q}$ be non-empty sets. The mapping $\Omega : \mathscr{M} \times \mathbb{Q} \to [0, 1]$ is called a $\mathbb{Q}$ fuzzy set in $\mathscr{M}$.

Definition 2.2. Let $(\mathbb{S}, \dagger, \odot)$ be semiring. The $\mathbb{Q}$ fuzzy set Ω: $\mathscr{M} \times \mathbb{Q} \to$ [0, 1] is called a $\mathbb{Q}$ fuzzy subsemiring of $\mathscr{M}$ if $\Omega_H(\kappa, \omega, \varrho^*) \geq \min\{\Omega_H(\kappa, \varrho^*), \Omega_H(\omega, \varrho^*)\}$, $\Omega_H(\kappa \cdot \omega, \varrho^*) \geq \min\{\Omega_H(\kappa, \varrho^*), \Omega_H(\omega, \varrho^*)\}$ for all $\kappa, \omega \in \mathscr{M}$ and $\varrho^* \in \mathbb{Q}$.

https://doi.org/10.54216/IJNS.220207
Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

Definition 2.3. An intuitionistic $\mathbb{Q}$ fuzzy subset defined on $\mathscr{M}$ and $\mathbb{Q}$ is of the form $H = \{<\kappa, \Lambda_H(\kappa,\varrho^*), \Phi_H(\kappa,\varrho^*)|\kappa \in \mathfrak{X} \ and \ \varrho^* \in \mathbb{Q}>\}$ where $\Lambda_H : \mathfrak{X}\times\mathbb{Q} \to [0,1]$ and $\Phi_H : \mathfrak{X}\times\mathbb{Q} \to [0,1]$ define the MD and NMD of the element $\kappa \in \mathfrak{X}$, respectively and every $\kappa \in \mathfrak{X}$ and $\varrho^* \in \mathbb{Q}$ satisfying $0 \le \Lambda_H(\kappa,\varrho^*) + \Phi_H(\kappa,\varrho^*) \le 1$.

Definition 2.4. Let H_1 and H_2 be any two intuitionistic $\mathbb{Q}$ fuzzy subsets of $\mathscr{M}$. Then
$H_1 \cap H_2 = \left\{ \langle \kappa, \min\{\Theta_{H_1}(\kappa,\varrho^*), \Theta_{H_2}(\kappa,\varrho^*)\}, \max\{\Phi_{H_1}(\kappa,\varrho^*), \Phi_{H_2}(\kappa,n^*)\}\rangle \right\}$,
$H_1 \cup H_2 = \left\{ \langle \kappa, \max\{\Theta_{H_1}(\kappa,\varrho^*), \Theta_{H_2}(\kappa,\varrho^*)\}, \min\{\Phi_{H_1}(\kappa,\varrho^*), \Phi_{H_2}(\kappa,\varrho^*)\}\rangle \right\}$.

Definition 2.5. Let H be an intuitionistic $\mathbb{Q}$ fuzzy subset of $\mathscr{M}$. For $t_1, t_2 \in [0,1]$ the level subset of H is the set $H_{(t_1,t_2)} = \{\kappa \in \mathscr{M} : \Theta_H(\kappa,\varrho^*) \ge t_1, \Theta_H(\kappa,\varrho^*) \le t_2, \varrho^* \in \mathbb{Q}\}$.

Definition 2.6. [21] A neutrosophic set H in the universe $\mathbb{G}$ is defined as $H = \{d, \Gamma^{\mathbb{T}}_H(\mathfrak{u}), \Gamma^{\mathbb{I}}_H(\mathfrak{u}), \Gamma^{\mathbb{F}}_H(\mathfrak{u})|\mathfrak{u} \in \mathbb{G}\}$, where $\Gamma^{\mathbb{T}}_H(\mathfrak{u})$, $\Gamma^{\mathbb{I}}_H(\mathfrak{u})$ $\Gamma^{\mathbb{F}}_H(\mathfrak{u})$ represents the degree of Truth-membership, degree of Indeterminacy membership and degree of Falsity-membership of H respectively. The mapping $\Gamma^{\mathbb{T}}_H, \Gamma^{\mathbb{I}}_H, \Gamma^{\mathbb{F}}_H : \mathbb{G} \to [0,1]$ and $0 \le \Gamma^{\mathbb{T}}_H(\mathfrak{u}) + \Gamma^{\mathbb{I}}_H(\mathfrak{u}) + \Gamma^{\mathbb{F}}_H(\mathfrak{u}) \le 3$.

Definition 2.7. [21] Let $H_1 = \langle \Gamma^{\mathbb{T}}_{H_1}, \Gamma^{\mathbb{I}}_{H_1}, \Gamma^{\mathbb{F}}_{H_1}\rangle$, $H_2 = \langle \Gamma^{\mathbb{T}}_{H_2}, \Gamma^{\mathbb{I}}_{H_2}, \Gamma^{\mathbb{F}}_{H_2}\rangle$ and $H_3 = \langle \Gamma^{\mathbb{T}}_{H_3}, \Gamma^{\mathbb{I}}_{H_3}, \Gamma^{\mathbb{F}}_{H_3}\rangle$ be three neutrosophic numbers over $\mathbb{G}$. Then
(i) ${H_1}^c = \langle \Gamma^{\mathbb{F}}_{H_1}, \Gamma^{\mathbb{I}}_{H_1}, \Gamma^{\mathbb{T}}_{H_1}\rangle$
(ii) $H_2 \bigvee H_3 = \left\langle \max(\Gamma^{\mathbb{T}}_{H_2}, \Gamma^{\mathbb{T}}_{H_3}), \min(\Gamma^{\mathbb{I}}_{H_2}, \Gamma^{\mathbb{I}}_{H_3}), \min(\Gamma^{\mathbb{F}}_{H_2}, \Gamma^{\mathbb{F}}_{H_3})\right\rangle$
(iii) $H_2 \bigwedge H_3 = \left\langle \min(\Gamma^{\mathbb{T}}_{H_2}, \Gamma^{\mathbb{T}}_{H_3}), \max(\Gamma^{\mathbb{I}}_{H_2}, \Gamma^{\mathbb{I}}_{H_3}), \max(\Gamma^{\mathbb{F}}_{H_2}, \Gamma^{\mathbb{F}}_{H_3})\right\rangle$
(iv) $H_2 \ge H_3$ iff $\Gamma^{\mathbb{T}}_{H_2} \ge \Gamma^{\mathbb{T}}_{H_3}$ and $\Gamma^{\mathbb{I}}_{H_2} \le \Gamma^{\mathbb{I}}_{H_3}$ and $\Gamma^{\mathbb{F}}_{H_2} \le \Gamma^{\mathbb{F}}_{H_3}$
(v) $H_2 = H_3$ iff $\Gamma^{\mathbb{T}}_{H_2} = \Gamma^{\mathbb{T}}_{H_3}$ and $\Gamma^{\mathbb{I}}_{H_2} = \Gamma^{\mathbb{I}}_{H_3}$ and $\Gamma^{\mathbb{F}}_{H_2} = \Gamma^{\mathbb{F}}_{H_3}$.

Definition 2.8. For any neutrosophic set $H = \{l, \Gamma^{\mathbb{T}}_H(l), \Gamma^{\mathbb{I}}_H(l), \Gamma^{\mathbb{F}}_H(l)\}$ of a set $\mathfrak{U}$, we defined a (ζ,ν)-cut of as the crisp subset $\{(z) \in (\mathfrak{U})|\Gamma^{\mathbb{T}}_H(l) \ge \zeta, \Gamma^{\mathbb{I}}_H(l) \ge \zeta, \Gamma^{\mathbb{F}}_H(l) \le \nu\}$ of $(\mathfrak{U})$.

Definition 2.9. [21] Let $\mathbb{H}$ and $\mathbb{I}$ be be two neutrosophic subsets of $\mathbb{S}$. The cartesian product of $\mathbb{H}$ and $\mathbb{I}$ denoted by $\mathbb{H}\times\mathbb{I}$ is given as $\mathbb{H}\times\mathbb{I} = \{\Gamma^{\mathbb{T}}_{\mathbb{H}\times\mathbb{I}}(l,g), \Gamma^{\mathbb{I}}_{\mathbb{H}\times\mathbb{I}}(l,g), \Gamma^{\mathbb{F}}_{\mathbb{H}\times\mathbb{I}}(l,g)| \ for \ all \ l,g \in \mathbb{S}\}$, where $\Gamma^{\mathbb{T}}_{\mathbb{H}\times\mathbb{I}}(z,y) = \min\{\Gamma^{\mathbb{T}}_{\mathbb{H}}(z), \Gamma^{\mathbb{T}}_{\mathbb{I}}(y)\}, \Gamma^{\mathbb{I}}_{\mathbb{H}\times\mathbb{I}}(l,g) = \frac{\Gamma^{\mathbb{I}}_{\mathbb{H}}(z)+\Gamma^{\mathbb{I}}_{\mathbb{I}}(l)}{2}, \Gamma^{\mathbb{F}}_{\mathbb{H}\times\mathbb{I}}(l,g) = \max\{\Gamma^{\mathbb{F}}_{\mathbb{H}}(z), \Gamma^{\mathbb{F}}_{\mathbb{I}}(l)\}$.

Definition 2.10. [10] A fuzzy subset H of a bisemiring $(\mathfrak{S}, \circledcirc_1, \circledcirc_2, \circledcirc_3)$ is said to be a fuzzy subbisemiring of $\mathfrak{S}$ if $\Gamma_H(\ell \circledcirc_1 g) \ge \min\{\Gamma_H(\ell), \Gamma_H(g)\}, \Gamma_H(\ell \circledcirc_2 g) \ge \min\{\Gamma_H(\ell), \Gamma_H(g)\}, \Gamma_H(\ell \circledcirc_3 g) \ge \min\{\Gamma_H(\ell), \Gamma_H(g)\}$, for all $\ell, g \in \mathfrak{S}$.

Definition 2.11. [10] A fuzzy subset H of a bisemiring $(\mathfrak{S}, \circledcirc_1, \circledcirc_2, \circledcirc_3)$ is said to be a fuzzy normal subbisemiring of $\mathfrak{S}$ if $\Gamma_H(\ell \circledcirc_1 g) = \Gamma_H(g \circledcirc_1 l), \Gamma_H(\ell \circledcirc_2 g) = \Gamma_H(g \circledcirc_2 l), \Gamma_H(\ell \circledcirc_3 g) = \Gamma_H(g \circledcirc_3 l)$, for all $\ell, g \in \mathfrak{S}$.

Definition 2.12. [18] Let $(\mathfrak{S}, \dagger, \bullet, \otimes)$ and $(\mathfrak{T}, *, \circledcirc, \odot)$ be two bisemirings. A function $\psi : \mathfrak{S} \to \mathfrak{T}$ is said to be a homomorphism if $\psi(l+g) = \psi(z) * \psi(g), \psi(l \bullet g) = \psi(z) \circledcirc \phi(g), \psi(l \otimes g) = \psi(l) \odot \psi(g)$, for all $l, g \in \mathfrak{S}$.

Definition 2.13. [15] Let $\mathbb{S}$ be the subbisemiring.The Bipolar valued fuzzy semiring A = $\langle f^p_A, f^n_A\rangle$ in $\mathbb{S}$ is said to be bipolar valued subbisemiring of $\mathbb{S}$ if it satisfies the following conditions :
(i) $f^p_A\,(x \otimes_1 y) \ge$ min $\{f^p_A(x), f^p_A(y)\}$,
(ii) $f^p_A\,(x \otimes_2 y) \ge$ min $\{f^p_A(x), f^p_A(y)\}$,
(iii) $f^p_A\,(x \otimes_3 y) \ge$ min $\{f^p_A(x), f^p_A(y)\}$,
(iv) $f^n_A\,(x \otimes_1 y) \le$ max $\{f^n_A(x), f^n_A(y)\}$,
(v) $f^n_A\,(x \otimes_2 y) \le$ max $\{f^n_A(x), f^n_A(y)\}$,
(vi) $f^n_A\,(x \otimes_3 y) \le$ max $\{f^n_A(x), f^n_A(y)\}$, $\forall$ x,y $\in \mathbb{S}$

Definition 2.14. A vague set $L = (\mathfrak{T}_L, \mathfrak{F}_L)$ of $\mathfrak{S}$ is said to be vague seimring if the following conditions are true. For all $s_1, s_2 \in \mathfrak{S}$,
(i)$\mathbb{V}_L\,(s_1 \oplus s_2) \ge$ min $\{\mathbb{V}_L\,(s_1)\,, \mathbb{V}_L\,(s_2)\}$ and
(ii)$\mathbb{V}_L\,(s_1 \cdot s_2) \ge min\,\{\mathbb{V}_L\,(s_1)\,, \mathbb{V}_L\,(s_2)\}$ and

Definition 2.15. [3] An interval valued fuzzy set $\mathbb{A}$ on the universe $\mathbb{X} \ne 0$ is a mapping $\mathbb{A} \to \mathbb{L}([0,1])$ such that the membership degree of $x \in \mathbb{X}$ is given by $\mathbb{A}(x) = [\underline{\mathbb{A}}(x), \overline{\mathbb{A}}(x)] \in \mathbb{L}([0,1])$, where $\underline{\mathbb{A}} : \mathbb{X} \to [0,1]$ and $\overline{\mathbb{A}} : \mathbb{X} \to [0,1]$ are mappings defining lower and upper bound of the membership interval $\mathbb{A}(x)$ respectively.

Definition 2.16. [17] An interval valued fuzzy set $\mathbb{A}$ on the universe $\mathbb{X} \ne 0$ is a mapping $\mathbb{A} \to \mathbb{L}([0,1])$ such that the membership degree of $x \in \mathbb{X}$ is given by $\mathbb{A}(x) = [\underline{\mathbb{A}}(x), \overline{\mathbb{A}}(x)] \in \mathbb{L}([0,1])$, where $\underline{\mathbb{A}} : \mathbb{X} \to [0,1]$ and $\overline{\mathbb{A}} : \mathbb{X} \to [0,1]$ are mappings defining lower and upper bound of the membership interval $\mathbb{A}(x)$ respectively.

3 Diophantine Q Neutrosophic Subbisemiring

Here $\mathbb{D}$ stands for bisemiring and DioQNSBS denotes diophantine Q-neutrosophic subbisemiring.

Definition 3.1. A Diophantine neutrosophic set H in $\mathbb{G}$ is of the form $\mathbb{H} = \Big\{(d), \Big(\Omega_H^{\mathbb{T}}(d), \Omega_H^{\mathbb{I}}(d), \Omega_H^{\mathbb{F}}(d)\Big), \Big(\Lambda_H(d), \Xi_H(d), \Phi_H(d)\Big)|u \in \mathbb{G}\Big\}$, where $\Omega_H^{\mathbb{T}}(d)$, $\Omega_H^{\mathbb{I}}(d)$ $\Omega_H^{\mathbb{F}}(d)$ represents the degree of Truth-membership, degree of Indeterminacy membership and degree of Falsity-membership of H respectively. The mapping $\Omega_H^{\mathbb{T}}, \Omega_H^{\mathbb{I}}, \Omega_H^{\mathbb{F}} : \mathfrak{U} \to [0,1]$ and $0 \leq (\Lambda_H(d) \cdot \Omega_H^{\mathbb{T}}(d)) + (\Xi_H(d) \cdot \Omega_H^{\mathbb{I}}(d)) + (\Phi_H(d) \cdot \Omega_H^{\mathbb{F}}(d)) \leq 2$. Since $H = \langle(\Omega_H^{\mathbb{T}}, \Omega_H^{\mathbb{I}}, \Omega_H^{\mathbb{F}}), (\Lambda_H, \Xi_H, \Phi_H)\rangle$ is called a Diophantine neutrosophic number.

Definition 3.2. A Diophantine Q neutrosophic subset H of $\mathfrak{S}$ is said to be a DioQNSBS of $\mathfrak{S}$ if the following criteria are satisfied.Then

$$\begin{cases} \Gamma_H^{\mathbb{T}}(\ell \circledcirc_1 \mathfrak{g}, \varrho^*) \geq \min\{\Gamma_H^{\mathbb{T}}(\ell, \varrho^*), \Gamma_H^{\mathbb{T}}(\mathfrak{g}, \varrho^*)\} \\ \Gamma_H^{\mathbb{T}}(\ell \circledcirc_2 \mathfrak{g}, \varrho^*) \geq \min\{\Gamma_H^{\mathbb{T}}(\ell, \varrho^*), \Gamma_H^{\mathbb{T}}(\mathfrak{g}, \varrho^*)\} \\ \Gamma_H^{\mathbb{T}}(\ell \circledcirc_3 \mathfrak{g}, \varrho^*) \geq \min\{\Gamma_H^{\mathbb{T}}(\ell, \varrho^*), \Gamma_H^{\mathbb{T}}(\mathfrak{g}, \varrho^*)\} \end{cases}$$

$$\begin{cases} \Gamma_H^{\mathbb{I}}(z \circledcirc_1 \mathfrak{g}, \varrho^*) \geq \frac{\Gamma_H^{\mathbb{I}}(l,\varrho^*)+\Gamma_H^{\mathbb{I}}(\mathfrak{g},\varrho^*)}{2} \\ \qquad\text{OR} \\ \Gamma_H^{\mathbb{I}}(z \circledcirc_2 \mathfrak{g}, \varrho^*) \geq \frac{\Gamma_H^{\mathbb{I}}(l,\varrho^*)+\Gamma_H^{\mathbb{I}}(\mathfrak{g},\varrho^*)}{2} \\ \qquad\text{OR} \\ \Gamma_H^{\mathbb{I}}(z \circledcirc_3 \mathfrak{g}, \varrho^*) \geq \frac{\Gamma_H^{\mathbb{I}}(l,\varrho^*)+\Gamma_H^{\mathbb{I}}(\mathfrak{g},\varrho^*)}{2} \end{cases}$$

$$\begin{cases} \Gamma_H^{\mathbb{F}}(\ell \circledcirc_1 \mathfrak{g}, \varrho^*) \leq \max\{\Gamma_H^{\mathbb{F}}(\ell, \varrho^*), \Gamma_H^{\mathbb{F}}(\mathfrak{g}, \varrho^*)\} \\ \Gamma_H^{\mathbb{F}}(\ell \circledcirc_2 \mathfrak{g}, \varrho^*) \leq \max\{\Gamma_H^{\mathbb{F}}(\ell, \varrho^*), \Gamma_H^{\mathbb{F}}(\mathfrak{g}, \varrho^*)\} \\ \Gamma_H^{\mathbb{F}}(\ell \circledcirc_3 \mathfrak{g}, \varrho^*) \leq \max\{\Gamma_H^{\mathbb{F}}(\ell, \varrho^*), \Gamma_H^{\mathbb{F}}(\mathfrak{g}, \varrho^*)\} \end{cases}$$

$$\begin{cases} (\Phi_H(\ell \circledcirc_1 \mathfrak{g}, \varrho^*) \geq \min\{(\Phi_H(\ell, \varrho^*), (\Phi_H(\mathfrak{g}, \varrho^*)\} \\ (\Phi_H(\ell \circledcirc_2 \mathfrak{g}, \varrho^*) \geq \min\{(\Phi_H(\ell, \varrho^*), (\Phi_H(\mathfrak{g}, \varrho^*)\} \\ (\Phi_H(\ell \circledcirc_3 \mathfrak{g}, \varrho^*) \geq \min\{(\Phi_H(\ell, \varrho^*), (\Phi_H(\mathfrak{g}, \varrho^*)\} \end{cases}$$

$$\begin{cases} \Xi_H(\ell \circledcirc_1 \mathfrak{g}, \varrho^*) \geq \frac{\Xi_H(\ell,\varrho^*)+\Xi_H(\mathfrak{g},\varrho^*)}{2} \\ \qquad\text{OR} \\ \Xi_H(\ell \circledcirc_2 \mathfrak{g}, \varrho^*) \geq \frac{\Xi_H(\ell,\varrho^*)+\Xi_H(\mathfrak{g},\varrho^*)}{2} \\ \qquad\text{OR} \\ \Xi_H(\ell \circledcirc_3 \mathfrak{g}, \varrho^*) \geq \frac{\Xi_H(\ell,\varrho^*)+\Xi_H(\mathfrak{g},\varrho^*)}{2} \end{cases}$$

$$\begin{cases} \Psi_H(\ell \circledcirc_1 \mathfrak{g}, \varrho^*) \leq \max\{\Psi_H(\ell, \varrho^*), \Psi_H(\mathfrak{g}, \varrho^*)\} \\ \Psi_H(\ell \circledcirc_2 \mathfrak{g}, \varrho^*) \leq \max\{\Psi_H(\ell, \varrho^*), \Psi_H(\mathfrak{g}, \varrho^*)\} \\ \Psi_H(\ell \circledcirc_3 \mathfrak{g}, \varrho^*) \leq \max\{\Psi_H(\ell, \varrho^*), \Psi_H(\mathfrak{g}, \varrho^*)\} \end{cases}$$

for all $\ell, \mathfrak{g} \in \mathbb{D}$.

Example 3.3. Let $\mathbb{D} = \{p_1, p_2, p_3, p_4\}$ be the bisemiring with the given cayley table:

https://doi.org/10.54216/IJNS.220207
Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

$\circledcirc_1$	p_1	p_2	p_3	p_4
p_1	p_1	p_1	p_1	p_1
p_2	p_1	p_2	p_1	p_2
p_3	p_1	p_1	p_3	p_3
p_4	p_1	p_2	p_3	p_4

$\bigcirc_2$	1	p_2	p_3	p_4
p_1	p_1	p_2	p_3	p_4
p_2	p_2	p_2	p_4	p_4
p_3	p_3	p_4	p_3	p_4
p_4	p_4	p_4	p_4	p_4

$\bigcirc_3$	p_1	p_2	p_3	p_4
p_1	p_1	p_1	p_1	p_1
p_2	p_1	p_2	p_3	p_4
p_3	p_4	p_4	p_4	p_4
p_4	p_4	p_4	p_4	p_4

	$(e,\varrho^*)=(e_1,\varrho^*)$	$(e,\varrho^*)=(e_2,\varrho^*)$	$(e,\varrho^*)=(e_3,\varrho^*)$	$(e,\varrho^*)=(e_4,\varrho^*)$
$\left(\Gamma^{\mathbb{T}}_H(e,\varrho^*),\Lambda_H(e,\varrho^*)\right)$	$(0.94,0.37)$	$(0.92,0.32)$	$(0.89,0.22)$	$(0.91,0.27)$
$\left(\Gamma^{\mathbb{I}}_H(e,\varrho^*),\Xi_H(e,\varrho^*)\right)$	$(0.77,0.22)$	$(0.75,0.17)$	$(0.70,0.07)$	$(0.72,0.12)$
$\left(\Gamma^{\mathbb{F}}_H(e,\varrho^*),\Phi_H(e,\varrho^*)\right)$	$(0.82,0.27)$	$(0.86,0.32)$	$(0.88,0.42)$	$(0.87,0.37)$

Clearly, H is a DioQNSBS of $\mathbb{D}$.

Theorem 3.4. *The arbitrary intersection of a family of $DioQNSBS$ is a DioQNSBS.*

Proof. Let $\{M_i : i\in I\}$ be a family of $DioQNSBS$ of a bisemiring $\mathbb{D}$ and $H=\bigcap_{i\in\mathbb{I}} M_i$.

Let l and g in $\mathbb{D}$. Then

$$\begin{aligned}\Gamma^{\mathbb{T}}_H(l\bigcirc_1 g,\varrho^*) &= \inf_{i\in\mathbb{I}}\ \Gamma^{\mathbb{T}}_{M_i}(l\bigcirc_1 g,\varrho^*)\\ &\geq \inf_{i\in\mathbb{I}}\ \min\{\Gamma^{\mathbb{T}}_{M_i}(l,\varrho^*),\Gamma^{\mathbb{T}}_{M_i}(g,\varrho^*)\}\\ &= \min\left\{\inf_{i\in\mathbb{I}}\ \Gamma^{\mathbb{T}}_{M_i}(l,\varrho^*),\inf_{i\in\mathbb{I}}\ \Gamma^{\mathbb{T}}_{M_i}(g,\varrho^*)\right\}\\ &= \min\{\Gamma^{\mathbb{T}}_H(l,\varrho^*),\Gamma^{\mathbb{T}}_H(g,\varrho^*)\}.\end{aligned}$$

Similarly, $\Gamma^{\mathbb{T}}_H(l\bigcirc_3 g,\varrho^*)\geq\min\{\Gamma^{\mathbb{T}}_H(l,g^*),\Gamma^{\mathbb{T}}_H(g,\varrho^*)\}$, $\Gamma^{\mathbb{T}}_H(l\bigcirc_3 g,\varrho^*)\geq\min\{\Gamma^{\mathbb{T}}_H(l,\varrho^*),\Gamma^{\mathbb{T}}_H(g,\varrho^*)\}$. Now,

$$\begin{aligned}\Gamma^{\mathbb{I}}_H(l\bigcirc_1 g,\varrho^*) &= \inf_{i\in\mathbb{I}}\ \Gamma^{\mathbb{I}}_{M_i}(l\bigcirc_1 g,\varrho^*)\\ &\geq \inf_{i\in\mathbb{I}}\ \frac{\Gamma^{\mathbb{I}}_{M_i}(l,\varrho^*)+\Gamma^{\mathbb{I}}_{M_i}(g,\varrho^*)}{2}\\ &= \frac{\inf_{i\in\mathbb{I}}\ \Gamma^{\mathbb{I}}_{M_i}(l,\varrho^*)+\inf_{i\in\mathbb{I}}\ \Gamma^{\mathbb{I}}_{M_i}(g,\varrho^*)}{2}\\ &= \frac{\Gamma^{\mathbb{I}}_H(l,\varrho^*)+\Gamma^{\mathbb{I}}_H(g,\varrho^*)}{2}.\end{aligned}$$

Similarly, $\Gamma^{\mathbb{I}}_H(l\bigcirc_3 g,\varrho^*)\geq\frac{\Gamma^{\mathbb{I}}_H(l,\varrho^*)+\Gamma^{\mathbb{I}}_H(g,\varrho^*)}{2}$ and $\Gamma^{\mathbb{I}}_H(l\bigcirc_3 g,\varrho^*)\geq\frac{\Gamma^{\mathbb{I}}_H(l,\varrho^*)+\Gamma^{\mathbb{I}}_H(g,\varrho^*)}{2}$. Now,

$$\begin{aligned}\Gamma^{\mathbb{F}}_H(l\bigcirc_1 g,\varrho^*) &= \sup_{i\in\mathbb{I}}\ \Gamma_{M_i}(z\bigcirc_1 g,\varrho^*)\\ &\leq \sup_{i\in\mathbb{I}}\ \max\{\Gamma_{M_i}(l,\varrho^*),\Gamma_{M_i}(g,\varrho^*)\}\\ &= \max\left\{\sup_{i\in\mathbb{I}}\ \Gamma_{M_i}(l,\varrho^*),\sup_{i\in\mathbb{I}}\ \Gamma_{M_i}(g,\varrho^*)\right\}\\ &= \max\{\Gamma^{\mathbb{F}}_H(l,\varrho^*),\Gamma^{\mathbb{F}}_H(g,\varrho^*)\}.\end{aligned}$$

Similarly, $\Gamma^{\mathbb{F}}_H(l\bigcirc_3 g,\varrho^*)\leq\max\{\Gamma^{\mathbb{F}}_H(l,\varrho^*),\Gamma^{\mathbb{F}}_H(g,\varrho^*)\}$ and
$\Gamma^{\mathbb{F}}_H(l\bigcirc_3 g,\varrho^*)\leq\max\{\Gamma^{\mathbb{F}}_H(l,\varrho^*),\Gamma^{\mathbb{F}}_H(g,\varrho^*)\}$.

$$\begin{aligned}\Lambda_H(l\bigcirc_1 g,\varrho^*) &= \inf_{i\in\mathbb{I}}\ \Lambda_{M_i}(l\bigcirc_1 g,\varrho^*)\\ &\geq \inf_{i\in\mathbb{I}}\ \min\{\Lambda_{M_i}(l,\varrho^*),\Lambda_{M_i}(g,\varrho^*)\}\\ &= \min\left\{\inf_{i\in\mathbb{I}}\ \Lambda_{M_i}(l,\varrho^*),\inf_{i\in\mathbb{I}}\ \Lambda_{M_i}(g,\varrho^*)\right\}\\ &= \min\{\Lambda_H(l,\varrho^*),\Lambda_H(g,\varrho^*)\}.\end{aligned}$$

https://doi.org/10.54216/IJNS.220207

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

Similarly, $\Lambda_H(l \bigcirc_3 g, \varrho^*) \geq \min\{\Lambda_H(l, \varrho^*), \Lambda_H(g, \varrho^*)\}$ and
$\Lambda_H(l \bigcirc_3 g, \varrho^*) \geq \min\{\Lambda_H(l, \varrho^*), \Lambda_H(g, \varrho^*)\}$. Now,

$$\begin{aligned}\Xi_H(l \bigcirc_1 g, \varrho^*) &= \inf_{i\in\mathbb{I}} \ \Xi_{M_i}(l \bigcirc_1 g, \varrho^*) \\ &\geq \inf_{i\in\mathbb{I}} \ \frac{\Xi_{M_i}(l, \varrho^*) + \Xi_{M_i}(g, \varrho^*)}{2} \\ &= \frac{\inf_{i\in\mathbb{I}} \ \Xi_{M_i}(l, \varrho^*) + \inf_{i\in\mathbb{I}} \ \Xi_{M_i}(g, \varrho^*)}{2} \\ &= \frac{\Xi_H(l, \varrho^*) + \Xi_H(g, \varrho^*)}{2}.\end{aligned}$$

Similarly, $\Xi_H(l \bigcirc_3 g, \varrho^*) \geq \frac{\Xi_H(l,\varrho^*)+\Xi_H(g,\varrho^*)}{2}$ and $\Xi_H(l \bigcirc_3 g, \varrho^*) \geq \frac{\Xi_H(l,\varrho^*)+\Xi_H(g,\varrho^*)}{2}$. Now,

$$\begin{aligned}\Phi_H(l \bigcirc_1 g, \varrho^*) &= \sup_{i\in\mathbb{I}} \ \Phi_{M_i}(l \bigcirc_1 g, \varrho^*) \\ &\leq \sup_{i\in\mathbb{I}} \ \max\{\Phi_{M_i}(l, \varrho^*), \Phi_{M_i}(g, \varrho^*)\} \\ &= \max\left\{\sup_{i\in\mathbb{I}} \ \Phi_{M_i}(l, \varrho^*), \sup_{i\in\mathbb{I}} \ \Phi_{M_i}(g, \varrho^*)\right\} \\ &= \max\{\Phi_H(l, \varrho^*), \Phi_H(g, \varrho^*)\}.\end{aligned}$$

Similarly, $\Phi_H(l \bigcirc_3 g, \varrho^*) \leq \max\{\Phi_H(l, \varrho^*), \Phi_H(g, \varrho^*)\}$ and
$\Phi_H(l \bigcirc_3 g, \varrho^*) \leq \max\{\Phi_H(l, \varrho^*), \Phi_H(g, \varrho^*)\}$. Hence H is a DioQNSBS of $\mathbb{D}$.

Theorem 3.5. *If H and I are any two $DioQNSBS^s$ of $\mathbb{D}_1$ and $\mathbb{D}_2$ respectively, then $H \times I$ is a DioQNSBS of $\mathbb{D}_1 \times \mathbb{D}_2$.*

Proof. Let H and I be two $DioQNSBS^s$ of $\mathbb{D}_1$ and $\mathbb{D}_2$ respectively. Let $l_1, l_2 \in \mathbb{D}_1$ and $g_1, g_2 \in \mathbb{D}_2$. Then (l_1, g_1) and (l_2, g_2, ϱ^*) are in $\mathbb{D}_1 \times \mathbb{D}_2$. Now

$$\begin{aligned}\Gamma^{\mathbb{T}}_{H\times I}[(l_1, g_1) \bigcirc_1 (l_2, g_2, \varrho^*)] &= \Gamma^{\mathbb{T}}_{H\times I}(l_1 \bigcirc_1 l_2, g_1 \bigcirc_1 g_2, \varrho^*) \\ &= \min\{\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 l_2, \varrho^*), \Gamma^{\mathbb{T}}_I(g_1 \bigcirc_1 g_2, \varrho^*)\} \\ &\geq \min\{\min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)\}, \min\{\Gamma^{\mathbb{T}}_I(g_1, \varrho^*), \Gamma^{\mathbb{T}}_I(g_2, \varrho^*)\}\} \\ &= \min\{\min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_I(g_1, \varrho^*)\}, \min\{\Gamma^{\mathbb{T}}_H(l_2, \varrho^*), \Gamma^{\mathbb{T}}_I(g_2, \varrho^*)\}\} \\ &= \min\{\Gamma^{\mathbb{T}}_{H\times I}(l_1, g_1, \varrho^*), \Gamma^{\mathbb{T}}_{H\times I}(l_2, g_2, \varrho^*)\}.\end{aligned}$$

Also $\Gamma^{\mathbb{T}}_{H\times I}[(l_1, g_1) \bigcirc_2 (l_2, g_2, \varrho^*)] \geq \min\{\Gamma^{\mathbb{T}}_{H\times I}(l_1, g_1, \varrho^*), \Gamma^{\mathbb{T}}_{H\times I}(l_2, g_2, \varrho^*)\}$,
$\Gamma^{\mathbb{T}}_{H\times I}[(l_1, g_1) \bigcirc_3 (l_2, g_2, \varrho^*)] \geq \min\{\Gamma^{\mathbb{T}}_{H\times I}(l_1, g_1, \varrho^*), \Gamma^{\mathbb{T}}_{H\times I}(l_2, g_2, \varrho^*)\}$. Now,

$$\begin{aligned}\Gamma^{\mathbb{I}}_{H\times I}[(l_1, g_1) \bigcirc_1 (l_2, g_2, \varrho^*)] &= \Gamma^{\mathbb{I}}_{H\times I}(l_1 \bigcirc_1 l_2, g_1 \bigcirc_1 g_2, \varrho^*) \\ &= \frac{\Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 l_2, \varrho^*) + \Gamma^{\mathbb{I}}_I(g_1 \bigcirc_1 g_2, \varrho^*)}{2} \\ &\geq \frac{1}{2}\left[\frac{\Gamma^{\mathbb{I}}_H(l_1, \varrho^*) + \Gamma^{\mathbb{I}}_H(l_2, \varrho^*)}{2} + \frac{\Gamma^{\mathbb{I}}_I(g_1, \varrho^*) + \Gamma^{\mathbb{I}}_I(g_2, \varrho^*)}{2}\right] \\ &= \frac{1}{2}\left[\frac{\Gamma^{\mathbb{I}}_H(l_1, \varrho^*) + \Gamma^{\mathbb{I}}_I(g_1, \varrho^*)}{2} + \frac{\Gamma^{\mathbb{I}}_H(l_2, \varrho^*) + \Gamma^{\mathbb{I}}_I(g_2, \varrho^*)}{2}\right] \\ &= \frac{1}{2}\Big[\Gamma^{\mathbb{I}}_{H\times I}(l_1, g_1, \varrho^*) + \Gamma^{\mathbb{I}}_{H\times I}(l_2, g_2, \varrho^*)\Big].\end{aligned}$$

Also $\Gamma^{\mathbb{I}}_{H\times I}[(l_1, g_1) \bigcirc_2 (l_2, g_2, \varrho^*)] \geq \frac{1}{2}\Big[\Gamma^{\mathbb{I}}_{H\times I}(l_1, g_1, \varrho^*) + \Gamma^{\mathbb{I}}_{H\times I}(l_2, g_2, \varrho^*)\Big]$ and

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

$\Gamma^{\mathbb{I}}_{H\times I}[(l_1,g_1)\bigcirc_3(l_2,g_2,\varrho^*)] \geq \frac{1}{2}\Big[\Gamma^{\mathbb{I}}_{H\times I}(l_1,g_1,\varrho^*)+\Gamma^{\mathbb{I}}_{H\times I}(l_2,g_2,\varrho^*)\Big]$. Now,

$$\begin{aligned}\Gamma^{\mathbb{F}}_{H\times I}[(l_1,g_1)\bigcirc_1(l_2,g_2,\varrho^*)] &= \Gamma^{\mathbb{F}}_{H\times I}(l_1\bigcirc_1 l_2, g_1\bigcirc_1 g_2,\varrho^*)\\ &= \max\{\Gamma^{\mathbb{F}}_{H}(l_1\bigcirc_1 l_2,\varrho^*),\Gamma^{\mathbb{F}}_{I}(g_1\bigcirc_1 g_2,\varrho^*)\}\\ &\leq \max\{\max\{\Gamma^{\mathbb{F}}_{H}(l_1,\varrho^*),\Gamma^{\mathbb{F}}_{H}(l_2,\varrho^*)\},\max\{\Gamma^{\mathbb{F}}_{I}(g_1,\varrho^*),\Gamma^{\mathbb{F}}_{I}(g_2,\varrho^*)\}\}\\ &= \max\{\max\{\Gamma^{\mathbb{F}}_{H}(l_1,\varrho^*),\Gamma^{\mathbb{F}}_{I}(g_1,\varrho^*)\},\max\{\Gamma^{\mathbb{F}}_{H}(l_2,\varrho^*),\Gamma^{\mathbb{F}}_{I}(g_2,\varrho^*)\}\}\\ &= \max\{\Gamma^{\mathbb{F}}_{H\times I}(l_1,g_1,\varrho^*),\Gamma^{\mathbb{F}}_{H\times I}(l_2,g_2,\varrho^*)\}.\end{aligned}$$

$\Gamma^{\mathbb{F}}_{H\times I}[(l_1,g_1)\bigcirc_3(l_2,g_2,\varrho^*)] \leq \max\{\Gamma^{\mathbb{F}}_{H\times I}(l_1,g_1,\varrho^*),\Gamma^{\mathbb{F}}_{H\times I}(l_2,g_2,\varrho^*)\}$.

$$\begin{aligned}\Lambda_{H\times I}[(l_1,g_1)\bigcirc_1(l_2,g_2,\varrho^*)] &= \Lambda_{H\times I}(l_1\bigcirc_1 l_2, g_1\bigcirc_1 g_2,\varrho^*)\\ &= \min\{\Lambda_{H}(l_1\bigcirc_1 l_2,\varrho^*),\Lambda_{I}(g_1\bigcirc_1 g_2,\varrho^*)\}\\ &\geq \min\{\min\{\Lambda_{H}(l_1,\varrho^*),\Lambda_{H}(l_2,\varrho^*)\},\min\{\Lambda_{I}(g_1,\varrho^*),\Lambda_{I}(g_2,\varrho^*)\}\}\\ &= \min\{\min\{\Lambda_{H}(l_1,\varrho^*),\Lambda_{I}(g_1,\varrho^*)\},\min\{\Lambda_{H}(l_2,\varrho^*),\Lambda_{I}(g_2,\varrho^*)\}\}\\ &= \min\{\Lambda_{H\times I}(l_1,g_1,\varrho^*),\Lambda_{H\times I}(l_2,g_2,\varrho^*)\}.\end{aligned}$$

Also $\Lambda_{H\times I}[(l_1,g_1)\bigcirc_2(l_2,g_2,\varrho^*)] \geq \min\{\Lambda_{H\times I}(l_1,g_1,\varrho^*),\Lambda_{H\times I}(l_2,g_2,\varrho^*)\}$, $\Lambda_{H\times I}[(l_1,g_1)\bigcirc_3(l_2,g_2,\varrho^*)] \geq \min\{\Lambda_{H\times I}(l_1,g_1,\varrho^*),\Lambda_{H\times I}(l_2,g_2,\varrho^*)\}$. Now,

$$\begin{aligned}\Xi_{H\times I}[(l_1,g_1)\bigcirc_1(l_2,g_2,\varrho^*)] &= \Xi_{H\times I}(l_1\bigcirc_1 l_2, g_1\bigcirc_1 g_2,\varrho^*)\\ &= \frac{\Xi_H(l_1\bigcirc_1 l_2,\varrho^*)+\Xi_I(g_1\bigcirc_1 g_2,\varrho^*)}{2}\\ &\geq \frac{1}{2}\left[\frac{\Xi_H(l_1,\varrho^*)+\Xi_H(l_2,\varrho^*)}{2}+\frac{\Xi_I(g_1,\varrho^*)+\Xi_I(g_2,\varrho^*)}{2}\right]\\ &= \frac{1}{2}\left[\frac{\Xi_H(l_1,\varrho^*)+\Xi_I(g_1,\varrho^*)}{2}+\frac{\Xi_H(l_2,\varrho^*)+\Xi_I(g_2,\varrho^*)}{2}\right]\\ &= \frac{1}{2}\Big[\Xi_{H\times I}(l_1,g_1,\varrho^*)+\Xi_{H\times I}(l_2,g_2,\varrho^*)\Big].\end{aligned}$$

Also $\Xi_{H\times I}[(l_1,g_1)\bigcirc_2(l_2,g_2,\varrho^*)] \geq \frac{1}{2}\Big[\Xi_{H\times I}(l_1,g_1,\varrho^*)+\Xi_{H\times I}(l_2,g_2,\varrho^*)\Big]$ and $\Xi_{H\times I}[(l_1,g_1)\bigcirc_3(l_2,g_2,\varrho^*)] \geq \frac{1}{2}\Big[\Xi_{H\times I}(l_1,g_1,\varrho^*)+\Xi_{H\times I}(l_2,g_2,\varrho^*)\Big]$. Now,

$$\begin{aligned}\Phi_{H\times I}[(l_1,g_1)\bigcirc_1(l_2,g_2,\varrho^*)] &= \Phi_{H\times I}(l_1\bigcirc_1 l_2, g_1\bigcirc_1 g_2,\varrho^*)\\ &= \max\{\Phi_{H}(l_1\bigcirc_1 l_2,\varrho^*),\Phi_{I}(g_1\bigcirc_1 g_2,\varrho^*)\}\\ &\leq \max\{\max\{\Phi_{H}(l_1,\varrho^*),\Phi_{H}(l_2,\varrho^*)\},\max\{\Phi_{I}(g_1,\varrho^*),\Phi_{I}(g_2,\varrho^*)\}\}\\ &= \max\{\max\{\Phi_{H}(l_1,\varrho^*),\Phi_{I}(g_1,\varrho^*)\},\max\{\Phi_{H}(l_2,\varrho^*),\Phi_{I}(g_2,\varrho^*)\}\}\\ &= \max\{\Phi_{H\times I}(l_1,g_1,\varrho^*),\Phi_{H\times I}(l_2,g_2,\varrho^*)\}.\end{aligned}$$

Also $\Phi_{H\times I}[(l_1,g_1)\bigcirc_2(l_2,g_2,\varrho^*)] \leq \max\{\Phi_{H\times I}(l_1,g_1,\varrho^*),\Phi_{H\times I}(l_2,g_2,\varrho^*)\}$, $\Phi_{H\times I}[(l_1,g_1)\bigcirc_3(l_2,g_2,\varrho^*)] \leq \max\{\Phi_{H\times I}(l_1,g_1,\varrho^*),\Phi_{H\times I}(l_2,g_2,\varrho^*)\}$. Hence $H\times I$ is a DioQNSBS of $\mathbb{D}$.

If $H_1, H_2, ..., H_n$ be the family of $DioQNSBS$ of $\mathbb{D}_1, \mathbb{D}_2, ..., \mathbb{D}_n$ respectively, then $H_1\times H_2\times ...\times H_n$ is a DioQNSBS of $\mathbb{D}_1\times\mathbb{D}_2\times ...\times\mathbb{D}_n$.

Definition 3.6. Let H be a diophantine Q-neutrosophic subset in $\mathbb{D}$, the SDioQNSR on $\mathbb{D}$ that is a diophantine Q-neutrosophic relation (DioQNSR)on H is defined as

$$\begin{cases}\Gamma^{\mathbb{T}}_{M}(l,g) = min\{\Gamma^{\mathbb{T}}_{A}(l,\varrho^*),\Gamma^{\mathbb{T}}_{A}(g,\varrho^*)\}\\ \Gamma^{\mathbb{I}}_{M}(l,g) = \frac{\Gamma^{\mathbb{I}}_{A}(l,n\varrho^*)+\Gamma^{\mathbb{I}}_{A}(g,\varrho^*)}{2}\\ \Gamma^{\mathbb{F}}_{M}(l,g) = max\{\Gamma^{\mathbb{F}}_{A}(l,\varrho^*),\Gamma^{\mathbb{F}}_{A}(g,\varrho^*)\}\end{cases}$$

https://doi.org/10.54216/IJNS.220207
Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

$$\begin{cases} \Lambda_M(l,g) = min\{\Lambda_A(l,\varrho^*), \Lambda_A(g,\varrho^*)\} \\ \Xi_M(l,g) = \frac{\Xi_A(l,\varrho^*)+\Xi_A(g,\varrho^*)}{2} \\ \Phi_M(l,n) = max\{\Phi_A(l,\varrho^*), \Phi_A g,\varrho^*)\} \end{cases}.$$

Theorem 3.7. *Let H be the DioQNSBS of $\mathbb{D}$ and M be the SDioQNSR of $\mathbb{D}$. Then H is a DioQNSBS of $\mathbb{D}$ if and only if M is a DioQNSBS of $\mathbb{D}\times\mathbb{D}$.*

Proof. Let H be the DioQNSBS of $\mathbb{D}$ and M be the SDioQNSR of $\mathbb{D}$. Then for any $l = (l_1, l_2, \varrho^*)$ and $g = (g_1, g_2, \varrho^*)$ are in $\mathbb{D}\times\mathbb{D}$. We have

$$\begin{aligned} \Gamma^{\mathbb{T}}_M(l \bigcirc_1 g, \varrho^*) &= \Gamma^{\mathbb{T}}_M[((l_1,l_2) \bigcirc_1 (g_1,g_2), \varrho^*] \\ &= \Gamma^{\mathbb{T}}_M(l_1 \bigcirc_1 g_1, l_2 \bigcirc_1 g_2, \varrho^*) \\ &= \min\{\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 g_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2 \bigcirc_1 g_2, \varrho^*)\} \\ &\geq \min\{\min\{\Gamma^{\mathbb{T}}_H(l_1,\varrho^*), \Gamma^{\mathbb{T}}_H(g_1,\varrho^*)\}, \min\{\Gamma^{\mathbb{T}}_H(l_2,\varrho^*), \Gamma^{\mathbb{T}}_H(g_2,\varrho^*)\}\} \\ &= \min\{\min\{\Gamma^{\mathbb{T}}_H(l_1,\varrho^*), \Gamma^{\mathbb{T}}_H(l_2,\varrho^*)\}, \min\{\Gamma^{\mathbb{T}}_H(g_1,\varrho^*), \Gamma^{\mathbb{T}}_H(g_2,\varrho^*)\}\} \\ &= \min\{\Gamma^{\mathbb{T}}_M(l_1,l_2,\varrho^*), \Gamma^{\mathbb{T}}_M(g_1,g_2,\varrho^*)\} \\ &= \min\{\Gamma^{\mathbb{T}}_M(l,\varrho^*), \Gamma^{\mathbb{T}}_M(g,\varrho^*)\}. \end{aligned}$$

Also, $\Gamma^{\mathbb{T}}_M(l \bigcirc_3 g, \varrho^*) \geq \min\{\Gamma^{\mathbb{T}}_M(l,\varrho^*), \Gamma^{\mathbb{T}}_M(g,\varrho^*)\}, \Gamma^{\mathbb{T}}_M(l \bigcirc_3 g, \varrho^*) \geq \min\{\Gamma^{\mathbb{T}}_M(l,\varrho^*), \Gamma^{\mathbb{T}}_M(g,\varrho^*)\}$.
Now,

$$\begin{aligned} \Gamma^{\mathbb{I}}_M(l \bigcirc_1 g, \varrho^*) &= \Gamma^{\mathbb{I}}_M[((l_1,l_2) \bigcirc_1 (g_1,g_2), \varrho^*] \\ &= \Gamma^{\mathbb{I}}_M(l_1 \bigcirc_1 g_1, l_2 \bigcirc_1 g_2, \varrho^*) \\ &= \frac{\Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 g_1, \varrho^*) + \Gamma^{\mathbb{I}}_H(l_2 \bigcirc_1 g_2, \varrho^*)}{2} \\ &\geq \frac{1}{2}\left[\frac{\Gamma^{\mathbb{I}}_H(l_1,\varrho^*) + \Gamma^{\mathbb{I}}_H(g_1,\varrho^*)}{2} + \frac{\Gamma^{\mathbb{I}}_H(l_2,\varrho^*) + \Gamma^{\mathbb{I}}_H(g_2,\varrho^*)}{2}\right] \\ &= \frac{1}{2}\left[\frac{\Gamma^{\mathbb{I}}_H(l_1,\varrho^*) + \Gamma^{\mathbb{I}}_H(l_2,\varrho^*)}{2} + \frac{\Gamma^{\mathbb{I}}_H(g_1,\varrho^*) + \Gamma^{\mathbb{I}}_H(g_2,\varrho^*)}{2}\right] \\ &= \frac{\Gamma^{\mathbb{I}}_M(l_1,l_2,\varrho^*) + \Gamma^{\mathbb{I}}_M(g_1,g_2,\varrho^*)}{2} \\ &= \frac{\Gamma^{\mathbb{I}}_M(l,\varrho^*) + \Gamma^{\mathbb{I}}_M(g,\varrho^*)}{2}. \end{aligned}$$

Also, $\Gamma^{\mathbb{I}}_M(l \bigcirc_3 g, \varrho^*) \geq \frac{\Gamma^{\mathbb{I}}_M(l,\varrho^*)+\Gamma^{\mathbb{I}}_M(g,\varrho^*)}{2}$ and $\Gamma^{\mathbb{I}}_M(l \bigcirc_3 g, \varrho^*) \geq \frac{\Gamma^{\mathbb{I}}_M(l,\varrho^*)+\Gamma^{\mathbb{I}}_M(g,\varrho^*)}{2}$.
Similarly, $\Gamma^{\mathbb{F}}_M(l \bigcirc_1 l, \varrho^*) \leq \max\{\Gamma^{\mathbb{F}}_M(l,\varrho^*), \Gamma^{\mathbb{F}}_M(g,\varrho^*)\}, \Gamma^{\mathbb{F}}_M(l \bigcirc_3 g, \varrho^*) \leq \max\{\Gamma^{\mathbb{F}}_M(l,\varrho^*), \Gamma^{\mathbb{F}}_M(g,\varrho^*)\}$ and
$\Gamma^{\mathbb{F}}_M(l \bigcirc_3 g, \varrho^*) \leq \max\{\Gamma^{\mathbb{F}}_M(l,\varrho^*), \Gamma^{\mathbb{F}}_M(g,\varrho^*)\}$.

$$\begin{aligned} \Lambda_M(l \bigcirc_1 g, \varrho^*) &= \Lambda_{HM}[((l_1,l_2) \bigcirc_1 (g_1,g_2), q^*] \\ &= \Lambda_M(l_1 \bigcirc_1 g_1, l_2 \bigcirc_1 g_2, \varrho^*) \\ &= \min\{\Lambda_H(l_1 \bigcirc_1 g_1, \varrho^*), \Lambda_H(l_2 \bigcirc_1 g_2, \varrho^*)\} \\ &\geq \min\{\min\{\Lambda_H(l_1,\varrho^*), \Lambda_H(g_1,\varrho^*)\}, \min\{\Lambda_H(l_2,\varrho^*), \Lambda_H(g_2,\varrho^*)\}\} \\ &= \min\{\min\{\Lambda_H(l_1,\varrho^*), \Lambda_H(l_2,\varrho^*)\}, \min\{\Lambda_H(g_1,\varrho^*), \Lambda_H(g_2,\varrho^*)\}\} \\ &= \min\{\Lambda_M(l_1,l_2,q^*), \Lambda_M(g_1,g_2,\varrho^*)\} \\ &= \min\{\Lambda_M(l,\varrho^*), \Lambda_M(g,\varrho^*)\}. \end{aligned}$$

https://doi.org/10.54216/IJNS.220207
Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

Also, $\Lambda_M(l \bigcirc_3 g, \varrho^*) \geq \min\{\Lambda_M(l, \varrho^*), \Lambda_M(g, \varrho^*)\}, \Lambda_M(l \bigcirc_3 g, \varrho^*) \geq \min\{\Lambda_M(l, \varrho^*), \Lambda_M(g, \varrho^*)\}$.
Now,

$$\begin{aligned}
\Xi_M(l \bigcirc_1 g, \varrho^*) &= \Xi_M[((l_1, l_2) \bigcirc_1 (g_1, g_2, \varrho^*)] \\
&= \Xi_M(l_1 \bigcirc_1 l_1, l_2 \bigcirc_1 g_2, \varrho^*) \\
&= \frac{\Xi_H(l_1 \bigcirc_1 g_1, \varrho^*) + \Xi_H(l_2 \bigcirc_1 l_2, \varrho^*)}{2} \\
&\geq \frac{1}{2}\left[\frac{\Xi_H(l_1, \varrho^*) + \Xi_H(g_1, \varrho^*)}{2} + \frac{\Xi_H(l_2, \varrho^*) + \Xi_H(g_2, \varrho^*)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\Xi_H(l_1, \varrho^*) + \Xi_H(l_2, \varrho^*)}{2} + \frac{\Xi_H(g_1, \varrho^*) + \Xi_H(g_2, \varrho^*)}{2}\right] \\
&= \frac{\Xi_M(l_1, l_2, q^*) + \Xi_M(g_1, g_2, \varrho^*)}{2} \\
&= \frac{\Xi_M(l, \varrho^*) + \Xi_M(g, \varrho^*)}{2}.
\end{aligned}$$

Also, $\Xi_M(l \bigcirc_3 g, \varrho^*) \geq \frac{\Xi_M(l,\varrho^*)+\Xi_M(g,\varrho^*)}{2}$ and $\Xi_M(l \bigcirc_3 g, \varrho^*) \geq \frac{\Xi_M(l,\varrho^*)+\Xi_M(g,\varrho^*)}{2}$.
Similarly, $\Phi_M(l \bigcirc_1 g, \varrho^*) \leq \max\{\Phi_M(l, \varrho^*), \Phi_M(g, \varrho^*)\}, \Phi_M(l \bigcirc_3 g, \varrho^*) \leq \max\{\Phi_M(l, \varrho^*), \Phi_M(g, \varrho^*)\}$ and
$\Phi_M(l \bigcirc_3 g, \varrho^*) \leq \max\{\Phi_M(l, \varrho^*), \Phi_M(g, \varrho^*)\}$. Hence M is a DioQNSBS of $\mathbb{D} \times \mathbb{D}$.

Conversely assume that M is a DioQNSBS of $\mathbb{D} \times \mathbb{D}$, then for any $l = (l_1, l_2)$ and $g = (g_1, g_2, \varrho^*)$ are in $\mathbb{D} \times \mathbb{D}$. We have
$\min\{\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 g_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2 \bigcirc_1 g_2, \varrho^*)\}$

$$\begin{aligned}
&= \Gamma^{\mathbb{T}}_M(l_1 \bigcirc_1 g_1, l_2 \bigcirc_1 g_2, \varrho^*) \\
&= \Gamma^{\mathbb{T}}_M[(l_1, l_2, \varrho^*) \bigcirc_1 (g_1, g_2, \varrho^*)] \\
&= \Gamma^{\mathbb{T}}_M(l \bigcirc_1 g, \varrho^*) \\
&\geq \min\{\Gamma^{\mathbb{T}}_M(l, \varrho^*), \Gamma^{\mathbb{T}}_M(g, \varrho^*)\} \\
&= \min\{\Gamma^{\mathbb{T}}_M(l_1, l_2, \varrho^*)\}, \Gamma^{\mathbb{T}}_M(g_1, g_2, \varrho^*)\} \\
&= \min\{\min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)\}, \min\{\Gamma^{\mathbb{T}}_H(g_1, \varrho^*), \Gamma^{\mathbb{T}}_H(g_2, \varrho^*)\}\}.
\end{aligned}$$

If $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 g_1, \varrho^*) \leq \Gamma^{\mathbb{T}}_H(l_2 \bigcirc_1 g_2, \varrho^*)$, then $\Gamma^{\mathbb{T}}_H(l_1, \varrho^*) \leq \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)$ and $\Gamma^{\mathbb{T}}_H(g_1, \varrho^*) \leq \Gamma^{\mathbb{T}}_H(g_2, \varrho^*)$. We get $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 g_1, \varrho^*) \geq \min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(g_1, \varrho^*)\}$ for all $l_1, g_1 \in \mathbb{D}$, and
$\min\{\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_2 g_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2 \bigcirc_2 g_2, \varrho^*)\} \geq \min\{\min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)\},$
$\min\{\Gamma^{\mathbb{T}}_H(g_1, \varrho^*), \Gamma^{\mathbb{T}}_H(g_2, \varrho^*)\}\}$
If $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_2 g_1, \varrho^*) \leq \Gamma^{\mathbb{T}}_H(l_2 \bigcirc_2 g_2, \varrho^*)$, then $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_2 g_1, \varrho^*) \geq \min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(g_1, \varrho^*)\}$.
$\min\{\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_3 g_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2 \bigcirc_3 g_2, \varrho^*)\} \geq \min\{\min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)\},$
$\min\{\Gamma^{\mathbb{T}}_H(g_1, \varrho^*), \Gamma^{\mathbb{T}}_H(g_2, \varrho^*)\}\}$.
If $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_3 g_1, \varrho^*) \leq \Gamma^{\mathbb{T}}_H(l_2 \bigcirc_3 g_2, \varrho^*)$, then $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_3 g_1, \varrho^*) \geq \min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(g_1, \varrho^*)\}$.
Now,

$$\begin{aligned}
\frac{1}{2}\Big[\Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 g_1, \varrho^*) + \Gamma^{\mathbb{I}}_H(l_2 \bigcirc_1 g_2, \varrho^*)\Big] &= \Gamma^{\mathbb{I}}_M(l_1 \bigcirc_1 g_1, l_2 \bigcirc_1 g_2, \varrho^*) \\
&= \Gamma^{\mathbb{I}}_M[(l_1, l_2, \varrho^*) \bigcirc_1 (g_1, g_2, \varrho^*)] \\
&= \Gamma^{\mathbb{I}}_M(l \bigcirc_1 g, \varrho^*) \\
&\geq \frac{\Gamma^{\mathbb{I}}_M(l, \varrho^*) + \Gamma^{\mathbb{I}}_M(g, \varrho^*)}{2} \\
&= \frac{\Gamma^{\mathbb{I}}_M(l_1, l_2, \varrho^*) + \Gamma^{\mathbb{I}}_M(g_1, g_2, \varrho^*)}{2} \\
&= \frac{1}{2}\left[\frac{\Gamma^{\mathbb{I}}_H(l_1, \varrho^*) + \Gamma^{\mathbb{I}}_H(l_2, \varrho^*)}{2} + \frac{\Gamma^{\mathbb{I}}_H(g_1, \varrho^*) + \Gamma^{\mathbb{I}}_H(g_2, \varrho^*)}{2}\right].
\end{aligned}$$

If $\Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 g_1, \varrho^*) \leq \Gamma^{\mathbb{I}}_H(l_2 \bigcirc_1 g_2, \varrho^*)$, then $\Gamma^{\mathbb{I}}_H(l_1, \varrho^*) \leq \Gamma^{\mathbb{I}}_H(l_2, \varrho^*)$ and $\Gamma^{\mathbb{I}}_H(g_1, \varrho^*) \leq \Gamma^{\mathbb{I}}_H(g_2, \varrho^*)$.
We get, $\Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 g_1, \varrho^*) \geq \frac{\Gamma^{\mathbb{I}}_H(l_1,\varrho^*)+\Gamma^{\mathbb{I}}_H(g_1,\varrho^*)}{2}$.

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

Similarly, $\Gamma^{\mathbb{I}}_{H}(l_1 \bigcirc_2 g_1, \varrho^*) \geq \frac{\Gamma^{\mathbb{I}}_{H}(l_1,\varrho^*)+\Gamma^{\mathbb{I}}_{H}(g_1,\varrho^*)}{2}$ and $\Gamma^{\mathbb{I}}_{H}(l_1 \bigcirc_3 g_1, \varrho^*) \geq \frac{\Gamma^{\mathbb{I}}_{H}(l_1,\varrho^*)+\Gamma^{\mathbb{I}}_{H}(g_1,\varrho^*)}{2}$.
Similarly to prove that
$\max\{\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_1 g_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(l_2 \bigcirc_1 g_2, \varrho^*)\}$
$\leq \max\{\max\{\Gamma^{\mathbb{F}}_{H}(l_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(l_2, \varrho^*)\}, \max\{\Gamma^{\mathbb{F}}_{H}(g_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(g_2, \varrho^*)\}\}$.
If $\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_1 g_1, \varrho^*) \geq \Gamma^{\mathbb{F}}_{H}(l_2 \bigcirc_1 g_2, \varrho^*)$, then $\Gamma^{\mathbb{F}}_{H}(l_1, \varrho^*) \geq \Gamma^{\mathbb{F}}_{H}(l_2, \varrho^*)$ and $\Gamma^{\mathbb{F}}_{H}(g_1, \varrho^*) \geq \Gamma^{\mathbb{F}}_{H}(g_2, \varrho^*)$.
We get, $\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_1 g_1, \varrho^*) \leq \max\{\Gamma^{\mathbb{F}}_{H}(l_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(g_1, \varrho^*)\}$.
$\max\{\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_2 g_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(l_2 \bigcirc_2 g_2, \varrho^*)\}$
$\leq \max\{\max\{\Gamma^{\mathbb{F}}_{H}(l_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(l_2, \varrho^*)\}, \max\{\Gamma^{\mathbb{F}}_{H}(g_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(g_2, \varrho^*)\}\}$.
If $\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_2 g_1, \varrho^*) \geq \Gamma^{\mathbb{F}}_{H}(l_2 \bigcirc_2 g_2, \varrho^*)$, then $\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_2 g_1, \varrho^*) \leq \max\{\Gamma^{\mathbb{F}}_{H}(l_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(g_1, \varrho^*)\}$.
$\max\{\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_3 g_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(l_2 \bigcirc_3 g_2, \varrho^*)\}$
$\leq \max\{\max\{\Gamma^{\mathbb{F}}_{H}(l_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(l_2, \varrho^*)\}, \max\{\Gamma^{\mathbb{F}}_{H}(g_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(g_2, \varrho^*)\}\}$
If $\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_3 g_1, \varrho^*) \geq \Gamma^{\mathbb{F}}_{H}(l_2 \bigcirc_3 g_2, \varrho^*)$, then $\Gamma^{\mathbb{F}}_{H}(l_1 \bigcirc_3 g_1, \varrho^*) \leq \max\{\Gamma^{\mathbb{F}}_{H}(l_1, \varrho^*), \Gamma^{\mathbb{F}}_{H}(g_1, \varrho^*)\}$.

$$\begin{aligned}
\min\{\Lambda_H(l_1 \bigcirc_1 g_1, \varrho^*), \Lambda_H(l_2 \bigcirc_1 g_2, \varrho^*)\} &= \Lambda_M(l_1 \bigcirc_1 g_1, l_2 \bigcirc_1 g_2, \varrho^*) \\
&= \Lambda_M[(l_1, l_2, q^*) \bigcirc_1 (g_1, g_2, \varrho^*)] \\
&= \Lambda_M(l \bigcirc_1 g, \varrho^*) \\
&\geq \min\{\Lambda_M(l, \varrho^*), \Lambda_M(g, \varrho^*)\} \\
&= \min\{\Lambda_M(l_1, l_2, \varrho^*)\}, \Lambda_M(g_1, g_2, \varrho^*)\} \\
&= \min\{\min\{\Lambda_H(l_1, \varrho^*), \Lambda_H(l_2, \varrho^*)\}, \min\{\Lambda_H(g_1, \varrho^*), \Lambda_H(g_2, \varrho^*)\}\}.
\end{aligned}$$

If $\Lambda_H(l_1 \bigcirc_1 g_1, \varrho^*) \leq \Lambda_H(l_2 \bigcirc_1 g_2, \varrho^*)$, then $\Lambda_H(l_1, \varrho^*) \leq \Lambda_H(l_2, \varrho^*)$ and $\Lambda_H(g_1, \varrho^*) \leq \Lambda_H(g_2, \varrho^*)$. We get $\Lambda_H(l_1 \bigcirc_1 g_1, \varrho^*) \geq \min\{\Lambda_H(l_1, \varrho^*), \Lambda_H(g_1, \varrho^*)\}$ for all $l_1, g_1 \in \mathbb{D}$, and
$\min\{\Lambda_H(l_1 \bigcirc_2 g_1, \varrho^*), \Lambda_H(l_2 \bigcirc_2 g_2, \varrho^*)\} \geq \min\{\min\{\Lambda_H(l_1, \varrho^*),$
$\Lambda_H(l_2, \varrho^*)\}, \min\{\Lambda_H(g_1, \varrho^*), \Lambda_H(g_2, \varrho^*)\}\}$
If $\Lambda_H(l_1 \bigcirc_2 g_1, \varrho^*) \leq \Lambda_H(l_2 \bigcirc_2 g_2, \varrho^*)$, then $\Lambda_H(l_1 \bigcirc_2 g_1, \varrho^*) \geq \min\{\Lambda_H(l_1, g^*), \Lambda_H(g_1, \varrho^*)\}$.
$\min\{\Lambda_H(l_1 \bigcirc_3 g_1, \varrho^*), \Lambda_H(l_2 \bigcirc_3 g_2, \varrho^*)\} \geq \min\{\min\{\Lambda_H(l_1, \varrho^*),$
$\Lambda_H(l_2, \varrho^*)\}, \min\{\Lambda_H(g_1, \varrho^*), \Lambda_H(g_2, \varrho^*)\}\}$.
If $\Lambda_H(l_1 \bigcirc_3 g_1, \varrho^*) \leq \Lambda_H(l_2 \bigcirc_3 g_2, \varrho^*)$, then $\Lambda_H(l_1 \bigcirc_3 g_1, \varrho^*) \geq \min\{\Lambda_H(l_1, \varrho^*), \Lambda_H(g_1, \varrho^*)\}$.
Now,

$$\begin{aligned}
\frac{1}{2}\Big[\Xi_H(l_1 \bigcirc_1 g_1, \varrho^*) + \Xi_H(l_2 \bigcirc_1 g_2, \varrho^*)\Big] &= \Xi_M(l_1 \bigcirc_1 g_1, l_2 \bigcirc_1 g_2, \varrho^*) \\
&= \Xi_M[(l_1, l_2, \varrho^*) \bigcirc_1 (g_1, g_2, \varrho^*)] \\
&= \Xi_M(l \bigcirc_1 g, \varrho^*) \\
&\geq \frac{\Xi_M(l, \varrho^*) + \Xi_M(g, \varrho^*)}{2} \\
&= \frac{\Xi_M(l_1, l_2, \varrho^*) + \Xi_M(g_1, g_2, \varrho^*)}{2} \\
&= \frac{1}{2}\left[\frac{\Xi_H(l_1, \varrho^*) + \Xi_H(l_2, \varrho^*)}{2} + \frac{\Xi_H(g_1, \varrho^*) + \Xi_H(g_2, \varrho^*)}{2}\right].
\end{aligned}$$

If $\Xi_H(l_1 \bigcirc_1 g_1, \varrho^*) \leq \Xi_H(l_2 \bigcirc_1 g_2, \varrho^*)$, then $\Xi_H(l_1, \varrho^*) \leq \Xi_H(l_2, \varrho^*)$ and $\Xi_H(g_1, \varrho^*) \leq \Xi_H(g_2, \varrho^*)$.
We get, $\Xi_H(l_1 \bigcirc_1 g_1, \varrho^*) \geq \frac{\Xi_H(l_1,\varrho^*)+\Xi_H(g_1,\varrho^*)}{2}$.
Similarly, $\Xi_H(l_1 \bigcirc_2 g_1, \varrho^*) \geq \frac{\Xi_H(l_1,\varrho^*)+\Xi_H(g_1,\varrho^*)}{2}$ and $\Xi_H(l_1 \bigcirc_3 g_1, \varrho^*) \geq \frac{\Xi_H(l_1,\varrho^*)+\Xi_H(g_1,\varrho^*)}{2}$.
Similarly to prove that
$\max\{\Phi_H(l_1 \bigcirc_1 g_1, \varrho^*), \Phi_H(l_2 \bigcirc_1 g_2, \varrho^*)\} \leq \max\{\max\{\Phi_H(l_1, \varrho^*),$
$\Phi_H(l_2, \varrho^*)\}, \max\{\Phi_H(g_1, \varrho^*), \Phi_H(g_2, \varrho^*)\}\}$.
If $\Phi_H(l_1 \bigcirc_1 g_1, \varrho^*) \geq \Phi_H(l_2 \bigcirc_1 g_2, \varrho^*)$, then $\Phi_H(l_1, \varrho^*) \geq \Phi_H(l_2, \varrho^*)$ and $\Phi_H(g_1, \varrho^*) \geq \Phi_H(g_2, \varrho^*)$.
We get, $\Phi_H(l_1 \bigcirc_1 g_1, \varrho^*) \leq \max\{\Phi_H(l_1, \varrho^*), \Phi_H(g_1, \varrho^*)\}$.
$\max\{\Phi_H(l_1 \bigcirc_2 g_1, \varrho^*), \Phi_H(l_2 \bigcirc_2 g_2, \varrho^*)\} \leq \max\{\max\{\Phi_H(l_1, \varrho^*),$
$\Phi_H(l_2, \varrho^*)\}, \max\{\Phi_H(g_1, \varrho^*), \Phi_H(g_2, \varrho^*)\}\}$.
If $\Phi_H(l_1 \bigcirc_2 g_1, \varrho^*) \geq \Phi_H(l_2 \bigcirc_2 g_2, \varrho^*)$, then $\Phi_H(l_1 \bigcirc_2 g_1, \varrho^*) \leq \max\{\Phi_H(l_1, \varrho^*), \Phi_H(g_1, \varrho^*)\}$.
$\max\{\Phi_H(l_1 \bigcirc_3 g_1, \varrho^*), \Phi_H(l_2 \bigcirc_3 g_2, \varrho^*)\} \leq \max\{\max\{\Phi_H(l_1, \varrho^*),$
$\Phi_H(l_2, \varrho^*)\}, \max\{\Phi_H(g_1, \varrho^*), \Phi_H(g_2, \varrho^*)\}\}$
If $\Phi_H(l_1 \bigcirc_3 g_1, \varrho^*) \geq \Phi_H(l_2 \bigcirc_3 g_2, \varrho^*)$, then $\Phi_H(l_1 \bigcirc_3 g_1, \varrho^*) \leq \max\{\Phi_H(l_1, \varrho^*), \Phi_H(g_1, \varrho^*)\}$.
Hence H is a DioQNSBS of $\mathbb{D}$.

https://doi.org/10.54216/IJNS.220207
Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

Theorem 3.8. *Let H be a diophantine Q-neutrosophic subset in $\mathbb{D}$. Then $H = \langle(\Gamma^{\mathbb{T}}_H, \Gamma^{\mathbb{I}}_H, \Gamma^{\mathbb{F}}_H), (\Lambda_H, \Xi_H, \Phi_H)\rangle$ is a DioQNSBS of $\mathbb{D}$ if and only if all non empty level set $H^{(t,s)}$ is a subbisemiring of $\mathbb{D}$ for $t, s \in [0, 1]$.*

Proof. Assume that H is a DioQNSBS of $\mathbb{D}$. For each $t, s \in [0, 1]$ and $l_1, l_2 \in H^{(t,s)}$. We have $\Gamma^{\mathbb{T}}_H(l_1, \varrho^*) \geq t, \Gamma^{\mathbb{T}}_H(l_2, \varrho^*) \geq t$, $\Gamma^{\mathbb{I}}_H(l_1, \varrho^*) \geq t, \Gamma^{\mathbb{I}}_H(l_2, \varrho^*) \geq t$, $\Gamma^{\mathbb{F}}_H(l_1, \varrho^*) \leq s, \Gamma^{\mathbb{F}}_H(l_2, \varrho^*) \leq s$ and $\Lambda_H(l_1, \varrho^*) \geq t, \Lambda_H(l_2, \varrho^*) \geq t, \Xi_H(l_1, \varrho^*) \geq t, \Xi_H(l_2, \varrho^*) \geq t$ and $\Phi_H(l_1, \varrho^*) \leq s, \Phi_H(l_2, \varrho^*) \leq s$. Now, $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 l_2) \geq \min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)\} \geq t$ and $\Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 l_2) \geq \frac{\Gamma^{\mathbb{I}}_H(l_1,\varrho^*)+\Gamma^{\mathbb{I}}_H(l_2,\varrho^*)}{2} \geq \frac{t+t}{2} = t$ and $\Gamma^{\mathbb{F}}_H(l_1 \bigcirc_1 l_2) \leq \max\{\Gamma^{\mathbb{F}}_H(l_1, \varrho^*), \Gamma^{\mathbb{F}}_H(l_2, \varrho^*)\} \leq s$. Similarly, $\Lambda_H(l_1 \bigcirc_1 l_2) \geq \min\{\Lambda_H(l_1, \varrho^*), \Lambda_H(l_2, \varrho^*)\} \geq t$ and $\Xi_H(l_1 \bigcirc_1 l_2) \geq \frac{\Xi_H(l_1,\varrho^*)+\Xi_H(l_2,\varrho^*)}{2} \geq \frac{t+t}{2} = t$ and $\Phi_H(l_1 \bigcirc_1 l_2) \leq \max\{\Phi_H(l_1, \varrho^*), \Phi_H(l_2, \varrho^*)\} \leq s$. This implies that $n_1 \bigcirc_1 l_2 \in H^{(t,s)}$. Similarly, $l_1 \bigcirc_2 l_2 \in H^{(t,s)}$ and $l_1 \bigcirc_3 l_2 \in H^{(t,s)}$. Then $H^{(t,s)}$ is a subbisemiring of $\mathbb{D}$ for each $t, s \in [0, 1]$.

Conversely assume that $H^{(t,s)}$ is a subbisemiring of $\mathbb{D}$ for each $t, s \in [0, 1]$. Suppose if there exist $l_1, l_2 \in \mathbb{D}$ such that $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 l_2, \varrho^*) < \min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)\}, \Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 l_2, \varrho^*) < \frac{\Gamma^{\mathbb{I}}_H(l_1,\varrho^*)+\Gamma^{\mathbb{I}}_H(l_2,\varrho^*)}{2}$, $\Gamma^{\mathbb{F}}_H(l_1 \bigcirc_1 l_2, q^*) > \max\{\Gamma^{\mathbb{F}}_H(l_1, \varrho^*), \Gamma^{\mathbb{F}}_H(l_2, \varrho^*)\}$ and $\Lambda_H(l_1 \bigcirc_1 l_2, \varrho^*) < \min\{\Lambda_H(l_1, \varrho^*), \Lambda_H(l_2, \varrho^*)\}$, $\Xi_H(l_1 \bigcirc_1 l_2, \varrho^*) < \frac{\Xi_H(l_1,\varrho^*)+\Xi_H(l_2,\varrho^*)}{2}$ and $\Phi_H(l_1 \bigcirc_1 l_2, \varrho^*) > \max\{\Phi_H(l_1, \varrho^*), \Phi_H(l_2, \varrho^*)\}$. Choose $t, s \in [0, 1]$ such that $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 l_2, \varrho^*) < t \leq \min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)\}$ and $\Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 l_2, \varrho^*) < t \leq \frac{\Gamma^{\mathbb{I}}_H(l_1,\varrho^*)+\Gamma^{\mathbb{I}}_H(l_2,\varrho^*)}{2}$ and $\Gamma^{\mathbb{F}}_H(l_1 \bigcirc_1 l_2, \varrho^*) > s \geq \max\{\Gamma^{\mathbb{F}}_H(l_1, \varrho^*), \Gamma^{\mathbb{F}}_H(l_2, \varrho^*)\}$. Then $l_1, l_2 \in H^{(t,s)}$, but $l_1 \bigcirc_1 l_2 \notin H^{(t,s)}$. This contradicts to that $H^{(t,s)}$ is a subbisemiring of $\mathbb{D}$. Hence $\Gamma^{\mathbb{T}}_H(l_1 \bigcirc_1 l_2, \varrho^*) \geq \min\{\Gamma^{\mathbb{T}}_H(l_1, \varrho^*), \Gamma^{\mathbb{T}}_H(l_2, \varrho^*)\}$, $\Gamma^{\mathbb{I}}_H(l_1 \bigcirc_1 l_2, \varrho^*) \geq \frac{\Gamma^{\mathbb{I}}_H(l_1,\varrho^*)+\Gamma^{\mathbb{I}}_H(l_2,\varrho^*)}{2}$ and $\Gamma^{\mathbb{F}}_H(l_1 \bigcirc_1 l_2, q^*)$
$\leq \max\{\Gamma^{\mathbb{F}}_H(l_1, \varrho^*), \Gamma^{\mathbb{F}}_H(l_2, \varrho^*)\}$. Choose $t, s \in [0, 1]$
such that $\Lambda_H(l_1 \bigcirc_1 l_2, \varrho^*) < t \leq \min\{\Lambda_H(l_1, \varrho^*), \Lambda_H(l_2, \varrho^*)\}$ and $\Xi_H(l_1 \bigcirc_1 l_2, \varrho^*) < t \leq \frac{\Xi_H(l_1,\varrho^*)+\Xi_H(l_2,\varrho^*)}{2}$ and $\Phi_H(l_1 \bigcirc_1 l_2, \varrho^*) > s \geq \max\{\Phi_H(l_1, \varrho^*), \Phi_H(l_2, \varrho^*)\}$. Then $l_1, l_2 \in H^{(t,s)}$, but $l_1 \bigcirc_1 l_2 \notin H^{(t,s)}$. This contradicts to that $H^{(t,s)}$ is a subbisemiring of $\mathbb{D}$.
Hence $\Lambda_H(l_1 \bigcirc_1 l_2, \varrho^*) \geq \min\{\Lambda_H(l_1, \varrho^*), \Lambda_H(l_2, \varrho^*)\}$, $\Xi_H(l_1 \bigcirc_1 l_2, \varrho^*) \geq \frac{\Xi_H(l_1,\varrho^*)+\Xi_H(l_2,\varrho^*)}{2}$ and $\Phi_H(l_1 \bigcirc_1 l_2, \varrho^*) \leq \max\{\Phi_H(l_1, \varrho^*), \Phi_H(l_2, \varrho^*)\}$. Similarly, $\bigcirc_2$ and $\bigcirc_3$ cases.
Hence $H = \langle(\Gamma^{\mathbb{T}}_H, \Gamma^{\mathbb{I}}_H, \Gamma^{\mathbb{F}}_H), (\Lambda_H, \Xi_H, \Phi_H)\rangle$ is a DioQNSBS of $\mathbb{D}$.

Definition 3.9. Let H be any DioQNSBS of $\mathbb{D}$, $a \in \mathbb{D}$ and P is any non-empty set. Then the pseudo diophantine Q-neutrosophic coset $(aH)^p$ is defined by

$$\begin{cases} ((a\Gamma^{\mathbb{T}}_H)^p)(l, \varrho^*) = p(a)\Gamma^{\mathbb{T}}_H(l, \varrho^*) \\ ((a\Gamma^{\mathbb{I}}_H)^p)(l, \varrho^*) = p(a)\Gamma^{\mathbb{I}}_H(l, \varrho^*) \\ ((a\Gamma^{\mathbb{F}}_H)^p)(l, \varrho^*) = p(a)\Gamma^{\mathbb{F}}_H(l, \varrho^*) \end{cases}$$

$$\begin{cases} ((a\Lambda_H)^p)(l, \varrho^*) = p(a)\Lambda_H(l, \varrho^*) \\ ((a\Xi_H)^p)(l, \varrho^*) = p(a)\Xi_H(l, \varrho^*) \\ ((a\Phi_H)^p)(l, \varrho^*) = p(a)\Phi_H(l, \varrho^*) \end{cases}$$

for every $l \in \mathbb{D}$ and for some $p \in P$.

Theorem 3.10. *Let H be any DioQNSBS of $\mathbb{D}$, then the pseudo diophantine Q-neutrosophic coset $(aH)^p$ is a DioQNSBS of $\mathbb{D}$, for every $a \in \mathbb{D}$.*

Proof. Let H be any DioQNSBS of $\mathbb{D}$ and for every $z, y \in \mathbb{D}$. Now, $((a\Gamma^{\mathbb{T}}_H)^p)(l \bigcirc_1 g, \varrho^*) = p(a)\ \Gamma^{\mathbb{T}}_H(l \bigcirc_1 g, \varrho^*) \geq p(a)\ min\{\Gamma^{\mathbb{T}}_H(l, \varrho^*), \Gamma^{\mathbb{T}}_H(g, \varrho^*)\} = min\{p(a)\ \Gamma^{\mathbb{T}}_H(l, \varrho^*), p(a)\ \Gamma^{\mathbb{T}}_H(g, \varrho^*)\} = min\{((a\Gamma^{\mathbb{T}}_H)^p)(l, \varrho^*), ((a\Gamma^{\mathbb{T}}_H)^p)(g, \varrho^*)\}$. Thus, $((a\Gamma^{\mathbb{T}}_H)^p)(l \bigcirc_1 g, \varrho^*) \geq \min\{((a\Gamma^{\mathbb{T}}_H)^p)(l, \varrho^*), ((a\Gamma^{\mathbb{T}}_H)^p)(g, \varrho^*)\}$. Now, $((a\Gamma^{\mathbb{I}}_H)^p)(l \bigcirc_1 g, \varrho^*) = p(a)\ \Gamma^{\mathbb{I}}_H(l \bigcirc_1 g, \varrho^*) \geq p(a)\left[\frac{\Gamma^{\mathbb{I}}_H(l,\varrho^*)+\Gamma^{\mathbb{I}}_H(g,\varrho^*)}{2}\right] = \frac{p(a)\ \Gamma^{\mathbb{I}}_H(l,\varrho^*)+p(a)\ \Gamma^{\mathbb{I}}_H(g,\varrho^*)}{2} = \frac{((a\Gamma^{\mathbb{I}}_H)^p)(l,\varrho^*)+((a\Gamma^{\mathbb{I}}_H)^p)(g,\varrho^*)}{2}$. Thus, $((a\Gamma^{\mathbb{I}}_H)^p)(l \bigcirc_1 g, \varrho^*) \geq \frac{((a\Gamma^{\mathbb{I}}_H)^p)(l,\varrho^*)+((a\Gamma^{\mathbb{I}}_H)^p)(g,\varrho^*)}{2}$. Now, $((a\Gamma^{\mathbb{F}}_H)^p)(l \bigcirc_1 g, \varrho^*) = p(a)\ \Gamma^{\mathbb{F}}_H(l \bigcirc_1 g, \varrho^*) \leq p(a)\ \max\{\Gamma^{\mathbb{F}}_H(l, \varrho^*), \Gamma^{\mathbb{F}}_H(g, \varrho^*)\} = \max\{p(a)\ \Gamma^{\mathbb{F}}_H(l, \varrho^*), p(a)\ \Gamma^{\mathbb{F}}_H(g, \varrho^*)\} =$

$\max\{((a\Gamma^{\mathbb{F}}_H)^p)(l, \varrho^*), ((a\Gamma^{\mathbb{F}}_H)^p)(g, \varrho^*)\}$.

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

Thus, $((a\Gamma^{\mathbb{F}}_H)^p)(l \bigcirc_1 g, \varrho^*) \leq \max\{((a\Gamma^{\mathbb{F}}_H)^p)(l, \varrho^*), ((a\Gamma^{\mathbb{F}}_H)^p)(g, \varrho^*)\}$. Now,

$$\begin{aligned} ((a\Lambda_H)^p)(l \bigcirc_1 g, \varrho^*) &= p(a)\ \Lambda_H(l \bigcirc_1 g, \varrho^*) \\ &\geq p(a)\ min\{\Lambda_H(l, \varrho^*), \Lambda_H(g, \varrho^*)\} \\ &= min\{p(a)\ \Lambda_H(l, \varrho^*), p(a)\ \Lambda_H(g, \varrho^*)\} \\ &= min\{((a\Lambda_H)^p)(l, \varrho^*), ((a\Lambda_H)^p)(g, \varrho^*)\}. \end{aligned}$$

Thus, $((a\Lambda_H)^p)(l \bigcirc_1 g, \varrho^*) \geq \min\{((a\Lambda_H)^p)(l, \varrho^*), ((a\Lambda_H)^p)(g, \varrho^*)\}$.
Now,

$$\begin{aligned} ((a\Xi_H)^p)(l \bigcirc_1 g, \varrho^*) &= p(a)\ \Xi_H(l \bigcirc_1 g, \varrho^*) \\ &\geq p(a)\left[\frac{\Xi_H(l, \varrho^*) + \Xi_H(g, \varrho^*)}{2}\right] \\ &= \frac{p(a)\ \Xi_H(l, \varrho^*) + p(a)\ \Xi_H(g, \varrho^*)}{2} \\ &= \frac{((a\Xi_H)^p)(l, \varrho^*) + ((a\Xi_H)^p)(g, \varrho^*)}{2}. \end{aligned}$$

Thus, $((a\Xi_H)^p)(l \bigcirc_1 g, \varrho^*) \geq \frac{((a\Xi_H)^p)(l, \varrho^*) + ((a\Xi_H)^p)(g, \varrho^*)}{2}$.
Now,

$$\begin{aligned} ((a\Phi_H)^p)(l \bigcirc_1 g, \varrho^*) &= p(a)\ \Phi_H(l \bigcirc_1 g, \varrho^*) \\ &\leq p(a)\ \max\{\Phi_H(l, \varrho^*), \Phi_H(g, \varrho^*)\} \\ &= \max\{p(a)\ \Phi_H(l, \varrho^*), p(a)\ \Phi_H(l, \varrho^*)\} \\ &= \max\{((a\Phi_H)^p)(l, \varrho^*), ((a\Phi_H)^p)(g, \varrho^*)\}. \end{aligned}$$

Thus, $((a\Phi_H)^p)(l \bigcirc_1 g, \varrho^*) \leq \max\{((a\Phi_H)^p)(l, \varrho^*), ((a\Phi_H)^p)(g, \varrho^*)\}$.
Similarly, $\bigcirc_2$ and $\bigcirc_3$ cases. Hence $(aH)^p$ is a DioQNSBS of $\mathbb{D}$.

4 Homomorphism

Definition 4.1. Let $(\mathbb{D}_1, \uplus_1, \uplus_2, \uplus_3)$ and $(\mathbb{S}_2, \bigcirc_1, \bigcirc_2, \bigcirc_3)$ be any two bisemirings. Let $\Theta : \mathbb{D}_1 \to \mathbb{D}_2$ be any function and H be any DioQNSBS in $\mathbb{D}_1$, M be any DioQNSBS in $\Theta(\mathbb{D}_1) = \mathbb{D}_2$. If $\Gamma_H = \langle(\Gamma^{\mathbb{T}}_H, \Gamma^{\mathbb{I}}_H, \Gamma^{\mathbb{F}}_H), (\Lambda_H, \Xi_H, \Phi_H)\rangle$ is a DioQNSS in $\mathbb{D}_1$, then Γ_M is a DioQNSS in $\mathbb{D}_2$, defined by

$$\Gamma^{\mathbb{T}}_M(g, \varrho^*) = \begin{cases} \sup \Gamma^{\mathbb{T}}_H(l, \varrho^*) & if\ \ l \in \Theta^{-1}y \\ 0 & otherwise \end{cases} \qquad \Gamma^{\mathbb{I}}_M(g, \varrho^*) = \begin{cases} \sup \Gamma^{\mathbb{I}}_H(l, \varrho^*) & if\ \ z \in \Theta^{-1}y \\ 0 & otherwise \end{cases}$$

$$\Gamma^{\mathbb{F}}_M(g, \varrho^*) = \begin{cases} \inf \Gamma^{\mathbb{F}}_H(l, \varrho^*) & if\ \ l \in \Theta^{-1}y \\ 1 & otherwise \end{cases}$$

$$\Lambda_M(g, \varrho^*) = \begin{cases} \sup \Lambda_H(l, \varrho^*) & if\ \ l \in \Theta^{-1}y \\ 0 & otherwise \end{cases} \qquad \Xi_M(g, \varrho^*) = \begin{cases} \sup \Xi_H(l, \varrho^*) & if\ \ l \in \Theta^{-1}g \\ 0 & otherwise \end{cases}$$

$$\Phi_M(g, \varrho^*) = \begin{cases} \inf \Phi_H(l, \varrho^*) & if\ \ l \in \Theta^{-1}g \\ 1 & otherwise \end{cases}$$

for all $l \in \mathbb{D}_1$ and $g \in \mathbb{D}_2$ is called the image of Γ_H under Θ.
Similarly, If $\Gamma_M = \langle(\Gamma^{\mathbb{T}}_M, \Gamma^{\mathbb{I}}_M, \Gamma^{\mathbb{F}}_M), (\Lambda_M, \Xi_M, \Phi_M)\rangle$ is a DioQNSS in $\mathbb{D}_2$, then DioQNSS $\Gamma_H = \Theta \circ \Gamma_M$ in $\mathbb{D}_1$ [ie, the DioQNSS defined by $\Gamma_H(l, \varrho^*) = \Gamma_M(\Theta(l, \varrho^*))$] is called the preimage of Γ_M under Θ.

Theorem 4.2. *Every homomorphic image of DioQNSBS of $\mathbb{D}_1$ is a DioQNSBS of $\mathbb{D}_2$.*

https://doi.org/10.54216/IJNS.220207
Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

Proof. Let $\Theta : \mathbb{D}_1 \to \mathbb{D}_2$ be any homomorphism. Then $\Theta(l \uplus_1 g, \varrho^*) = \Theta(l, \varrho^*) \bigcirc_1 \Theta(g, \varrho^*), \Theta(l \uplus_2 g) = \Theta(l, \varrho^*) \bigcirc_2 \Theta(g, \varrho^*)$ and $\Theta(l \uplus_3 g) = \Theta(l, \varrho^*) \bigcirc_3 \Theta(g, \varrho^*)$ for all $x, y \in \mathbb{D}_1$. Let $M = \Theta(H)$, H is any DioQNSBS of $\mathbb{D}_1$. Let $\Theta(l, \varrho^*), \Theta(g, \varrho^*) \in \mathbb{D}_2$. Let $l \in \Theta^{-1}(\Theta(l, \varrho^*))$ and $g \in \Theta^{-1}(\Theta(g, \varrho^*))$ be such that $\Gamma^{\mathbb{T}}_H(l, \varrho^*) = \sup_{(l', \varrho^*) \in \Theta^{-1}(\Theta(l,\varrho^*))} \Gamma^{\mathbb{T}}_H((l', \varrho^*))$ and $\Gamma^{\mathbb{T}}_H(g, \varrho^*) = \sup_{(l', \varrho^*) \in \Theta^{-1}(\Theta(g,\varrho^*))} \Gamma^{\mathbb{T}}_H((l', \varrho^*))$. Now,

$$\begin{aligned}
\Gamma^{\mathbb{T}}_M(\Theta(l, \varrho^*) \bigcirc_1 \Theta(g, \varrho^*)) &= \sup_{(l'', \varrho^*) \in \Theta^{-1}(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*))} \Gamma^{\mathbb{T}}_H((l'', \varrho^*)) \\
&= \sup_{(l'', \varrho^*) \in \Theta^{-1}(\Theta(l \uplus_1 g)} \Gamma^{\mathbb{T}}_H((l'', \varrho^*)) \\
&= \Gamma^{\mathbb{T}}_H(l \uplus_1 g, \varrho^*) \\
&\geq \min\{\Gamma^{\mathbb{T}}_H(l, \varrho^*), \Gamma^{\mathbb{T}}_H(g, \varrho^*)\} \\
&= \min\{\Gamma^{\mathbb{T}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{T}}_M \Theta(g, \varrho^*)\}.
\end{aligned}$$

Thus, $\Gamma^{\mathbb{T}}_M(\Theta(l, \varrho^*) \bigcirc_1 \Theta(g, \varrho^*)) \geq \min\{\Gamma^{\mathbb{T}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{T}}_M \Theta(g, \varrho^*)\}$.
Similarly, $\Gamma^{\mathbb{T}}_M(\Theta(l, \varrho^*) \bigcirc_2 \Theta(g, \varrho^*)) \geq \min\{\Gamma^{\mathbb{T}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{T}}_M \Theta(g, \varrho^*)\}$ and
$\Gamma^{\mathbb{T}}_M(\Theta(l, \varrho^*) \bigcirc_3 \Theta(g, \varrho^*)) \geq \min\{\Gamma^{\mathbb{T}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{T}}_M \Theta(g, \varrho^*)\}$.
Let $z \in \Theta^{-1}(\Theta(l, \varrho^*))$ and $y \in \Theta^{-1}(\Theta(g, \varrho^*))$ be such that $\Gamma^{\mathbb{I}}_H(l, \varrho^*) = \sup_{(l', \varrho^*) \in \Theta^{-1}(\Theta(l,\varrho^*))} \Gamma^{\mathbb{I}}_H((l', \varrho^*))$ and $\Gamma^{\mathbb{I}}_H(g, \varrho^*) = \sup_{(l', \varrho^*) \in \Theta^{-1}(\Theta(g,\varrho^*))} \Gamma^{\mathbb{I}}_H((l', q^*))$. Now,

$$\begin{aligned}
\Gamma^{\mathbb{I}}_M(\Theta(l, \varrho^*) \bigcirc_1 \Theta(g, \varrho^*)) &= \sup_{(l'', \varrho^*) \in \Theta^{-1}(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*))} \Gamma^{\mathbb{I}}_H((l'', \varrho^*)) \\
&= \sup_{(l'', \varrho^*) \in \Theta^{-1}(\Theta(l \uplus_1 g)} \Gamma^{\mathbb{I}}_H((l'', \varrho^*)) \\
&= \Gamma^{\mathbb{I}}_H(l \uplus_1 g, \varrho^*) \\
&\geq \frac{\Gamma^{\mathbb{I}}_H(l, \varrho^*) + \Gamma^{\mathbb{I}}_H(g, \varrho^*)}{2} \\
&= \frac{\Gamma^{\mathbb{I}}_M \Theta(l, \varrho^*) + \Gamma^{\mathbb{I}}_M \Theta(g, \varrho^*)}{2}.
\end{aligned}$$

Thus, $\Gamma^{\mathbb{I}}_M(\Theta(l, \varrho^*) \bigcirc_1 \Theta(g, \varrho^*)) \geq \frac{\Gamma^{\mathbb{I}}_M \Theta(l,\varrho^*) + \Gamma^{\mathbb{I}}_M \Theta(g,\varrho^*)}{2}$.
Similarly, $\Gamma^{\mathbb{I}}_M(\Theta(l, \varrho^*) \bigcirc_2 \Theta(g, \varrho^*)) \geq \frac{\Gamma^{\mathbb{I}}_M \Theta(l,\varrho^*) + \Gamma^{\mathbb{I}}_M \Theta(g,\varrho^*)}{2}$ and
$\Gamma^{\mathbb{I}}_M(\Theta(l, \varrho^*) \bigcirc_3 \Theta(g, \varrho^*)) \geq \frac{\Gamma^{\mathbb{I}}_M \Theta(l,\varrho^*) + \Gamma^{\mathbb{I}}_M \Theta(g,\varrho^*)}{2}$.
Let $\Theta(l, \varrho^*), \Theta(g, \varrho^*) \in \mathbb{D}_2$. Let $z \in \Theta^{-1}(\Theta(l, \varrho^*))$ and $g \in \Theta^{-1}(\Theta(g, \varrho^*))$ be such that
$\Gamma^{\mathbb{F}}_H(l, \varrho^*) = \inf_{(l', \varrho^*) \in \Theta^{-1}(\Theta(l,\varrho^*))} \Gamma^{\mathbb{F}}_H((l', \varrho^*))$ and $\Gamma^{\mathbb{F}}_H(g, \varrho^*) = \inf_{(l', \varrho^*) \in \Theta^{-1}(\Theta(g,\varrho^*))} \Gamma^{\mathbb{F}}_H((l', \varrho^*))$.
Now,

$$\begin{aligned}
\Gamma^{\mathbb{F}}_M(\Theta(z, q^*) \bigcirc_1 \Theta(g, \varrho^*)) &= \inf_{(l'', \varrho^*) \in \Theta^{-1}(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*))} \Gamma^{\mathbb{F}}_H((l'', \varrho^*)) \\
&= \inf_{(l'', \varrho^*) \in \Theta^{-1}(\Theta(l \uplus_1 g)} \Gamma^{\mathbb{F}}_H((l'', \varrho^*)) \\
&= \Gamma^{\mathbb{F}}_H(l \uplus_1 g, \varrho^*) \\
&\leq \max\{\Gamma^{\mathbb{F}}_H(l, \varrho^*), \Gamma^{\mathbb{F}}_H(g, \varrho^*)\} \\
&= \max\{\Gamma^{\mathbb{F}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{F}}_M \Theta(g, \varrho^*)\}.
\end{aligned}$$

Thus, $\Gamma^{\mathbb{F}}_M(\Theta(l, \varrho^*) \bigcirc_1 \Theta(g, \varrho^*)) \leq \max\{\Gamma^{\mathbb{F}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{F}}_M \Theta(g, \varrho^*)\}$.
Similarly, $\Gamma^{\mathbb{F}}_M(\Theta(l, \varrho^*) \bigcirc_2 \Theta(g, \varrho^*)) \leq \max\{\Gamma^{\mathbb{F}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{F}}_M \Theta(g, \varrho^*)\}$ and
$\Gamma^{\mathbb{F}}_M(\Theta(l, \varrho^*) \bigcirc_3 \Theta(g, \varrho^*)) \leq \max\{\Gamma^{\mathbb{F}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{F}}_M \Theta(g, \varrho^*)\}$.
Let $l \in \Theta^{-1}(\Theta(l, \varrho^*))$ and $g \in \Theta^{-1}(\Theta(g, \varrho^*))$ be such that $\Lambda_H(l, \varrho^*) = \sup_{(l', \varrho^*) \in \Theta^{-1}(\Theta(l,\varrho^*))} \Lambda_H((l', \varrho^*))$ and

https://doi.org/10.54216/IJNS.220207

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

$\Lambda_H(g,\varrho^*) = \sup_{(l',\varrho^*)\in\Theta^{-1}(\Theta(g,\varrho^*))} \Lambda_H((l',\varrho^*))$. Now,

$$\begin{aligned}
\Lambda_M(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*)) &= \sup_{(l'',\varrho^*)\in\Theta^{-1}(\Theta(l,\varrho^*)\bigcirc_1\Theta(g,\varrho^*))} \Lambda_H((l'',\varrho^*)) \\
&= \sup_{(l'',\varrho^*)\in\Theta^{-1}(\Theta(l\uplus_1 g)} \Lambda_H((l'',\varrho^*)) \\
&= \Lambda_H(l\uplus_1 g,\varrho^*) \\
&\geq \min\{\Lambda_H(l,\varrho^*),\Lambda_H(g,\varrho^*)\} \\
&= \min\{\Lambda_M\Theta(l,\varrho^*),\Lambda_M\Theta(g,\varrho^*)\}.
\end{aligned}$$

Thus, $\Lambda_M(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*)) \geq \min\{\Lambda_M\Theta(l,\varrho^*),\Lambda_M\Theta(g,\varrho^*)\}$.
Similarly, $\Lambda_M(\Theta(l,\varrho^*) \bigcirc_2 \Theta(g,\varrho^*)) \geq \min\{\Lambda_M\Theta(l,\varrho^*),\Lambda_M\Theta(g,\varrho^*)\}$ and
$\Lambda_M(\Theta(l,\varrho^*) \bigcirc_3 \Theta(g,\varrho^*)) \geq \min\{\Lambda_M\Theta(l,\varrho^*),\Lambda_M\Theta(g,\varrho^*)\}$.
Let $l \in \Theta^{-1}(\Theta(l,\varrho^*))$ and $g \in \Theta^{-1}(\Theta(g,\varrho^*))$ be such that $\Xi_H(l,\varrho^*) = \sup_{(l',\varrho^*)\in\Theta^{-1}(\Theta(l,\varrho^*))} \Xi_H((l',\varrho^*))$ and

$\Xi_H(g,\varrho^*) = \sup_{(l',\varrho^*)\in\Theta^{-1}(\Theta(g,\varrho^*))} \Xi_H((l',\varrho^*))$. Now,

$$\begin{aligned}
\Xi_M(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*)) &= \sup_{(l'',\varrho^*)\in\Theta^{-1}(\Theta(l,\varrho^*)\bigcirc_1\Theta(g,\varrho^*))} \Xi_H((l'',\varrho^*)) \\
&= \sup_{(l'',\varrho^*)\in\Theta^{-1}(\Theta(l\uplus_1 g)} \Xi_H((l'',\varrho^*)) \\
&= \Xi_H(z\uplus_1 y,q^*) \\
&\geq \frac{\Xi_H(l,\varrho^*)+\Xi_H(g,\varrho^*)}{2} \\
&= \frac{\Xi_M\Theta(l,\varrho^*)+\Xi_M\Theta(g,\varrho^*)}{2}.
\end{aligned}$$

Thus, $\Xi_M(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*)) \geq \frac{\Xi_M\Theta(l,\varrho^*)+\Xi_M\Theta(g,\varrho^*)}{2}$.
Similarly, $\Xi_M(\Theta(l,\varrho^*)\bigcirc_2\Theta(g,\varrho^*)) \geq \frac{\Xi_M\Theta(l,\varrho^*)+\Xi_M\Theta(g,\varrho^*)}{2}$ and $\Xi_M(\Theta(g,\varrho^*)\bigcirc_3\Theta(g,\varrho^*)) \geq \frac{\Xi_M\Theta(l,\varrho^*)+\Xi_M\Theta(g,\varrho^*)}{2}$.
Let $\Theta(l,\varrho^*),\Theta(g,\varrho^*) \in \mathbb{D}_2$. Let $l \in \Theta^{-1}(\Theta(l,\varrho^*))$ and $g \in \Theta^{-1}(\Theta(g,\varrho^*))$ be such that
$\Phi_H(l,\varrho^*) = \inf_{(l',\varrho^*)\in\Theta^{-1}(\Theta(l,\varrho^*))} \Phi_H((l',\varrho^*))$ and $\Phi_H(l,\varrho^*) = \inf_{(l',\varrho^*)\in\Theta^{-1}(\Theta(l,\varrho^*))} \Phi_H((l',\varrho^*))$. Now,

$$\begin{aligned}
\Phi_M(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*)) &= \inf_{(l'',\varrho^*)\in\Theta^{-1}(\Theta(l,\varrho^*)\bigcirc_1\Theta(g,\varrho^*))} \Phi_H((l'',\varrho^*)) \\
&= \inf_{(l'',\varrho^*)\in\Theta^{-1}(\Theta(l\uplus_1 g)} \Phi_H((l'',\varrho^*)) \\
&= \Phi_H l\uplus_1 g,\varrho^*) \\
&\leq \max\{\Phi_H(l,\varrho^*),\Phi_H(g,\varrho^*)\} \\
&= \max\{\Phi_M\Theta(l,\varrho^*),\Phi_M\Theta(g,\varrho^*)\}.
\end{aligned}$$

Thus, $\Phi_M(\Theta(l,\varrho^*) \bigcirc_1 \Theta(g,\varrho^*)) \leq \max\{\Phi_M\Theta(l,\varrho^*),\Phi_M\Theta(g,\varrho^*)\}$.
Similarly, $\Phi_M(\Theta(l,\varrho^*) \bigcirc_2 \Theta(g,\varrho^*)) \leq \max\{\Phi_M\Theta(l,\varrho^*),\Phi_M\Theta(g,\varrho^*)\}$ and
$\Phi_M(\Theta(l,\varrho^*) \bigcirc_3 \Theta(g,\varrho^*)) \leq \max\{\Phi_M\Theta(l,\varrho^*),\Phi_M\Theta(g,\varrho^*)\}$. Hence M is a DioQNSBS of $\mathbb{D}_2$.

Theorem 4.3. *Let $(\mathbb{D}_1,\biguplus_1,\biguplus_2,\biguplus_3)$ and $(\mathbb{D}_2,\bigodot_1,\bigodot_2,\bigodot_3)$ be any two bisemirings. The homomorphic preimage of DioQNSBS of $\mathbb{D}_2$ is a DioQNSBS of $\mathbb{D}_1$.*

Proof. Let $\Theta:\mathbb{D}_1\to\mathbb{D}_2$ be any homomorphism. Then $\Theta(l\biguplus_1 g,\varrho^*) = \Theta(l,\varrho^*)\bigodot_1\Theta(g,\varrho^*), \Theta(l\biguplus_2 g) = \Theta(l,\varrho^*)\bigodot_2\Theta(g,\varrho^*)$ and $\Theta(l\biguplus_3 g) = \Theta(l,\varrho^*)\bigodot_3\Theta(g,\varrho^*)$ for all $l,g\in\mathbb{D}_1$. Let $M=\Theta(H)$, where M is any DioQNSBS of $\mathbb{D}_2$. Let $l,g\in\mathbb{D}_1$.
Now, $\Gamma^{\mathbb{T}}_H(l\biguplus_1 g,\varrho^*) = \Gamma^{\mathbb{T}}_M(\Theta(l\biguplus_1 g,\varrho^*)) = \Gamma^{\mathbb{T}}_M(\Theta(l,\varrho^*)\bigodot_1\Theta(g,\varrho^*)) \geq \min\{\Gamma^{\mathbb{T}}_M\Theta(l,\varrho^*),\Gamma^{\mathbb{T}}_M\Theta(g,\varrho^*)\} = \min\{\Gamma^{\mathbb{T}}_H(l,\varrho^*),\Gamma^{\mathbb{T}}_H(g,\varrho^*)\}$. Thus, $\Gamma^{\mathbb{T}}_H(l\biguplus_1 g,\varrho^*) \geq \min\{\Gamma^{\mathbb{T}}_H(l,\varrho^*),\Gamma^{\mathbb{T}}_H(g,\varrho^*)\}$.
Now, $\Gamma^{\mathbb{I}}_H(l\biguplus_1 g,\varrho^*) = \Gamma^{\mathbb{I}}_M(\Theta(l\biguplus_1 g,\varrho^*)) = \Gamma^{\mathbb{I}}_M(\Theta(l,\varrho^*)\bigodot_1\Theta(g,\varrho^*))$
$\geq \frac{\Gamma^{\mathbb{I}}_M\Theta(l,\varrho^*)+\Gamma^{\mathbb{I}}_M\Theta(g,\varrho^*)}{2} = \frac{\Gamma^{\mathbb{I}}_H(l,\varrho^*)+\Gamma^{\mathbb{I}}_H(g,\varrho^*)}{2}$. Thus, $\Gamma^{\mathbb{I}}_H(l\biguplus_1 g,\varrho^*) \geq \frac{\Gamma^{\mathbb{I}}_H(l,\varrho^*)+\Gamma^{\mathbb{I}}_H(g,\varrho^*)}{2}$.

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

Now, $\Gamma^{\mathbb{F}}_H(l \uplus_1 g, \varrho^*) = \Gamma^{\mathbb{F}}_M(\Theta(l \uplus_1 g, \varrho^*)) = \Gamma^{\mathbb{F}}_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \leq \max\{\Gamma^{\mathbb{F}}_M \Theta(l, \varrho^*), \Gamma^{\mathbb{F}}_M \Theta(g, \varrho^*)\} = \max\{\Gamma^{\mathbb{F}}_H(l, \varrho^*), \Gamma^{\mathbb{F}}_H(g, \varrho^*)\}$. Thus, $\Gamma^{\mathbb{F}}_H(l \uplus_1 g, \varrho^*) \leq \max\{\Gamma^{\mathbb{F}}_H(l, \varrho^*), \Gamma^{\mathbb{F}}_H(g, \varrho^*)\}$.
Now, $\Lambda_H(l \uplus_1 g, \varrho^*) = \Lambda_M(\Theta(l \uplus_1 g, \varrho^*)) = \Lambda_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*))$
$\geq \min\{\Lambda_M \Theta(l, \varrho^*), \Lambda_M \Theta(g, \varrho^*)\} = \min\{\Lambda_H(l, \varrho^*), \Lambda_H(g, \varrho^*)\}$.
Thus, $\Lambda_H(l \uplus_1 g, \varrho^*) \geq \min\{\Lambda_H(l, \varrho^*), \Lambda_H(g, \varrho^*)\}$.
Now, $\Xi_H(l \uplus_1 g, \varrho^*) = \Xi_M(\Theta(l \uplus_1 g, \varrho^*)) = \Xi_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*))$
$\geq \frac{\Xi_M \Theta(l, \varrho^*) + \Xi_M \Theta(g, \varrho^*)}{2} = \frac{\Xi_H(l, \varrho^*) + \Xi_H(g, \varrho^*)}{2}$.
Thus, $\Xi_H(l \uplus_1 g, \varrho^*) \geq \frac{\Xi_H(l, \varrho^*) + \Xi_H(g, g^*)}{2}$.
Now, $\Phi_H(l \uplus_1 g, \varrho^*) = \Phi_M(\Theta(l \uplus_1 g, \varrho^*)) = \Phi_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \leq \max\{\Phi_M \Theta(l, \varrho^*), \Phi_M \Theta(g, \varrho^*)\} = \max\{\Phi_H(l, \varrho^*), \Phi_H(g, \varrho^*)\}$.
Thus, $\Phi_H(l \uplus_1 g, \varrho^*) \leq \max\{\Phi_H(l, \varrho^*), \Phi_H(g, \varrho^*)\}$. Similarly to prove two other operations, hence H is a DioQNSBS of $\mathbb{D}_1$.

Theorem 4.4. *If $\Theta : \mathbb{D}_1 \to \mathbb{D}_2$ is a homomorphism, then $\Theta(H_{(t,s)})$ is a level subbisemiring of DioQNSBS M of $\mathbb{D}_2$.*

Proof. Let $\Theta : \mathbb{D}_1 \to \mathbb{D}_2$ be any homomorphism. Then $\Theta(l \uplus_1 g, \varrho^*) = \Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*), \Theta(l \uplus_2 g) = \Theta(l, \varrho^*) \bigodot_2 \Theta(g, \varrho^*)$ and $\Theta(l \uplus_3 g) = \Theta(l, \varrho^*) \bigodot_3 \Theta(g, \varrho^*)$ for all $l, g \in \mathbb{D}_1$. Let $M = \Theta(H)$, H is a DioQNSBS of $\mathbb{D}_1$. By Theorem 4.2, M is a DioQNSBS of $\mathbb{D}_2$. Let $H_{(t,s)}$ be any level subbisemiring of H. Suppose that $l, g \in H_{(t,s)}$. Then $\Theta(l \uplus_1 g, \varrho^*), \Theta(l \uplus_2 g)$ and $\Theta(l \uplus_3 g) \in H_{(t,s)}$. Now, $\Gamma^{\mathbb{T}}_M(\Theta(l, \varrho^*)) = \Gamma^{\mathbb{T}}_H(l, \varrho^*) \geq t, \Gamma^{\mathbb{T}}_M(\Theta(g, \varrho^*)) = \Gamma^{\mathbb{T}}_H(g, \varrho^*) \geq t$. Thus, $\Gamma^{\mathbb{T}}_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \geq \Gamma^{\mathbb{T}}_H(l \uplus_1 g, \varrho^*) \geq t$. Now, $\Gamma^{\mathbb{I}}_M(\Theta(l, \varrho^*)) = \Gamma^{\mathbb{I}}_H(l, \varrho^*) \geq t, \Gamma^{\mathbb{I}}_M(\Theta(g, \varrho^*)) = \Gamma^{\mathbb{I}}_H(g, \varrho^*) \geq t$. Thus, $\Gamma^{\mathbb{I}}_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \geq \Gamma^{\mathbb{I}}_H(l \uplus_1 g, \varrho^*) \geq t$. Now, $\Gamma^{\mathbb{F}}_M(\Theta(l, \varrho^*)) = \Gamma^{\mathbb{F}}_H(n, \varrho^*) \leq s, \Gamma^{\mathbb{F}}_M(\Theta(n, \varrho^*)) = \Gamma^{\mathbb{F}}_H(g, \varrho^*) \leq s$. Thus, $\Gamma^{\mathbb{F}}_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \leq \Gamma^{\mathbb{F}}_H(l \uplus_1 g, \varrho^*) \leq s$, for all $\Theta(l, \varrho^*), \Theta(g, \varrho^*) \in \mathbb{D}_2$. Now, $\Lambda_{HM}(\Theta(l, \varrho^*)) = \Lambda_H(l, \varrho^*) \geq t, \Lambda_M(\Theta(g, \varrho^*)) = \Lambda(n, \varrho^*) \geq t$. Thus, $\Lambda_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \geq \Lambda_H(l \uplus_1 g, \varrho^*) \geq t$. Now, $\Xi_M(\Theta(l, \varrho^*)) = \Xi_H(l, \varrho^*) \geq t, \Xi_M(\Theta(g, \varrho^*)) = \Xi_H(g, \varrho^*) \geq t$. Thus, $\Xi_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \geq \Xi_H(l \uplus_1 g, \varrho^*) \geq t$. Now, $\Phi_M(\Theta(l, \varrho^*)) = \Phi_H(l, \varrho^*) \leq s, \Phi_M(\Theta(g, \varrho^*)) = \Phi_H(g, \varrho^*) \leq s$. Thus, $\Phi_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \leq \Phi_H(l \uplus_1 g, \varrho^*) \leq s$, for all $\Theta(l, \varrho^*), \Theta(g, \varrho^*) \in \mathbb{D}_2$. Similarly to prove other operations, hence $\Theta(H_{(t,s)})$ is a level subbisemiring of DioQNSBS M of $\mathbb{D}_2$.

Theorem 4.5. *If $\Theta : \mathbb{D}_1 \to \mathbb{D}_2$ is any homomorphism, then $H_{(t,s)}$ is a level subbisemiring of DioQNSBS H of $\mathbb{D}_1$.*

Proof. Let $\Theta : \mathbb{D}_1 \to \mathbb{D}_2$ be any homomorphism. Then $\Theta(l \uplus_1 g, \varrho^*) = \Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*), \Theta(l \uplus_2 g) = \Theta(l, \varrho^*) \bigodot_2 \Theta(g, \varrho^*)$ and $\Theta(l \uplus_3 g) = \Theta(l, \varrho^*) \bigodot_3 \Theta(g, \varrho^*)$ for all $l, g \in \mathbb{D}_1$. Let $M = \Theta(H)$, M is a DioQNSBS of $\mathbb{D}_2$. By Theorem 4.3, H is a DioQNSBS of $\mathbb{D}_1$. Let $\Theta(H_{(t,s)})$ be a level subbisemiring of M. Suppose that $\Theta(l, \varrho^*), \Theta(g, \varrho^*) \in \Theta(H_{(t,s)})$. Then $\Theta(l \uplus_1 g, \varrho^*), \Theta(l \uplus_2 g)$ and $\Theta(l \uplus_3 g) \in \Theta(H_{(t,s)})$. Now, $\Gamma^{\mathbb{T}}_H(l, \varrho^*) = \Gamma^{\mathbb{T}}_M(\Theta(l, \varrho^*)) \geq t, \Gamma^{\mathbb{T}}_H(g, \varrho^*) = \Gamma^{\mathbb{T}}_M(\Theta(g, \varrho^*)) \geq t$. Thus, $\Gamma^{\mathbb{T}}_H(l \uplus_1 g, \varrho^*) \geq \min\{\Gamma^{\mathbb{T}}_H(l, \varrho^*), \Gamma^{\mathbb{T}}_H(g, \varrho^*)\} \geq t$. Now, $\Gamma^{\mathbb{I}}_H(l, \varrho^*) = \Gamma^{\mathbb{I}}_M(\Theta(l, \varrho^*)) \geq t, \Gamma^{\mathbb{I}}_H(g, \varrho^*) = \Gamma^{\mathbb{I}}_M(\Theta(g, \varrho^*)) \geq t$. Thus, $\Gamma^{\mathbb{I}}_H(l \uplus_1 g, \varrho^*) \geq \frac{\Gamma^{\mathbb{I}}_H(l, \varrho^*) + \Gamma^{\mathbb{I}}_H(g, \varrho^*)}{2} \geq t$. Now, $\Gamma^{\mathbb{F}}_H(l, \varrho^*) = \Gamma^{\mathbb{F}}_M(\Theta(l, \varrho^*)) \leq s, \Gamma^{\mathbb{F}}_H(g, \varrho^*) = \Gamma^{\mathbb{F}}_M(\Theta(g, \varrho^*)) \leq s$. Thus, $\Gamma^{\mathbb{F}}_H(l \uplus_1 g, \varrho^*) = \Gamma^{\mathbb{F}}_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \leq \max\{\Gamma^{\mathbb{F}}_H(l, \varrho^*), \Gamma^{\mathbb{F}}_H(g, \varrho^*)\} \leq s$, for all $l, g \in \mathbb{D}_1$. Now, $\Lambda_H(l, \varrho^*) = \Lambda_{HM}(\Theta(l, \varrho^*)) \geq t, \Lambda_H(g, \varrho^*) = \Lambda_M(\Theta(g, \varrho^*)) \geq t$. Thus, $\Lambda_H(l \uplus_1 g, \varrho^*) \geq \min\{\Lambda_H(l, \varrho^*), \Lambda_H(g, \varrho^*)\} \geq t$. Now, $\Xi_H(l, \varrho^*) = \Xi_M(\Theta(l, \varrho^*)) \geq t, \Xi_H(g, \varrho^*) = \Xi_M(\Theta(g, \varrho^*)) \geq t$. Thus, $\Xi_H(l \uplus_1 g, \varrho^*) \geq \frac{\Xi_H(l, \varrho^*) + \Xi_H(g, \varrho^*)}{2} \geq t$. Now, $\Phi_H(l, \varrho^*) = \Phi_M(\Theta(l, \varrho^*)) \leq s, \Phi_H(g, \varrho^*) = \Phi_M(\Theta(g, \varrho^*)) \leq s$.
Thus, $\Phi_H(l \uplus_1 g, \varrho^*) = \Phi_M(\Theta(l, \varrho^*) \bigodot_1 \Theta(g, \varrho^*)) \leq \max\{\Phi_H(l, \varrho^*), \Phi_H(g, \varrho^*)\} \leq s$, for all $l, g \in \mathbb{D}_1$. Similarly to prove other two operations, hence $H_{(t,s)}$ is a level subbisemiring of DioQNSBS H of $\mathbb{D}_1$.

5 Conclusion:

The concepts DioQNSBS is developed. We proposed the extension of fuzzy subbisemiring over bisemiring. Presenting a homomorphism fuzzy subbisemirings of subbisemirings to DioQNSBSs of subbisemirings is the major objective of this work. Consequently, in the future, we should consider the applications of interval valued soft neutrosophic subbisemirings.

Acknowledgments:

The author is obliged and thankful to the reviewer for the numerous and significant suggestions that raised the consistency of the ideas presented in this paper.

References

[1] Ahsan.J, Saifullah.K, Khan.F, Fuzzy semirings,Fuzzy Sets and systems,60,(1993),309-320.

[2] Atanassov. K, Intuitionistic fuzzy sets, Fuzzy Sets and Systems,20(1),(1986),87-96.

[3] Faward Hussian,Raja Muhammad Hashism, Ajab Khan,Muhammad Naeem, Generalization of bisemirings, International Journal of Computer Science and Information Security,14(9),(2016), 275–289. London.

[4] Golan.S.J, Semirings and their Applications, Kluwer Academic Publishers, 1999;

[5] Iseki.K, Ideal theory of semiring, 1956; Proceedings of the Japan Academy, Vol. 32. pp. 554-559.

[6] Iseki.K, Ideals in semirings, Proceedings of the Japan Academy, 34,(1958),29-31.

[7] Iski.K, Quasi-ideals in semirings without zero,Proceedings of the Japan Academy,34(2),(1958),79-81.

[8] Palanikumar.M, Arulmozhi.K, (α, β)-neutrosophic subbisemiring of bisemiring,Neutrosophic Sets and Systems,48,(2022),368-385.

[9] Palanikumar.M, Arulmozhi.K,On intuitionistic fuzzy normal subbisemiring of bisemiring,Nonlinear Studies, 28(3),(2021),717-721.

[10] Palanikumar.M, Arulmozhi.K,Jana.C,Multiple attribute decision-making approach for Pythagorean neutrosophic normal interval-valued aggregation operators, Comp. Appl. Math. 41(90), (2022), 1-27.

[11] Palanikumar.M, Arulmozhi.K,On Pythagorean normal subbisemiring of bisemiring,Annals of Communications in Mathematics,4,(2021), 63-72

[12] Palanikumar.M, Arulmozhi.K,On new approach towards cubic vague subbisemirings in bisemirings,Annals of Communications in Mathematics,4,(2021), 237-248

[13] Palanikumar.M, Iampan. A, Spherical fermatean interval valued fuzzy soft set based on multi criteria group decision making, International Journal of Innovative Computing, Information and Control,18(2),(2022),607-619.

[14] Palanikumar.M,Arulmozhi.K, Iampan.A, Interval valued Neutrosophic Subbisemiring of Bisemiring,International Journal of Neutrosophic Science,19(1),(2022), 116-131.

[15] Palanikumar.M, Iampan.A, Novel approach to decision making based on type-II generalized fermatean bipolar fuzzy soft sets,International Journal of Innovative Computing, Information and Control,18(3),(2022), 769-781.

[16] Palanikumar.M,Arulmozhi.K, Iampan.A, Interval valued Neutrosophic Subbisemiring of Bisemiring,International Journal of Neutrosophic Science,19(1),(2022), 116-131.

[17] Palanikumar.M,Arulmozhi.K, Iampan.A, Said Broumi,New algebraic extension of interval valued Q-neutrosophic normal subbisemirings of bisemirings,International Journal of Neutrosophic Science,20(1),(2023), 106-118.

[18] Palanikumar.M, Iampan.A, Arulmozhi.K,Iranian.D,Seethalakshmy,Raghavendran.R,New approach towards $(\varsigma_1, \varsigma_2)$-interval valued Q_1 neutrosophic subbisemirings of bisemirings and its extension,International Journal of Neutrosophic Science,20(1),(2023), 49-58.

[19] Riaz.M, Hashmi,M.R, Linear diophantine fuzzy set and its applications towards multi attribute decision making problems, Journal of Intelligent and Fuzzy Systems,37(4),(2019),5417-5439.

[20] Sen.M.K, Ghosh.S, An introduction to bisemirings,Southeast Asian Bulletin of Mathematics,28(3),(2001), 547-559.

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023

[21] Smarandache.F, A unifying field in logics Neutrosophy Neutrosophic Probability,Set and Logic, Rehoboth American Research Press,(1999).

[22] Vandiver, H. S, Note on a simple type of algebra in which the cancellation law of addition does not hold, Yulletin of the American Mathematical Society, 40(12), (1934), 914-920.

[23] Zadeh.L.A, Fuzzy sets, Information and Control,8,(1965),38-353.

https://doi.org/10.54216/IJNS.220207

Received: April 21, 2023 Revised: June 20, 2023 Accepted: September 19, 2023