An Approach To Symbolic n-Plithogenic Square Real Matrices For 9$\leq n \leq$ 12

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Abstract

The concept of symbolic n-plithogenic algebraic matrices as symmetric structures with n+1 symmetric classical components with the special definition of the multiplication operation. This paper is dedicated to studying the properties of symbolic 10, and 9-plithogenic real square matrices and 11, 12-plithogenic real matrices from algebraic point of view, where algorithms for computing the eigenvalues and determinants will be proved. Also, the inverse of a symbolic n-plithogenic matrix for the special values n=10, n=9, n=11, and n=12 will be presented.

Keywords: symbolic 9-plithogenic matrix; symbolic 10-plithogenic matrix; symbolic 11-plithogenic matrix; symbolic 12-plithogenic matrix symbolic plithogenic eigenvalue; symbolic plithogenic eigenvector.

1. Introduction

Matrix theory is one of the most important and broad theories in mathematics, as it plays a central role in the representation of groups and spaces, as well as in Game Theory and computer science.
In previous research works, several generalizations of matrices have been made using fuzzy sets, neutrosophic, and also plithogenic [1-2].
The different patterns of these matrices have been studied, and this is through many related algebraic concepts such as diagonalization, algebraic equations, and linear representation [3-5].
The symbolic n-plithogenic sets are considered as fertile ground for the construction of algebraic extensions for other classical structures such as rings, vector spaces, modules, and equations [6-13,19-22].
The symbolic n-plithogenic matrices for some special values of n have been handled by many authors, see [14-16, 22-23].
We refer to the fact that symbolic n-plithogenic algebraic structures are very similar to neutrosophic and refined neutrosophic structures, with special multiplication definition, see [23-33].
This motivates us to study the n-plithogenic real square matrices for n=9, n=10, n=11, and n=12, where algorithms for computing the eigenvalues and determinants will be proved. Also, the inverse of these matrices will be presented.
For results about symbolic 2-plithogenic matrices, check [14], for 3-plithogenic matrices, check [15], and for other types, check [16-18].

2. Main Discussion

Definition:
The square symbolic 9-plithogenic matrix is defined as follows:
$U = U_0 + \sum_{i=1}^{9} U_i P_i\,; (U_i)_{n\times n}$ is a square matrix of real entries.
Example.
Consider the symbolic 9-plithogenic matrix:

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$U = \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P_1 + \begin{pmatrix} 8 & -1 \\ 3 & 1 \end{pmatrix} P_2 + \begin{pmatrix} 9 & -1 \\ 9 & -8 \end{pmatrix} P_3 + \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix} P_4 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_5 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_6 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_7 + \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} P_8 + \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} P_9.$

Definition.

Let $U = U_0 + \sum_{i=1}^{9} U_i P_i$ be a symbolic 9-plithogenic matrix of size $n \times n$, hence:

$$\begin{aligned}\det U = \det(U_0) &+ \left[\det\left(\sum_{i=0}^{1} U_i\right) - \det(U_0)\right] P_1 + \left[\det\left(\sum_{i=0}^{2} U_i\right) - \det\left(\sum_{i=0}^{1} U_i\right)\right] P_2 \\ &+ \left[\det\left(\sum_{i=0}^{3} U_i\right) - \det\left(\sum_{i=0}^{2} U_i\right)\right] P_3 + \left[\det\left(\sum_{i=0}^{4} U_i\right) - \det\left(\sum_{i=0}^{3} U_i\right)\right] P_4 \\ &+ \left[\det\left(\sum_{i=0}^{5} U_i\right) - \det\left(\sum_{i=0}^{4} U_i\right)\right] P_5 + \left[\det\left(\sum_{i=0}^{6} U_i\right) - \det\left(\sum_{i=0}^{5} U_i\right)\right] P_6 \\ &+ \left[\det\left(\sum_{i=0}^{7} U_i\right) - \det\left(\sum_{i=0}^{6} U_i\right)\right] P_7 + \left[\det\left(\sum_{i=0}^{8} U_i\right) - \det\left(\sum_{i=0}^{7} U_i\right)\right] P_8 \\ &+ \left[\det\left(\sum_{i=0}^{9} U_i\right) - \det\left(\sum_{i=0}^{8} U_i\right)\right] P_9\end{aligned}$$

Theorem1.

Let $U = U_0 + \sum_{i=1}^{9} U_i P_i$ be a symbolic 9-plithogenic matrix of size $n \times n$, hence:

1. U is invertible if and only if $\det U$ is an invertible symbolic 9-plithogenic real number.
2. $U^{-1} = {U_0}^{-1} + \left[(\sum_{i=0}^{1} U_i)^{-1} - {U_0}^{-1}\right] P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}] P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}] P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}] P_4 + \left[(\sum_{i=1}^{5} U_i)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - (\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}] P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}] P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}] P_9$

Definition.

Let $g = g_0 + \sum_{i=1}^{9} g_i P_i$ be a symbolic 9-plithogenic real number and $U = U_0 + \sum_{i=1}^{9} U_i P_i$ be a symbolic 9-plithogenic square real matrix, then g is called symbolic 9-plithogenic eigenvalue if and only if $UY = gY$.

Y is called a symbolic 9-plithogenic eigenvector.

Theorem2.

Let $g = g_0 + \sum_{i=1}^{9} g_i P_i \in 9 - SP_R$, $Y = Y_0 + \sum_{i=1}^{9} Y_i P_i$ be a symbolic 9-plithogenic real vector, then g is eigenvalue of $U = U_0 + \sum_{i=1}^{9} U_i P_i$ with Y as the corresponding eigenvector if and only if:

$\sum_{i=0}^{j} g_i$ is eigen value of $\sum_{i=0}^{j} U_i$ with $\sum_{i=0}^{j} Y_i$ as eigen vector with $0 \le j \le 9$.

Theorem3.

$$\begin{aligned}U^n = {U_0}^n + P_1\left[\left(\sum_{i=0}^{1} U_i\right)^n - {U_0}^n\right] &+ \left[\left(\sum_{i=0}^{2} U_i\right)^n - \left(\sum_{i=0}^{1} U_i\right)^n\right] P_2 + \left[\left(\sum_{i=1}^{3} U_i\right)^n - \left(\sum_{i=0}^{2} U_i\right)^n\right] P_3 \\ &+ \left[\left(\sum_{i=1}^{4} U_i\right)^n - \left(\sum_{i=0}^{3} U_i\right)^n\right] P_4 + \left[\left(\sum_{i=1}^{5} U_i\right)^n - \left(\sum_{i=0}^{4} U_i\right)^n\right] P_5 + \left[\left(\sum_{i=1}^{6} U_i\right)^n - \left(\sum_{i=0}^{5} U_i\right)^n\right] P_6 \\ &+ \left[\left(\sum_{i=1}^{7} U_i\right)^n - \left(\sum_{i=0}^{6} U_i\right)^n\right] P_7 + \left[\left(\sum_{i=1}^{8} U_i\right)^n - \left(\sum_{i=0}^{7} U_i\right)^n\right] P_8 + \left[\left(\sum_{i=1}^{9} U_i\right)^n - \left(\sum_{i=0}^{8} U_i\right)^n\right] P_9\end{aligned}$$

Theorem4.

Let $U = U_0 + \sum_{i=1}^{9} U_i P_i$ be a square 9-plithogenic invertible real matrix, then:

1). $\det(U^{-1}) = (\det U)^{-1}$

2). $\det U^t = \det U$

3). $\det(U.B) = \det U . \det B\, ; B = B_0 + \sum_{i=1}^{9} B_i P_i$.

Definition.

Let $U = U_0 + \sum_{i=1}^{9} U_i P_i$ be a symbolic 9-plithogenic real square matrix, then:

A is called orthogonal if and only if $U^t = U^{-1}$.

Theorem5.

U is orthogonal if and only if $\sum_{i=0}^{j} U_i\, ;\ 0 \le j \le 9$ are orthogonal.

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Definition.
Let $U = U_0 + \sum_{i=1}^{9} U_i P_i$ be a symbolic 9-plithogenic complex square matrix, then U is called Hermit matrix if $U^* = (\overline{U})^t = U^{-1}$.
Theorem6.
U is Hermit matrix if and only if $\sum_{i=0}^{j} U_i$; $0 \le j \le 9$ are Hermit matrices.
Proof of theorem1.
1). Let $U = U_0 + \sum_{i=1}^{9} U_i P_i$, then U is invertible if and only if there exists $V = V_0 + \sum_{i=1}^{9} V_i P_i$ such that: $U \times V = U_{n\times n}$, hence:

$$\begin{cases} U_0 V_0 = U_{n\times n} \\ \sum_{i=0}^{1} U_i \sum_{i=0}^{1} V_i - U_0 V_0 = O_{n\times n} \\ \sum_{i=0}^{2} U_i \sum_{i=0}^{2} V_i - \sum_{i=0}^{1} U_i \sum_{i=0}^{1} V_i = O_{n\times n} \\ \sum_{i=0}^{3} U_i \sum_{i=0}^{3} V_i - \sum_{i=0}^{2} U_i \sum_{i=0}^{2} V_i = O_{n\times n} \\ \sum_{i=0}^{4} U_i \sum_{i=0}^{4} V_i - \sum_{i=0}^{3} U_i \sum_{i=0}^{3} V_i = O_{n\times n} \\ \sum_{i=0}^{5} U_i \sum_{i=0}^{5} V_i - \sum_{i=0}^{4} U_i \sum_{i=0}^{4} V_i = O_{n\times n} \\ \sum_{i=0}^{6} U_i \sum_{i=0}^{6} V_i - \sum_{i=0}^{5} U_i \sum_{i=0}^{5} V_i = O_{n\times n} \\ \sum_{i=0}^{7} U_i \sum_{i=0}^{7} V_i - \sum_{i=0}^{6} U_i \sum_{i=0}^{6} V_i = O_{n\times n} \\ \sum_{i=0}^{8} U_i \sum_{i=0}^{8} V_i - \sum_{i=0}^{7} U_i \sum_{i=0}^{7} V_i = O_{n\times n} \\ \sum_{i=0}^{9} U_i \sum_{i=0}^{9} V_i - \sum_{i=0}^{8} U_i \sum_{i=0}^{8} V_i = O_{n\times n} \end{cases}$$

This implies that:

$$\begin{cases} U_0 V_0 = U_{n\times n} \\ \sum_{i=0}^{j} U_i \sum_{i=0}^{j} V_i = U_{n\times n} \end{cases} ;\ 1 \le j \le 9$$

Hence $\det\left(\sum_{i=0}^{j} U_i\right) \ne 0$ for all $1 \le j \le 9$, so that $\det(U)$ is invertible in $9 - SP_R$.
2). It holds directly as follows:
$\sum_{i=0}^{j} V_i = \left(\sum_{i=0}^{j} U_i\right)^{-1}$ for $1 \le j \le 9$, hence:

$$U^{-1} = {U_0}^{-1} + \left[(\sum_{i=0}^{1} U_i)^{-1} - {U_0}^{-1}\right]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[\left(\sum_{i=1}^{5} U_i\right)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - \left(\sum_{i=0}^{5} U_i\right)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9.$$

Proof of theorem2.
It is clear that g is an eigen value of U with Y as an eigen vector if and only if:
$A.Y = g.Y$, which is equivalent to:

$$\begin{cases} U_0 Y_0 = g_0 Y_0 \\ \sum_{i=0}^{j} U_i \sum_{i=0}^{j} Y_i = \sum_{i=0}^{j} g_i \sum_{i=0}^{j} Y_i \end{cases} ;\ 1 \le j \le 9$$

Which is equivalent to:

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$\sum_{i=0}^{j} g_i$ is an eigen value of $\sum_{i=0}^{j} U_i$ with $\sum_{i=0}^{j} Y_i$ as an eigen vector for all $1 \leq j \leq 9$.

Proof of theorem3.

It holds easily.

Proof of theorem4.

1). $detU^{-1} = det(U_0{}^{-1}) + P_1[det(\sum_{i=0}^{1} U_i)^{-1} - det(U_0{}^{-1})] + [det(\sum_{i=0}^{2} U_i)^{-1} - det(\sum_{i=0}^{1} U_i)^{-1}]P_2 + [det(\sum_{i=1}^{3} U_i)^{-1} - det(\sum_{i=0}^{2} U_i)^{-1}]P_3 + [det(\sum_{i=1}^{4} U_i)^{-1} - det(\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[det(\sum_{i=1}^{5} U_i)^{-1} - det(\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[det(\sum_{i=1}^{6} U_i)^{-1} - det(\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [det(\sum_{i=1}^{7} U_i)^{-1} - det(\sum_{i=0}^{6} U_i)^{-1}]P_7 + [det(\sum_{i=1}^{8} U_i)^{-1} - det(\sum_{i=0}^{7} U_i)^{-1}]P_8 + [det(\sum_{i=1}^{9} U_i)^{-1} - det(\sum_{i=0}^{8} U_i)^{-1}]P_9 = (detU)^{-1}$.

2). $U^t = U_0{}^t + U_1{}^t P_1 + U_2{}^t P_2 + U_3{}^t P_3 + U_4{}^t P_4 + U_5{}^t P_5 + U_6{}^t P_6 + U_7{}^t P_7 + U_8{}^t P_8 + U_9{}^t P_9$.

$detU^t = det(U_0{}^t) + [det(\sum_{i=0}^{1} U_i{}^t) - det(U_0{}^t)]P_1 + [det(\sum_{i=0}^{2} U_i{}^t) - det(\sum_{i=0}^{1} U_i{}^t)]P_2 + [det(\sum_{i=0}^{3} U_i{}^t) - det(\sum_{i=0}^{2} U_i{}^t)]P_3 + [det(\sum_{i=0}^{4} U_i{}^t) - det(\sum_{i=0}^{3} U_i{}^t)]P_4 + [det(\sum_{i=0}^{5} U_i{}^t) - det\ (\sum_{i=0}^{4} U_i{}^t)]P_5 + [det(\sum_{i=0}^{6} U_i{}^t) - det\ (\sum_{i=0}^{5} U_i{}^t)]P_6 + [det(\sum_{i=0}^{7} U_i{}^t) - det\ (\sum_{i=0}^{6} U_i{}^t)]P_7 + [det(\sum_{i=0}^{8} U_i{}^t) - det\ (\sum_{i=0}^{7} U_i{}^t)]P_8 + [det(\sum_{i=0}^{9} U_i{}^t) - det\ (\sum_{i=0}^{8} U_i{}^t)]P_9 = \det(U_0) + [\det(\sum_{i=0}^{1} U_i) - \det(U_0)]P_1 + [\det(\sum_{i=0}^{2} U_i) - \det(\sum_{i=0}^{1} U_i)]P_2 + [\det(\sum_{i=0}^{3} U_i) - \det(\sum_{i=0}^{2} U_i)]P_3 + [\det(\sum_{i=0}^{4} U_i) - \det(\sum_{i=0}^{3} U_i)]P_4 + [\det(\sum_{i=0}^{5} U_i) - \det(\sum_{i=0}^{4} U_i)]P_5 + [\det(\sum_{i=0}^{6} U_i) - \det(\sum_{i=0}^{5} U_i)]P_6 + [\det(\sum_{i=0}^{7} U_i) - \det(\sum_{i=0}^{6} U_i)]P_7 + [\det(\sum_{i=0}^{8} U_i) - \det(\sum_{i=0}^{7} U_i)]P_8 + [\det(\sum_{i=0}^{9} U_i) - \det(\sum_{i=0}^{8} U_i)]P_9 = detU$.

3). we have:

$U.B = U_0B_0 + [\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i - U_0B_0]P_1 + [\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i - \sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i]P_2 + [\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i - \sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i]P_3 + [\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i - \sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i]P_4 + [\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i - \sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i]P_5 + [\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i - \sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i]P_6 + [\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i - \sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i]P_7 + [\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i - \sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i]P_8 + [\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i - \sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i]P_9$.

$det(U.B) = det(U_0B_0) + [det(\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i) - det(U_0B_0)]P_1 + [det(\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i) - det(\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i)]P_2 + [det(\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i) - det(\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i)]P_3 + [det(\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i) - det(\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i)]P_4 + [det(\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i) - det\ (\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i)]P_5 + [det(\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i) - det\ (\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i)]P_6 + [det(\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i) - det\ (\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i)]P_7 + [det(\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i) - det\ (\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i)]P_8 + [det(\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i) - det\ (\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i)]P_9 = det(U_0)det(B_0) + [det(\sum_{i=0}^{j} U_i).det(\sum_{i=0}^{j} B_i) - det(\sum_{i=1}^{j-1} U_{i-1}).det(\sum_{i=1}^{j-1} B_{i-1})]P_i = det(U)det(B); 1 \leq j \leq 9$.

Proof of theorem5.

U is orthogonal if and only if $U^t = U^{-1}$, hence:

$U_0{}^t + \sum_{i=1}^{9} U_i{}^t P_i = U_0{}^{-1} + [(\sum_{i=0}^{1} U_i)^{-1} - U_0{}^{-1}]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[(\sum_{i=1}^{5} U_i)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - (\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9$, thus:

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$$\begin{cases} {U_0}^t = {U_0}^{-1} \\ {U_1}^t = \left(\sum_{i=0}^{1} U_i\right)^{-1} - {U_0}^{-1} \\ {U_2}^t = \left(\sum_{i=0}^{2} U_i\right)^{-1} - \left(\sum_{i=0}^{1} U_i\right)^{-1} \\ {U_3}^t = \left(\sum_{i=0}^{3} U_i\right)^{-1} - \left(\sum_{i=0}^{2} U_i\right)^{-1} \\ {U_4}^t = \left(\sum_{i=0}^{4} U_i\right)^{-1} - \left(\sum_{i=0}^{3} U_i\right)^{-1} \\ {U_5}^t = \left(\sum_{i=0}^{5} U_i\right)^{-1} - \left(\sum_{i=0}^{4} U_i\right)^{-1} \\ {U_6}^t = \left(\sum_{i=0}^{6} U_i\right)^{-1} - \left(\sum_{i=0}^{5} U_i\right)^{-1} \\ {U_7}^t = \left(\sum_{i=0}^{7} U_i\right)^{-1} - \left(\sum_{i=0}^{6} U_i\right)^{-1} \\ {U_8}^t = \left(\sum_{i=0}^{8} U_i\right)^{-1} - \left(\sum_{i=0}^{7} U_i\right)^{-1} \\ {U_9}^t = \left(\sum_{i=0}^{9} U_i\right)^{-1} - \left(\sum_{i=0}^{8} U_i\right)^{-1} \end{cases}$$

This implies that:

$$\begin{cases} {U_0}^t = {U_0}^{-1} \\ \sum_{i=0}^{1} {U_i}^t = \left(\sum_{i=0}^{1} U_i\right)^{-1} \\ \sum_{i=0}^{2} {U_i}^t = \left(\sum_{i=0}^{2} U_i\right)^{-1} \\ \sum_{i=0}^{3} {U_i}^t = \left(\sum_{i=0}^{3} U_i\right)^{-1} \\ \sum_{i=0}^{4} {U_i}^t = \left(\sum_{i=0}^{4} U_i\right)^{-1} \\ \sum_{i=0}^{5} {U_i}^t = \left(\sum_{i=0}^{5} U_i\right)^{-1} \\ \sum_{i=0}^{6} {U_i}^t = \left(\sum_{i=0}^{6} U_i\right)^{-1} \\ \sum_{i=0}^{7} {U_i}^t = \left(\sum_{i=0}^{7} U_i\right)^{-1} \\ \sum_{i=0}^{8} {U_i}^t = \left(\sum_{i=0}^{8} U_i\right)^{-1} \\ \sum_{i=0}^{9} {U_i}^t = \left(\sum_{i=0}^{9} U_i\right)^{-1} \end{cases},.$$

Definition:

The square symbolic 10-plithogenic matrix is defined as follows:

$U = U_0 + \sum_{i=1}^{10} U_i P_i$; $(U_i)_{n\times n}$ is square matrix of real entries.

Example.

Consider the symbolic 10-plithogenic matrix:

$U = \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P_1 + \begin{pmatrix} 8 & -1 \\ 3 & 1 \end{pmatrix} P_2 + \begin{pmatrix} 9 & -1 \\ 9 & -8 \end{pmatrix} P_3 + \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix} P_4 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_5 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_6 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_7 + \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} P_8 + \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} P_{10}.$

Definition.

Let $U = U_0 + \sum_{i=1}^{10} U_i P_i$ be a symbolic 10-plithogenic matrix of size $n \times n$, hence:

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$$\det U = \det(U_0) + \left[\det\left(\sum_{i=0}^{1} U_i\right) - \det(U_0)\right] P_1 + \left[\det\left(\sum_{i=0}^{2} U_i\right) - \det\left(\sum_{i=0}^{1} U_i\right)\right] P_2 + \left[\det\left(\sum_{i=0}^{3} U_i\right) - \det\left(\sum_{i=0}^{2} U_i\right)\right] P_3 + \left[\det\left(\sum_{i=0}^{4} U_i\right) - \det\left(\sum_{i=0}^{3} U_i\right)\right] P_4 + \left[\det\left(\sum_{i=0}^{5} U_i\right) - \det\left(\sum_{i=0}^{4} U_i\right)\right] P_5 + \left[\det\left(\sum_{i=0}^{6} U_i\right) - \det\left(\sum_{i=0}^{5} U_i\right)\right] P_6 + \left[\det\left(\sum_{i=0}^{7} U_i\right) - \det\left(\sum_{i=0}^{6} U_i\right)\right] P_7 + \left[\det\left(\sum_{i=0}^{8} U_i\right) - \det\left(\sum_{i=0}^{7} U_i\right)\right] P_8 + \left[\det\left(\sum_{i=0}^{9} U_i\right) - \det\left(\sum_{i=0}^{8} U_i\right)\right] P_9 + \left[\det\left(\sum_{i=0}^{10} U_i\right) - \det\left(\sum_{i=0}^{9} U_i\right)\right] P_{10}$$

Theorem6.

Let $U = U_0 + \sum_{i=1}^{10} U_i P_i$ be a symbolic 10-plithogenic matrix of size $n \times n$, hence:

1. U is invertible if and only if $\det U$ is an invertible symbolic 10-plithogenic real number.
2. $U^{-1} = {U_0}^{-1} + [(\sum_{i=0}^{1} U_i)^{-1} - {U_0}^{-1}]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[(\sum_{i=1}^{5} U_i)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right]P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - (\sum_{i=0}^{5} U_i)^{-1}\right]P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}]P_{10}$

Definition.

Let $g = g_0 + \sum_{i=1}^{10} g_i P_i$ be a symbolic 10-plithogenic real number and $U = U_0 + \sum_{i=1}^{10} U_i P_i$ be a symbolic 10-plithogenic square real matrix, then g is called symbolic 10-plithogenic eigen value if and only if $UY = gY$.
Y is called symbolic 10-plithogenic eigenvector.

Theorem7.

Let $g = g_0 + \sum_{i=1}^{10} g_i P_i \in 10 - SP_R, Y = Y_0 + \sum_{i=1}^{10} Y_i P_i$ be a symbolic 10-plithogenic real vector, then g is eigen value of $U = U_0 + \sum_{i=1}^{10} U_i P_i$ with Y as the corresponding eigen vector if and only if:
$\sum_{i=0}^{j} g_i$ is eigen value of $\sum_{i=0}^{j} U_i$ with $\sum_{i=0}^{j} Y_i$ as eigen vector with $0 \le j \le 10$.

Theorem8.

$$U^n = {U_0}^n + P_1\left[\left(\sum_{i=0}^{1} U_i\right)^n - {U_0}^n\right] + \left[\left(\sum_{i=0}^{2} U_i\right)^n - \left(\sum_{i=0}^{1} U_i\right)^n\right]P_2 + \left[\left(\sum_{i=1}^{3} U_i\right)^n - \left(\sum_{i=0}^{2} U_i\right)^n\right]P_3 + \left[\left(\sum_{i=1}^{4} U_i\right)^n - \left(\sum_{i=0}^{3} U_i\right)^n\right]P_4 + \left[\left(\sum_{i=1}^{5} U_i\right)^n - \left(\sum_{i=0}^{4} U_i\right)^n\right]P_5 + \left[\left(\sum_{i=1}^{6} U_i\right)^n - \left(\sum_{i=0}^{5} U_i\right)^n\right]P_6 + \left[\left(\sum_{i=1}^{7} U_i\right)^n - \left(\sum_{i=0}^{6} U_i\right)^n\right]P_7 + \left[\left(\sum_{i=1}^{8} U_i\right)^n - \left(\sum_{i=0}^{7} U_i\right)^n\right]P_8 + \left[\left(\sum_{i=1}^{9} U_i\right)^n - \left(\sum_{i=0}^{8} U_i\right)^n\right]P_9 + \left[\left(\sum_{i=1}^{10} U_i\right)^n - \left(\sum_{i=0}^{9} U_i\right)^n\right]P_{10}$$

Theorem9.

Let $U = U_0 + \sum_{i=1}^{10} U_i P_i$ be a square 10-plithogenic invertible real matrix, then:
1). $\det(U^{-1}) = (\det U)^{-1}$
2). $\det U^t = \det U$
3). $\det(U.B) = \det U . \det B ; B = B_0 + \sum_{i=1}^{10} B_i P_i$.

Definition.

Let $U = U_0 + \sum_{i=1}^{10} U_i P_i$ be a symbolic 10-plithogenic real square matrix, then:
A is called orthogonal if and only if $U^t = U^{-1}$.

Theorem10.

U is orthogonal if and only if $\sum_{i=0}^{j} U_i ;\ 0 \le j \le 10$ are orthogonal.

Definition.

Let $U = U_0 + \sum_{i=1}^{10} U_i P_i$ be a symbolic 10-plithogenic complex square matrix, then U is called Hermit matrix if $U^* = (\overline{U})^t = U^{-1}$.

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Theorem11.

U is Hermit matrix if and only if $\sum_{i=0}^{j} U_i$; $0 \le j \le 10$ are Hermit matrices.

Proof of theorem6.

1). Let $U = U_0 + \sum_{i=1}^{10} U_i P_i$, then U is invertible if and only if there exists $V = V_0 + \sum_{i=1}^{10} V_i P_i$ such that: $U \times V = U_{n\times n}$, hence:

$$\begin{cases} U_0V_0 = U_{n\times n} \\ \sum_{i=0}^{1} U_i \sum_{i=0}^{1} V_i - U_0V_0 = O_{n\times n} \\ \sum_{i=0}^{2} U_i \sum_{i=0}^{2} V_i - \sum_{i=0}^{1} U_i \sum_{i=0}^{1} V_i = O_{n\times n} \\ \sum_{i=0}^{3} U_i \sum_{i=0}^{3} V_i - \sum_{i=0}^{2} U_i \sum_{i=0}^{2} V_i = O_{n\times n} \\ \sum_{i=0}^{4} U_i \sum_{i=0}^{4} V_i - \sum_{i=0}^{3} U_i \sum_{i=0}^{3} V_i = O_{n\times n} \\ \sum_{i=0}^{5} U_i \sum_{i=0}^{5} V_i - \sum_{i=0}^{4} U_i \sum_{i=0}^{4} V_i = O_{n\times n} \\ \sum_{i=0}^{6} U_i \sum_{i=0}^{6} V_i - \sum_{i=0}^{5} U_i \sum_{i=0}^{5} V_i = O_{n\times n} \\ \sum_{i=0}^{7} U_i \sum_{i=0}^{7} V_i - \sum_{i=0}^{6} U_i \sum_{i=0}^{6} V_i = O_{n\times n} \\ \sum_{i=0}^{8} U_i \sum_{i=0}^{8} V_i - \sum_{i=0}^{7} U_i \sum_{i=0}^{7} V_i = O_{n\times n} \\ \sum_{i=0}^{9} U_i \sum_{i=0}^{9} V_i - \sum_{i=0}^{8} U_i \sum_{i=0}^{8} V_i = O_{n\times n} \\ \sum_{i=0}^{10} U_i \sum_{i=0}^{10} V_i - \sum_{i=0}^{9} U_i \sum_{i=0}^{9} V_i = O_{n\times n} \end{cases}$$

This implies that:

$$\begin{cases} U_0V_0 = U_{n\times n} \\ \sum_{i=0}^{j} U_i \sum_{i=0}^{j} V_i = U_{n\times n} \end{cases} ;\ 1 \le j \le 10$$

Hence $\det\left(\sum_{i=0}^{j} U_i\right) \neq 0$ for all $1 \le j \le 10$, so that $\det(U)$ is invertible in $10 - SP_R$.

2). It holds directly as follows:

$\sum_{i=0}^{j} V_i = \left(\sum_{i=0}^{j} U_i\right)^{-1}$ for $1 \le j \le 10$, hence:

$$U^{-1} = U_0^{-1} + \left[(\textstyle\sum_{i=0}^{1} U_i)^{-1} - U_0^{-1}\right]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[\left(\sum_{i=1}^{5} U_i\right)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right]P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - \left(\sum_{i=0}^{5} U_i\right)^{-1}\right]P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}]P_{10}.$$

Proof of theorem7.

It is clear that g is an eigen value of U with Y as an eigen vector if and only if:

$A.Y = g.Y$, which is equivalent to:

$$\begin{cases} U_0Y_0 = g_0Y_0 \\ \sum_{i=0}^{j} U_i \sum_{i=0}^{j} Y_i = \sum_{i=0}^{j} g_i \sum_{i=0}^{j} Y_i \end{cases} ;\ 1 \le j \le 10$$

Which is equivalent to:

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$\sum_{i=0}^{j} g_i$ is an eigen value of $\sum_{i=0}^{j} U_i$ with $\sum_{i=0}^{j} Y_i$ as an eigen vector for all $1 \leq j \leq 10$.

Proof of theorem8.

It holds easily.

Proof of theorem9.

1). $detU^{-1} = det(U_0{}^{-1}) + P_1[det(\sum_{i=0}^{1} U_i)^{-1} - det(U_0{}^{-1})] + [det(\sum_{i=0}^{2} U_i)^{-1} - det(\sum_{i=0}^{1} U_i)^{-1}]P_2 + [det(\sum_{i=1}^{3} U_i)^{-1} - det(\sum_{i=0}^{2} U_i)^{-1}]P_3 + [det(\sum_{i=1}^{4} U_i)^{-1} - det(\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[det\left(\sum_{i=1}^{5} U_i\right)^{-1} - det(\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[det(\sum_{i=1}^{6} U_i)^{-1} - det\left(\sum_{i=0}^{5} U_i\right)^{-1}\right] P_6 + [det(\sum_{i=1}^{7} U_i)^{-1} - det(\sum_{i=0}^{6} U_i)^{-1}]P_7 + [det(\sum_{i=1}^{8} U_i)^{-1} - det(\sum_{i=0}^{7} U_i)^{-1}]P_8 + [det(\sum_{i=1}^{9} U_i)^{-1} - det(\sum_{i=0}^{8} U_i)^{-1}]P_9 + [det(\sum_{i=1}^{10} U_i)^{-1} - det(\sum_{i=0}^{9} U_i)^{-1}]P_{10} = (detU)^{-1}$.

2). $U^t = U_0{}^t + U_1{}^t P_1 + U_2{}^t P_2 + U_3{}^t P_3 + U_4{}^t P_4 + U_5{}^t P_5 + U_6{}^t P_6 + U_7{}^t P_7 + U_8{}^t P_8 + U_9{}^t P_9 + U_{10}{}^t P_{10}$.

$detU^t = det(U_0{}^t) + [det(\sum_{i=0}^{1} U_i{}^t) - det(U_0{}^t)]P_1 + [det(\sum_{i=0}^{2} U_i{}^t) - det(\sum_{i=0}^{1} U_i{}^t)]P_2 + [det(\sum_{i=0}^{3} U_i{}^t) - det(\sum_{i=0}^{2} U_i{}^t)]P_3 + [det(\sum_{i=0}^{4} U_i{}^t) - det(\sum_{i=0}^{3} U_i{}^t)]P_4 + [det(\sum_{i=0}^{5} U_i{}^t) - det\ (\sum_{i=0}^{4} U_i{}^t)]P_5 + [det(\sum_{i=0}^{6} U_i{}^t) - det\ (\sum_{i=0}^{5} U_i{}^t)]P_6 + [det(\sum_{i=0}^{7} U_i{}^t) - det\ (\sum_{i=0}^{6} U_i{}^t)]P_7 + [det(\sum_{i=0}^{8} U_i{}^t) - det\ (\sum_{i=0}^{7} U_i{}^t)]P_8 + [det(\sum_{i=0}^{9} U_i{}^t) - det\ (\sum_{i=0}^{8} U_i{}^t)]P_9 + [det(\sum_{i=0}^{10} U_i{}^t) - det\ (\sum_{i=0}^{9} U_i{}^t)]P_{10} = \det(U_0) + [\det(\sum_{i=0}^{1} U_i) - \det(U_0)]P_1 + [\det(\sum_{i=0}^{2} U_i) - \det(\sum_{i=0}^{1} U_i)]P_2 + [\det(\sum_{i=0}^{3} U_i) - \det(\sum_{i=0}^{2} U_i)]P_3 + [\det(\sum_{i=0}^{4} U_i) - \det(\sum_{i=0}^{3} U_i)]P_4 + [\det(\sum_{i=0}^{5} U_i) - \det(\sum_{i=0}^{4} U_i)]P_5 + [\det(\sum_{i=0}^{6} U_i) - \det(\sum_{i=0}^{5} U_i)]P_6 + [\det(\sum_{i=0}^{7} U_i) - \det(\sum_{i=0}^{6} U_i)]P_7 + [\det(\sum_{i=0}^{8} U_i) - \det(\sum_{i=0}^{7} U_i)]P_8 + [\det(\sum_{i=0}^{9} U_i) - \det(\sum_{i=0}^{8} U_i)]P_9 + [\det(\sum_{i=0}^{10} U_i) - \det(\sum_{i=0}^{9} U_i)]P_{10} = detU$.

3). we have:

$U.B = U_0B_0 + [\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i - U_0B_0]P_1 + [\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i - \sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i]P_2 + [\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i - \sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i]P_3 + [\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i - \sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i]P_4 + [\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i - \sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i]P_5 + [\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i - \sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i]P_6 + [\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i - \sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i]P_7 + [\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i - \sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i]P_8 + [\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i - \sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i]P_9 + [\sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i - \sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i]P_{10}$.

$det(U.B) = det(U_0B_0) + [det(\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i) - det(U_0B_0)]P_1 + [det(\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i) - det(\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i)]P_2 + [det(\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i) - det(\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i)]P_3 + [det(\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i) - det(\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i)]P_4 + [det(\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i) - det\ (\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i)]P_5 + [det(\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i) - det\ (\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i)]P_6 + [det(\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i) - det\ (\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i)]P_7 + [det(\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i) - det\ (\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i)]P_8 + [det(\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i) - det\ (\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i)]P_9 + [det(\sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i) - det\ (\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i)]P_{10} = det(U_0)det(B_0) + [det(\sum_{i=0}^{j} U_i).det(\sum_{i=0}^{j} B_i) - det(\sum_{i=1}^{j-1} U_{i-1}).det(\sum_{i=1}^{j-1} B_{i-1})]P_i = det(U)det(B); 1 \leq j \leq 10$.

Proof of theorem10.

U is orthogonal if and only if $U^t = U^{-1}$, hence:

$U_0{}^t + \sum_{i=1}^{10} U_i{}^t P_i = U_0{}^{-1} + [(\sum_{i=0}^{1} U_i)^{-1} - U_0{}^{-1}]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[\left(\sum_{i=1}^{5} U_i\right)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - \left(\sum_{i=0}^{5} U_i\right)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}]P_{10}$, thus:

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$$\begin{cases} {U_0}^t = {U_0}^{-1} \\ {U_1}^t = \left(\sum_{i=0}^{1} U_i\right)^{-1} - {U_0}^{-1} \\ {U_2}^t = \left(\sum_{i=0}^{2} U_i\right)^{-1} - \left(\sum_{i=0}^{1} U_i\right)^{-1} \\ {U_3}^t = \left(\sum_{i=0}^{3} U_i\right)^{-1} - \left(\sum_{i=0}^{2} U_i\right)^{-1} \\ {U_4}^t = \left(\sum_{i=0}^{4} U_i\right)^{-1} - \left(\sum_{i=0}^{3} U_i\right)^{-1} \\ {U_5}^t = \left(\sum_{i=0}^{5} U_i\right)^{-1} - \left(\sum_{i=0}^{4} U_i\right)^{-1} \\ {U_6}^t = \left(\sum_{i=0}^{6} U_i\right)^{-1} - \left(\sum_{i=0}^{5} U_i\right)^{-1} \\ {U_7}^t = \left(\sum_{i=0}^{7} U_i\right)^{-1} - \left(\sum_{i=0}^{6} U_i\right)^{-1} \\ {U_8}^t = \left(\sum_{i=0}^{8} U_i\right)^{-1} - \left(\sum_{i=0}^{7} U_i\right)^{-1} \\ {U_9}^t = \left(\sum_{i=0}^{9} U_i\right)^{-1} - \left(\sum_{i=0}^{8} U_i\right)^{-1} \\ {U_{10}}^t = \left(\sum_{i=0}^{10} U_i\right)^{-1} - \left(\sum_{i=0}^{9} U_i\right)^{-1} \end{cases}$$

This implies that:

$$\begin{cases} {U_0}^t = {U_0}^{-1} \\ \sum_{i=0}^{1} {U_i}^t = \left(\sum_{i=0}^{1} U_i\right)^{-1} \\ \sum_{i=0}^{2} {U_i}^t = \left(\sum_{i=0}^{2} U_i\right)^{-1} \\ \sum_{i=0}^{3} {U_i}^t = \left(\sum_{i=0}^{3} U_i\right)^{-1} \\ \sum_{i=0}^{4} {U_i}^t = \left(\sum_{i=0}^{4} U_i\right)^{-1} \\ \sum_{i=0}^{5} {U_i}^t = \left(\sum_{i=0}^{5} U_i\right)^{-1}, \\ \sum_{i=0}^{6} {U_i}^t = \left(\sum_{i=0}^{6} U_i\right)^{-1} \\ \sum_{i=0}^{7} {U_i}^t = \left(\sum_{i=0}^{7} U_i\right)^{-1} \\ \sum_{i=0}^{8} {U_i}^t = \left(\sum_{i=0}^{8} U_i\right)^{-1} \\ \sum_{i=0}^{9} {U_i}^t = \left(\sum_{i=0}^{9} U_i\right)^{-1} \\ \sum_{i=0}^{10} {U_i}^t = \left(\sum_{i=0}^{10} U_i\right)^{-1} \end{cases}$$

Definition:

The square symbolic 11-plithogenic matrix is defined as follows:

$U = U_0 + \sum_{i=1}^{11} U_i P_i$; $(U_i)_{n\times n}$ is square matrix of real entries.

Example.

Consider the symbolic 11-plithogenic matrix:

$U = \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P_1 + \begin{pmatrix} 8 & -1 \\ 3 & 1 \end{pmatrix} P_2 + \begin{pmatrix} 9 & -1 \\ 9 & -8 \end{pmatrix} P_3 + \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix} P_4 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_5 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_6 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_7 + \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} P_8 + \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} P_{10} + \begin{pmatrix} 8 & -1 \\ 3 & 1 \end{pmatrix} P_{11}.$

Definition.

Let $U = U_0 + \sum_{i=1}^{11} U_i P_i$ be a symbolic 11-plithogenic matrix of size $n \times n$, hence:

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$$\det U = \det(U_0) + \left[\det\left(\sum_{i=0}^{1} U_i\right) - \det(U_0)\right] P_1 + \left[\det\left(\sum_{i=0}^{2} U_i\right) - \det\left(\sum_{i=0}^{1} U_i\right)\right] P_2 + \left[\det\left(\sum_{i=0}^{3} U_i\right) - \det\left(\sum_{i=0}^{2} U_i\right)\right] P_3 + \left[\det\left(\sum_{i=0}^{4} U_i\right) - \det\left(\sum_{i=0}^{3} U_i\right)\right] P_4 + \left[\det\left(\sum_{i=0}^{5} U_i\right) - \det\left(\sum_{i=0}^{4} U_i\right)\right] P_5 + \left[\det\left(\sum_{i=0}^{6} U_i\right) - \det\left(\sum_{i=0}^{5} U_i\right)\right] P_6 + \left[\det\left(\sum_{i=0}^{7} U_i\right) - \det\left(\sum_{i=0}^{6} U_i\right)\right] P_7 + \left[\det\left(\sum_{i=0}^{8} U_i\right) - \det\left(\sum_{i=0}^{7} U_i\right)\right] P_8 + \left[\det\left(\sum_{i=0}^{9} U_i\right) - \det\left(\sum_{i=0}^{8} U_i\right)\right] P_9 + \left[\det\left(\sum_{i=0}^{10} U_i\right) - \det\left(\sum_{i=0}^{9} U_i\right)\right] P_{10} + \left[\det\left(\sum_{i=0}^{11} U_i\right) - \det\left(\sum_{i=0}^{10} U_i\right)\right] P_{11}$$

Theorem12.

Let $U = U_0 + \sum_{i=1}^{11} U_i P_i$ be a symbolic 11-plithogenic matrix of size $n \times n$, hence:

3. U is invertible if and only if $\det U$ is an invertible symbolic 11-plithogenic real number.
4. $U^{-1} = U_0^{-1} + \left[(\sum_{i=0}^{1} U_i)^{-1} - U_0^{-1}\right] P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}] P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}] P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}] P_4 + \left[(\sum_{i=1}^{5} U_i)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - (\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}] P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}] P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}] P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}] P_{10} + [(\sum_{i=1}^{11} U_i)^{-1} - (\sum_{i=0}^{10} U_i)^{-1}] P_{11}$

Definition.

Let $g = g_0 + \sum_{i=1}^{11} g_i P_i$ be a symbolic 11-plithogenic real number and $U = U_0 + \sum_{i=1}^{11} U_i P_i$ be a symbolic 11-plithogenic square real matrix, then g is called symbolic 11-plithogenic eigen value if and only if $UY = gY$.
Y is called symbolic 11-plithogenic eigenvector.

Theorem13.

Let $g = g_0 + \sum_{i=1}^{11} g_i P_i \in 11 - SP_R$, $Y = Y_0 + \sum_{i=1}^{11} Y_i P_i$ be a symbolic 11-plithogenic real vector, then g is eigen value of $U = U_0 + \sum_{i=1}^{11} U_i P_i$ with Y as the corresponding eigen vector if and only if:
$\sum_{i=0}^{j} g_i$ is eigen value of $\sum_{i=0}^{j} U_i$ with $\sum_{i=0}^{j} Y_i$ as eigen vector with $0 \le j \le 11$.

Theorem14.

Let $U = U_0 + \sum_{i=1}^{11} U_i P_i$ be a square 11-plithogenic invertible real matrix, then:
1). $\det(U^{-1}) = (\det U)^{-1}$
2). $\det U^t = \det U$
3). $\det(U.B) = \det U . \det B\,; B = B_0 + \sum_{i=1}^{11} B_i P_i$.

Definition.

Let $U = U_0 + \sum_{i=1}^{11} U_i P_i$ be a symbolic 11-plithogenic real square matrix, then:
A is called orthogonal if and only if $U^t = U^{-1}$.

Theorem15.

U is orthogonal if and only if $\sum_{i=0}^{j} U_i\,;\ 0 \le j \le 11$ are orthogonal.

Proof of theorem12.

1). Let $U = U_0 + \sum_{i=1}^{11} U_i P_i$, then U is invertible if and only if there exists $V = V_0 + \sum_{i=1}^{11} V_i P_i$ such that:
$U \times V = U_{n\times n}$, hence:

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$$\begin{cases} U_0V_0 = U_{n\times n} \\ \sum_{i=0}^{1} U_i \sum_{i=0}^{1} V_i - U_0V_0 = O_{n\times n} \\ \sum_{i=0}^{2} U_i \sum_{i=0}^{2} V_i - \sum_{i=0}^{1} U_i \sum_{i=0}^{1} V_i = O_{n\times n} \\ \sum_{i=0}^{3} U_i \sum_{i=0}^{3} V_i - \sum_{i=0}^{2} U_i \sum_{i=0}^{2} V_i = O_{n\times n} \\ \sum_{i=0}^{4} U_i \sum_{i=0}^{4} V_i - \sum_{i=0}^{3} U_i \sum_{i=0}^{3} V_i = O_{n\times n} \\ \sum_{i=0}^{5} U_i \sum_{i=0}^{5} V_i - \sum_{i=0}^{4} U_i \sum_{i=0}^{4} V_i = O_{n\times n} \\ \sum_{i=0}^{6} U_i \sum_{i=0}^{6} V_i - \sum_{i=0}^{5} U_i \sum_{i=0}^{5} V_i = O_{n\times n} \\ \sum_{i=0}^{7} U_i \sum_{i=0}^{7} V_i - \sum_{i=0}^{6} U_i \sum_{i=0}^{6} V_i = O_{n\times n} \\ \sum_{i=0}^{8} U_i \sum_{i=0}^{8} V_i - \sum_{i=0}^{7} U_i \sum_{i=0}^{7} V_i = O_{n\times n} \\ \sum_{i=0}^{9} U_i \sum_{i=0}^{9} V_i - \sum_{i=0}^{8} U_i \sum_{i=0}^{8} V_i = O_{n\times n} \\ \sum_{i=0}^{10} U_i \sum_{i=0}^{10} V_i - \sum_{i=0}^{9} U_i \sum_{i=0}^{9} V_i = O_{n\times n} \\ \sum_{i=0}^{11} U_i \sum_{i=0}^{11} V_i - \sum_{i=0}^{10} U_i \sum_{i=0}^{10} V_i = O_{n\times n} \end{cases}$$

This implies that:

$$\begin{cases} U_0V_0 = U_{n\times n} \\ \sum_{i=0}^{j} U_i \sum_{i=0}^{j} V_i = U_{n\times n} \end{cases} ;\ 1 \le j \le 11$$

Hence $\det\left(\sum_{i=0}^{j} U_i\right) \neq 0$ for all $1 \le j \le 11$, so that $\det(U)$ is invertible in $11 - SP_R$.

2). It holds directly as follows:

$\sum_{i=0}^{j} V_i = \left(\sum_{i=0}^{j} U_i\right)^{-1}$ for $1 \le j \le 11$, hence:

$$U^{-1} = U_0^{-1} + \left[(\sum_{i=0}^{1} U_i)^{-1} - U_0^{-1}\right]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[\left(\sum_{i=1}^{5} U_i\right)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - \left(\sum_{i=0}^{5} U_i\right)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}]P_{10} + [(\sum_{i=1}^{11} U_i)^{-1} - (\sum_{i=0}^{10} U_i)^{-1}]P_{11}.$$

Proof of theorem13.

It is clear that g is an eigen value of U with Y as an eigen vector if and only if:

$A.Y = g.Y$, which is equivalent to:

$$\begin{cases} U_0Y_0 = g_0Y_0 \\ \sum_{i=0}^{j} U_i \sum_{i=0}^{j} Y_i = \sum_{i=0}^{j} g_i \sum_{i=0}^{j} Y_i \end{cases} ;\ 1 \le j \le 11$$

Which is equivalent to:

$\sum_{i=0}^{j} g_i$ is an eigen value of $\sum_{i=0}^{j} U_i$ with $\sum_{i=0}^{j} Y_i$ as an eigen vector for all $1 \le j \le 11$.

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Proof of theorem14.

1). $detU^{-1} = det(U_0^{-1}) + P_1[det(\sum_{i=0}^{1} U_i)^{-1} - det(U_0^{-1})] + [det(\sum_{i=0}^{2} U_i)^{-1} - det(\sum_{i=0}^{1} U_i)^{-1}]P_2 + [det(\sum_{i=1}^{3} U_i)^{-1} - det(\sum_{i=0}^{2} U_i)^{-1}]P_3 + [det(\sum_{i=1}^{4} U_i)^{-1} - det(\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[det(\sum_{i=1}^{5} U_i)^{-1} - det(\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[det(\sum_{i=1}^{6} U_i)^{-1} - det(\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [det(\sum_{i=1}^{7} U_i)^{-1} - det(\sum_{i=0}^{6} U_i)^{-1}]P_7 + [det(\sum_{i=1}^{8} U_i)^{-1} - det(\sum_{i=0}^{7} U_i)^{-1}]P_8 + [det(\sum_{i=1}^{9} U_i)^{-1} - det(\sum_{i=0}^{8} U_i)^{-1}]P_9 + [det(\sum_{i=1}^{10} U_i)^{-1} - det(\sum_{i=0}^{9} U_i)^{-1}]P_{10} + [det(\sum_{i=1}^{11} U_i)^{-1} - det(\sum_{i=0}^{10} U_i)^{-1}]P_{11} = (detU)^{-1}$.

2). $U^t = U_0{}^t + U_1{}^tP_1 + U_2{}^tP_2 + U_3{}^tP_3 + U_4{}^tP_4 + U_5{}^tP_5 + U_6{}^tP_6 + U_7{}^tP_7 + U_8{}^tP_8 + U_9{}^tP_9 + U_{10}{}^tP_{10} + U_{11}{}^tP_{11}$.

$detU^t = det(U_0{}^t) + [det(\sum_{i=0}^{1} U_i{}^t) - det(U_0{}^t)]P_1 + [det(\sum_{i=0}^{2} U_i{}^t) - det(\sum_{i=0}^{1} U_i{}^t)]P_2 + [det(\sum_{i=0}^{3} U_i{}^t) - det(\sum_{i=0}^{2} U_i{}^t)]P_3 + [det(\sum_{i=0}^{4} U_i{}^t) - det(\sum_{i=0}^{3} U_i{}^t)]P_4 + [det(\sum_{i=0}^{5} U_i{}^t) - det\ (\sum_{i=0}^{4} U_i{}^t)]P_5 + [det(\sum_{i=0}^{6} U_i{}^t) - det\ (\sum_{i=0}^{5} U_i{}^t)]P_6 + [det(\sum_{i=0}^{7} U_i{}^t) - det\ (\sum_{i=0}^{6} U_i{}^t)]P_7 + [det(\sum_{i=0}^{8} U_i{}^t) - det\ (\sum_{i=0}^{7} U_i{}^t)]P_8 + [det(\sum_{i=0}^{9} U_i{}^t) - det\ (\sum_{i=0}^{8} U_i{}^t)]P_9 + [det(\sum_{i=0}^{10} U_i{}^t) - det\ (\sum_{i=0}^{9} U_i{}^t)]P_{10} + [det(\sum_{i=0}^{11} U_i{}^t) - det\ (\sum_{i=0}^{10} U_i{}^t)]P_{11} = \det(U_0) + [\det(\sum_{i=0}^{1} U_i) - \det(U_0)]P_1 + [\det(\sum_{i=0}^{2} U_i) - \det(\sum_{i=0}^{1} U_i)]P_2 + [\det(\sum_{i=0}^{3} U_i) - \det(\sum_{i=0}^{2} U_i)]P_3 + [\det(\sum_{i=0}^{4} U_i) - \det(\sum_{i=0}^{3} U_i)]P_4 + [\det(\sum_{i=0}^{5} U_i) - \det(\sum_{i=0}^{4} U_i)]P_5 + [\det(\sum_{i=0}^{6} U_i) - \det(\sum_{i=0}^{5} U_i)]P_6 + [\det(\sum_{i=0}^{7} U_i) - \det(\sum_{i=0}^{6} U_i)]P_7 + [\det(\sum_{i=0}^{8} U_i) - \det(\sum_{i=0}^{7} U_i)]P_8 + [\det(\sum_{i=0}^{9} U_i) - \det(\sum_{i=0}^{8} U_i)]P_9 + [\det(\sum_{i=0}^{10} U_i) - \det(\sum_{i=0}^{9} U_i)]P_{10} + [\det(\sum_{i=0}^{11} U_i) - \det(\sum_{i=0}^{10} U_i)]P_{11} = detU$.

3). we have:

$U.B = U_0B_0 + [\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i - U_0B_0]P_1 + [\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i - \sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i]P_2 + [\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i - \sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i]P_3 + [\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i - \sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i]P_4 + [\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i - \sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i]P_5 + [\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i - \sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i]P_6 + [\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i - \sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i]P_7 + [\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i - \sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i]P_8 + [\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i - \sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i]P_9 + [\sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i - \sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i]P_{10} + [\sum_{i=0}^{11} U_i \sum_{i=0}^{11} B_i - \sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i]P_{11}$.

$det(U.B) = det(U_0B_0) + [det(\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i) - det(U_0B_0)]P_1 + [det(\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i) - det(\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i)]P_2 + [det(\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i) - det(\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i)]P_3 + [det(\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i) - det(\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i)]P_4 + [det(\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i) - det\ (\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i)]P_5 + [det(\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i) - det\ (\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i)]P_6 + [det(\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i) - det\ (\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i)]P_7 + [det(\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i) - det\ (\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i)]P_8 + [det(\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i) - det\ (\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i)]P_9 + [det(\sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i) - det\ (\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i)]P_{10} + [det(\sum_{i=0}^{11} U_i \sum_{i=0}^{11} B_i) - det\ (\sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i)]P_{11} = det(U_0)det(B_0) + [det(\sum_{i=0}^{j} U_i).det(\sum_{i=0}^{j} B_i) - det(\sum_{i=1}^{j-1} U_{i-1}).det(\sum_{i=1}^{j-1} B_{i-1})]P_i = det(U)det(B); 1 \leq j \leq 11$.

Proof of theorem15.

U is orthogonal if and only if $U^t = U^{-1}$, hence:

$U_0{}^t + \sum_{i=1}^{11} U_i{}^tP_i = U_0{}^{-1} + [(\sum_{i=0}^{1} U_i)^{-1} - U_0{}^{-1}]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[(\sum_{i=1}^{5} U_i)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - (\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}]P_{10} + [(\sum_{i=1}^{11} U_i)^{-1} - (\sum_{i=0}^{10} U_i)^{-1}]P_{11}$, thus:

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$$\begin{cases} {U_0}^t = {U_0}^{-1} \\ {U_1}^t = \left(\sum_{i=0}^{1} U_i\right)^{-1} - {U_0}^{-1} \\ {U_2}^t = \left(\sum_{i=0}^{2} U_i\right)^{-1} - \left(\sum_{i=0}^{1} U_i\right)^{-1} \\ {U_3}^t = \left(\sum_{i=0}^{3} U_i\right)^{-1} - \left(\sum_{i=0}^{2} U_i\right)^{-1} \\ {U_4}^t = \left(\sum_{i=0}^{4} U_i\right)^{-1} - \left(\sum_{i=0}^{3} U_i\right)^{-1} \\ {U_5}^t = \left(\sum_{i=0}^{5} U_i\right)^{-1} - \left(\sum_{i=0}^{4} U_i\right)^{-1} \\ {U_6}^t = \left(\sum_{i=0}^{6} U_i\right)^{-1} - \left(\sum_{i=0}^{5} U_i\right)^{-1} \\ {U_7}^t = \left(\sum_{i=0}^{7} U_i\right)^{-1} - \left(\sum_{i=0}^{6} U_i\right)^{-1} \\ {U_8}^t = \left(\sum_{i=0}^{8} U_i\right)^{-1} - \left(\sum_{i=0}^{7} U_i\right)^{-1} \\ {U_9}^t = \left(\sum_{i=0}^{9} U_i\right)^{-1} - \left(\sum_{i=0}^{8} U_i\right)^{-1} \\ {U_{10}}^t = \left(\sum_{i=0}^{10} U_i\right)^{-1} - \left(\sum_{i=0}^{9} U_i\right)^{-1} \\ {U_{11}}^t = \left(\sum_{i=0}^{11} U_i\right)^{-1} - \left(\sum_{i=0}^{10} U_i\right)^{-1} \end{cases}$$

This implies that:

$$\begin{cases} {U_0}^t = {U_0}^{-1} \\ \sum_{i=0}^{1} {U_i}^t = \left(\sum_{i=0}^{1} U_i\right)^{-1} \\ \sum_{i=0}^{2} {U_i}^t = \left(\sum_{i=0}^{2} U_i\right)^{-1} \\ \sum_{i=0}^{3} {U_i}^t = \left(\sum_{i=0}^{3} U_i\right)^{-1} \\ \sum_{i=0}^{4} {U_i}^t = \left(\sum_{i=0}^{4} U_i\right)^{-1} \\ \sum_{i=0}^{5} {U_i}^t = \left(\sum_{i=0}^{5} U_i\right)^{-1} \\ \sum_{i=0}^{6} {U_i}^t = \left(\sum_{i=0}^{6} U_i\right)^{-1}, \\ \sum_{i=0}^{7} {U_i}^t = \left(\sum_{i=0}^{7} U_i\right)^{-1} \\ \sum_{i=0}^{8} {U_i}^t = \left(\sum_{i=0}^{8} U_i\right)^{-1} \\ \sum_{i=0}^{9} {U_i}^t = \left(\sum_{i=0}^{9} U_i\right)^{-1} \\ \sum_{i=0}^{10} {U_i}^t = \left(\sum_{i=0}^{10} U_i\right)^{-1} \\ \sum_{i=0}^{11} {U_i}^t = \left(\sum_{i=0}^{11} U_i\right)^{-1} \end{cases}$$

Definition:

The square symbolic 12-plithogenic matrix is defined as follows:

$U = U_0 + \sum_{i=1}^{12} U_i P_i$; $(U_i)_{n\times n}$ is square matrix of real entries.

Example.

Consider the symbolic 12-plithogenic matrix:

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$U = \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P_1 + \begin{pmatrix} 8 & -1 \\ 3 & 1 \end{pmatrix} P_2 + \begin{pmatrix} 9 & -1 \\ 9 & -8 \end{pmatrix} P_3 + \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix} P_4 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_5 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_6 + \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} P_7 + \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} P_8 + \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} P_{10} + \begin{pmatrix} 8 & -1 \\ 3 & 1 \end{pmatrix} P_{11} + \begin{pmatrix} 9 & -1 \\ 9 & -8 \end{pmatrix} P_{12}.$

Definition.

Let $U = U_0 + \sum_{i=1}^{12} U_i P_i$ be a symbolic 12-plithogenic matrix of size $n \times n$, hence:

$$\det U = \det(U_0) + \left[\det\left(\sum_{i=0}^{1} U_i\right) - \det(U_0)\right] P_1 + \left[\det\left(\sum_{i=0}^{2} U_i\right) - \det\left(\sum_{i=0}^{1} U_i\right)\right] P_2 + \left[\det\left(\sum_{i=0}^{3} U_i\right) - \det\left(\sum_{i=0}^{2} U_i\right)\right] P_3 + \left[\det\left(\sum_{i=0}^{4} U_i\right) - \det\left(\sum_{i=0}^{3} U_i\right)\right] P_4 + \left[\det\left(\sum_{i=0}^{5} U_i\right) - \det\left(\sum_{i=0}^{4} U_i\right)\right] P_5 + \left[\det\left(\sum_{i=0}^{6} U_i\right) - \det\left(\sum_{i=0}^{5} U_i\right)\right] P_6 + \left[\det\left(\sum_{i=0}^{7} U_i\right) - \det\left(\sum_{i=0}^{6} U_i\right)\right] P_7 + \left[\det\left(\sum_{i=0}^{8} U_i\right) - \det\left(\sum_{i=0}^{7} U_i\right)\right] P_8 + \left[\det\left(\sum_{i=0}^{9} U_i\right) - \det\left(\sum_{i=0}^{8} U_i\right)\right] P_9 + \left[\det\left(\sum_{i=0}^{10} U_i\right) - \det\left(\sum_{i=0}^{9} U_i\right)\right] P_{10} + \left[\det\left(\sum_{i=0}^{11} U_i\right) - \det\left(\sum_{i=0}^{10} U_i\right)\right] P_{11} + \left[\det\left(\sum_{i=0}^{12} U_i\right) - \det\left(\sum_{i=0}^{11} U_i\right)\right] P_{12}$$

Theorem16.

Let $U = U_0 + \sum_{i=1}^{12} U_i P_i$ be a symbolic 12-plithogenic matrix of size $n \times n$, hence:

1. U is invertible if and only if $\det U$ is an invertible symbolic 12-plithogenic real number.
2. $U^{-1} = {U_0}^{-1} + [(\sum_{i=0}^{1} U_i)^{-1} - {U_0}^{-1}]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[(\sum_{i=1}^{5} U_i)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - (\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}]P_{10} + [(\sum_{i=1}^{11} U_i)^{-1} - (\sum_{i=0}^{10} U_i)^{-1}]P_{11} + [(\sum_{i=1}^{12} U_i)^{-1} - (\sum_{i=0}^{11} U_i)^{-1}]P_{12}$

Definition.

Let $g = g_0 + \sum_{i=1}^{12} g_i P_i$ be a symbolic 12-plithogenic real number and $U = U_0 + \sum_{i=1}^{12} U_i P_i$ be a symbolic 12-plithogenic square real matrix, then g is called symbolic 12-plithogenic eigen value if and only if $UY = gY$.
Y is called symbolic 12-plithogenic eigenvector.

Theorem17.

Let $g = g_0 + \sum_{i=1}^{12} g_i P_i \in 12 - SP_R$, $Y = Y_0 + \sum_{i=1}^{12} Y_i P_i$ be a symbolic 12-plithogenic real vector, then g is eigen value of $U = U_0 + \sum_{i=1}^{12} U_i P_i$ with Y as the corresponding eigen vector if and only if:
$\sum_{i=0}^{j} g_i$ is eigen value of $\sum_{i=0}^{j} U_i$ with $\sum_{i=0}^{j} Y_i$ as eigen vector with $0 \le j \le 12$.

Theorem18.

$$U^n = {U_0}^n + P_1\left[\left(\sum_{i=0}^{1} U_i\right)^n - {U_0}^n\right] + \left[\left(\sum_{i=0}^{2} U_i\right)^n - \left(\sum_{i=0}^{1} U_i\right)^n\right] P_2 + \left[\left(\sum_{i=1}^{3} U_i\right)^n - \left(\sum_{i=0}^{2} U_i\right)^n\right] P_3 + \left[\left(\sum_{i=1}^{4} U_i\right)^n - \left(\sum_{i=0}^{3} U_i\right)^n\right] P_4 + \left[\left(\sum_{i=1}^{5} U_i\right)^n - \left(\sum_{i=0}^{4} U_i\right)^n\right] P_5 + \left[\left(\sum_{i=1}^{6} U_i\right)^n - \left(\sum_{i=0}^{5} U_i\right)^n\right] P_6 + \left[\left(\sum_{i=1}^{7} U_i\right)^n - \left(\sum_{i=0}^{6} U_i\right)^n\right] P_7 + \left[\left(\sum_{i=1}^{8} U_i\right)^n - \left(\sum_{i=0}^{7} U_i\right)^n\right] P_8 + \left[\left(\sum_{i=1}^{9} U_i\right)^n - \left(\sum_{i=0}^{8} U_i\right)^n\right] P_9 + \left[\left(\sum_{i=1}^{10} U_i\right)^n - \left(\sum_{i=0}^{9} U_i\right)^n\right] P_{10} + \left[\left(\sum_{i=1}^{11} U_i\right)^n - \left(\sum_{i=0}^{10} U_i\right)^n\right] P_{11} + \left[\left(\sum_{i=1}^{12} U_i\right)^n - \left(\sum_{i=0}^{11} U_i\right)^n\right] P_{12}$$

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Theorem19.

Let $U=U_0+\sum_{i=1}^{12}U_iP_i$ be a square 12-plithogenic invertible real matrix, then:

1). $\det(U^{-1})=(\det U)^{-1}$

2). $\det U^t=\det U$

3). $\det(U.B)=\det U.\det B\,;B=B_0+\sum_{i=1}^{12}B_iP_i$.

Proof of theorem16.

1). Let $U=U_0+\sum_{i=1}^{12}U_iP_i$, then U is invertible if and only if there exists $V=V_0+\sum_{i=1}^{12}V_iP_i$ such that: $U\times V=U_{n\times n}$, hence:

$$\begin{cases} U_0V_0=U_{n\times n} \\ \sum_{i=0}^{1}U_i\sum_{i=0}^{1}V_i-U_0V_0=O_{n\times n} \\ \sum_{i=0}^{2}U_i\sum_{i=0}^{2}V_i-\sum_{i=0}^{1}U_i\sum_{i=0}^{1}V_i=O_{n\times n} \\ \sum_{i=0}^{3}U_i\sum_{i=0}^{3}V_i-\sum_{i=0}^{2}U_i\sum_{i=0}^{2}V_i=O_{n\times n} \\ \sum_{i=0}^{4}U_i\sum_{i=0}^{4}V_i-\sum_{i=0}^{3}U_i\sum_{i=0}^{3}V_i=O_{n\times n} \\ \sum_{i=0}^{5}U_i\sum_{i=0}^{5}V_i-\sum_{i=0}^{4}U_i\sum_{i=0}^{4}V_i=O_{n\times n} \\ \sum_{i=0}^{6}U_i\sum_{i=0}^{6}V_i-\sum_{i=0}^{5}U_i\sum_{i=0}^{5}V_i=O_{n\times n} \\ \sum_{i=0}^{7}U_i\sum_{i=0}^{7}V_i-\sum_{i=0}^{6}U_i\sum_{i=0}^{6}V_i=O_{n\times n} \\ \sum_{i=0}^{8}U_i\sum_{i=0}^{8}V_i-\sum_{i=0}^{7}U_i\sum_{i=0}^{7}V_i=O_{n\times n} \\ \sum_{i=0}^{9}U_i\sum_{i=0}^{9}V_i-\sum_{i=0}^{8}U_i\sum_{i=0}^{8}V_i=O_{n\times n} \\ \sum_{i=0}^{10}U_i\sum_{i=0}^{10}V_i-\sum_{i=0}^{9}U_i\sum_{i=0}^{9}V_i=O_{n\times n} \\ \sum_{i=0}^{11}U_i\sum_{i=0}^{11}V_i-\sum_{i=0}^{10}U_i\sum_{i=0}^{10}V_i=O_{n\times n} \\ \sum_{i=0}^{12}U_i\sum_{i=0}^{12}V_i-\sum_{i=0}^{11}U_i\sum_{i=0}^{11}V_i=O_{n\times n} \end{cases}$$

This implies that:

$$\begin{cases} U_0V_0=U_{n\times n} \\ \sum_{i=0}^{j}U_i\sum_{i=0}^{j}V_i=U_{n\times n} \end{cases};\ 1\le j\le 12$$

Hence $\det\left(\sum_{i=0}^{j}U_i\right)\neq 0$ for all $1\le j\le 12$, so that $\det(U)$ is invertible in $12-SP_R$.

2). It holds directly as follows:

$\sum_{i=0}^{j}V_i=\left(\sum_{i=0}^{j}U_i\right)^{-1}$ for $1\le j\le 12$, hence:

$$U^{-1}={U_0}^{-1}+\left[\left(\sum_{i=0}^{1}U_i\right)^{-1}-{U_0}^{-1}\right]P_1+\left[\left(\sum_{i=0}^{2}U_i\right)^{-1}-\left(\sum_{i=0}^{1}U_i\right)^{-1}\right]P_2+\left[\left(\sum_{i=1}^{3}U_i\right)^{-1}-\left(\sum_{i=0}^{2}U_i\right)^{-1}\right]P_3+\left[\left(\sum_{i=1}^{4}U_i\right)^{-1}-\left(\sum_{i=0}^{3}U_i\right)^{-1}\right]P_4+\left[\left(\sum_{i=1}^{5}U_i\right)^{-1}-\left(\sum_{i=0}^{4}U_i\right)^{-1}\right]P_5+\left[\left(\sum_{i=1}^{6}U_i\right)^{-1}-\left(\sum_{i=0}^{5}U_i\right)^{-1}\right]P_6+\left[\left(\sum_{i=1}^{7}U_i\right)^{-1}-\left(\sum_{i=0}^{6}U_i\right)^{-1}\right]P_7+\left[\left(\sum_{i=1}^{8}U_i\right)^{-1}-\left(\sum_{i=0}^{7}U_i\right)^{-1}\right]P_8+\left[\left(\sum_{i=1}^{9}U_i\right)^{-1}-\right.$$

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$(\sum_{i=0}^{8} U_i)^{-1}]P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}]P_{10} + [(\sum_{i=1}^{11} U_i)^{-1} - (\sum_{i=0}^{10} U_i)^{-1}]P_{11} + [(\sum_{i=1}^{12} U_i)^{-1} - (\sum_{i=0}^{11} U_i)^{-1}]P_{12}$.

Proof of theorem17.

It is clear that g is an eigen value of U with Y as an eigen vector if and only if:

$A.Y = g.Y$, which is equivalent to:

$$\begin{cases} U_0Y_0 = g_0Y_0 \\ \sum_{i=0}^{j} U_i \sum_{i=0}^{j} Y_i = \sum_{i=0}^{j} g_i \sum_{i=0}^{j} Y_i \end{cases}; \ 1 \le j \le 12$$

Which is equivalent to:

$\sum_{i=0}^{j} g_i$ is an eigen value of $\sum_{i=0}^{j} U_i$ with $\sum_{i=0}^{j} Y_i$ as an eigen vector for all $1 \le j \le 12$.

Proof of theorem18.

1). $detU^{-1} = det(U_0{}^{-1}) + P_1[det(\sum_{i=0}^{1} U_i)^{-1} - det(U_0{}^{-1})] + [det(\sum_{i=0}^{2} U_i)^{-1} - det(\sum_{i=0}^{1} U_i)^{-1}]P_2 + [det(\sum_{i=1}^{3} U_i)^{-1} - det(\sum_{i=0}^{2} U_i)^{-1}]P_3 + [det(\sum_{i=1}^{4} U_i)^{-1} - det(\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[det(\sum_{i=1}^{5} U_i)^{-1} - det(\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[det(\sum_{i=1}^{6} U_i)^{-1} - det(\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [det(\sum_{i=1}^{7} U_i)^{-1} - det(\sum_{i=0}^{6} U_i)^{-1}]P_7 + [det(\sum_{i=1}^{8} U_i)^{-1} - det(\sum_{i=0}^{7} U_i)^{-1}]P_8 + [det(\sum_{i=1}^{9} U_i)^{-1} - det(\sum_{i=0}^{8} U_i)^{-1}]P_9 + [det(\sum_{i=1}^{10} U_i)^{-1} - det(\sum_{i=0}^{9} U_i)^{-1}]P_{10} + [det(\sum_{i=1}^{11} U_i)^{-1} - det(\sum_{i=0}^{10} U_i)^{-1}]P_{11} + [det(\sum_{i=1}^{12} U_i)^{-1} - det(\sum_{i=0}^{11} U_i)^{-1}]P_{12} = (detU)^{-1}$.

2). $U^t = U_0{}^t + U_1{}^tP_1 + U_2{}^tP_2 + U_3{}^tP_3 + U_4{}^tP_4 + U_5{}^tP_5 + U_6{}^tP_6 + U_7{}^tP_7 + U_8{}^tP_8 + U_9{}^tP_9 + U_{10}{}^tP_{10} + U_{11}{}^tP_{11} + U_{12}{}^tP_{12}$.

$detU^t = det(U_0{}^t) + [det(\sum_{i=0}^{1} U_i{}^t) - det(U_0{}^t)]P_1 + [det(\sum_{i=0}^{2} U_i{}^t) - det(\sum_{i=0}^{1} U_i{}^t)]P_2 + [det(\sum_{i=0}^{3} U_i{}^t) - det(\sum_{i=0}^{2} U_i{}^t)]P_3 + [det(\sum_{i=0}^{4} U_i{}^t) - det(\sum_{i=0}^{3} U_i{}^t)]P_4 + [det(\sum_{i=0}^{5} U_i{}^t) - det\ (\sum_{i=0}^{4} U_i{}^t)]P_5 + [det(\sum_{i=0}^{6} U_i{}^t) - det\ (\sum_{i=0}^{5} U_i{}^t)]P_6 + [det(\sum_{i=0}^{7} U_i{}^t) - det\ (\sum_{i=0}^{6} U_i{}^t)]P_7 + [det(\sum_{i=0}^{8} U_i{}^t) - det\ (\sum_{i=0}^{7} U_i{}^t)]P_8 + [det(\sum_{i=0}^{9} U_i{}^t) - det\ (\sum_{i=0}^{8} U_i{}^t)]P_9 + [det(\sum_{i=0}^{10} U_i{}^t) - det\ (\sum_{i=0}^{9} U_i{}^t)]P_{10} + [det(\sum_{i=0}^{11} U_i{}^t) - det\ (\sum_{i=0}^{10} U_i{}^t)]P_{11} + [det(\sum_{i=0}^{12} U_i{}^t) - det\ (\sum_{i=0}^{11} U_i{}^t)]P_{12} = \det(U_0) + [\det(\sum_{i=0}^{1} U_i) - \det(U_0)]P_1 + [\det(\sum_{i=0}^{2} U_i) - \det(\sum_{i=0}^{1} U_i)]P_2 + [\det(\sum_{i=0}^{3} U_i) - \det(\sum_{i=0}^{2} U_i)]P_3 + [\det(\sum_{i=0}^{4} U_i) - \det(\sum_{i=0}^{3} U_i)]P_4 + [\det(\sum_{i=0}^{5} U_i) - \det(\sum_{i=0}^{4} U_i)]P_5 + [\det(\sum_{i=0}^{6} U_i) - \det(\sum_{i=0}^{5} U_i)]P_6 + [\det(\sum_{i=0}^{7} U_i) - \det(\sum_{i=0}^{6} U_i)]P_7 + [\det(\sum_{i=0}^{8} U_i) - \det(\sum_{i=0}^{7} U_i)]P_8 + [\det(\sum_{i=0}^{9} U_i) - \det(\sum_{i=0}^{8} U_i)]P_9 + [\det(\sum_{i=0}^{10} U_i) - \det(\sum_{i=0}^{9} U_i)]P_{10} + [\det(\sum_{i=0}^{11} U_i) - \det(\sum_{i=0}^{10} U_i)]P_{11} + [\det(\sum_{i=0}^{12} U_i) - \det(\sum_{i=0}^{11} U_i)]P_{12} = detU$.

3). we have:

$U.B = U_0B_0 + [\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i - U_0B_0]P_1 + [\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i - \sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i]P_2 + [\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i - \sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i]P_3 + [\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i - \sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i]P_4 + [\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i - \sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i]P_5 + [\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i - \sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i]P_6 + [\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i - \sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i]P_7 + [\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i - \sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i]P_8 + [\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i - \sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i]P_9 + [\sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i - \sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i]P_{10} + [\sum_{i=0}^{11} U_i \sum_{i=0}^{11} B_i - \sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i]P_{11} + [\sum_{i=0}^{12} U_i \sum_{i=0}^{12} B_i - \sum_{i=0}^{11} U_i \sum_{i=0}^{11} B_i]P_{12}$.

$det(U.B) = det(U_0B_0) + [det(\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i) - det(U_0B_0)]P_1 + [det(\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i) - det(\sum_{i=0}^{1} U_i \sum_{i=0}^{1} B_i)]P_2 + [det(\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i) - det(\sum_{i=0}^{2} U_i \sum_{i=0}^{2} B_i)]P_3 + [det(\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i) - det(\sum_{i=0}^{3} U_i \sum_{i=0}^{3} B_i)]P_4 + [det(\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i) - det\ (\sum_{i=0}^{4} U_i \sum_{i=0}^{4} B_i)]P_5 + [det(\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i) - det\ (\sum_{i=0}^{5} U_i \sum_{i=0}^{5} B_i)]P_6 + [det(\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i) - det\ (\sum_{i=0}^{6} U_i \sum_{i=0}^{6} B_i)]P_7 + [det(\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i) - det\ (\sum_{i=0}^{7} U_i \sum_{i=0}^{7} B_i)]P_8 + [det(\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i) - det\ (\sum_{i=0}^{8} U_i \sum_{i=0}^{8} B_i)]P_9 + [det(\sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i) - det\ (\sum_{i=0}^{9} U_i \sum_{i=0}^{9} B_i)]P_{10} + [det(\sum_{i=0}^{11} U_i \sum_{i=0}^{11} B_i) - det\ (\sum_{i=0}^{10} U_i \sum_{i=0}^{10} B_i)]P_{11} + [det(\sum_{i=0}^{12} U_i \sum_{i=0}^{12} B_i) - det\ (\sum_{i=0}^{11} U_i \sum_{i=0}^{11} B_i)]P_{12} = det(U_0)det(B_0) + [det(\sum_{i=0}^{j} U_i).det(\sum_{i=0}^{j} B_i) - det(\sum_{i=1}^{j-1} U_{i-1}).det(\sum_{i=1}^{j-1} B_{i-1})]P_i = det(U)det(B); 1 \le j \le 12$.

Proof of theorem19.

U is orthogonal if and only if $U^t = U^{-1}$, hence:

$U_0{}^t + \sum_{i=1}^{11} U_i{}^tP_i = U_0{}^{-1} + [(\sum_{i=0}^{1} U_i)^{-1} - U_0{}^{-1}]P_1 + [(\sum_{i=0}^{2} U_i)^{-1} - (\sum_{i=0}^{1} U_i)^{-1}]P_2 + [(\sum_{i=1}^{3} U_i)^{-1} - (\sum_{i=0}^{2} U_i)^{-1}]P_3 + [(\sum_{i=1}^{4} U_i)^{-1} - (\sum_{i=0}^{3} U_i)^{-1}]P_4 + \left[(\sum_{i=1}^{5} U_i)^{-1} - (\sum_{i=0}^{4} U_i)^{-1}\right] P_5 + \left[(\sum_{i=1}^{6} U_i)^{-1} - (\sum_{i=0}^{5} U_i)^{-1}\right] P_6 + [(\sum_{i=1}^{7} U_i)^{-1} - (\sum_{i=0}^{6} U_i)^{-1}]P_7 + [(\sum_{i=1}^{8} U_i)^{-1} - (\sum_{i=0}^{7} U_i)^{-1}]P_8 + [(\sum_{i=1}^{9} U_i)^{-1} - (\sum_{i=0}^{8} U_i)^{-1}]P_9 + [(\sum_{i=1}^{10} U_i)^{-1} - (\sum_{i=0}^{9} U_i)^{-1}]P_{10} + [(\sum_{i=1}^{11} U_i)^{-1} - (\sum_{i=0}^{10} U_i)^{-1}]P_{11} + [(\sum_{i=1}^{12} U_i)^{-1} - (\sum_{i=0}^{11} U_i)^{-1}]P_{12}$, thus:

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$$\begin{cases} U_0{}^t = U_0{}^{-1} \\ U_1{}^t = \left(\sum_{i=0}^{1} U_i\right)^{-1} - U_0{}^{-1} \\ U_2{}^t = \left(\sum_{i=0}^{2} U_i\right)^{-1} - \left(\sum_{i=0}^{1} U_i\right)^{-1} \\ U_3{}^t = \left(\sum_{i=0}^{3} U_i\right)^{-1} - \left(\sum_{i=0}^{2} U_i\right)^{-1} \\ U_4{}^t = \left(\sum_{i=0}^{4} U_i\right)^{-1} - \left(\sum_{i=0}^{3} U_i\right)^{-1} \\ U_5{}^t = \left(\sum_{i=0}^{5} U_i\right)^{-1} - \left(\sum_{i=0}^{4} U_i\right)^{-1} \\ U_6{}^t = \left(\sum_{i=0}^{6} U_i\right)^{-1} - \left(\sum_{i=0}^{5} U_i\right)^{-1} \\ U_7{}^t = \left(\sum_{i=0}^{7} U_i\right)^{-1} - \left(\sum_{i=0}^{6} U_i\right)^{-1} \\ U_8{}^t = \left(\sum_{i=0}^{8} U_i\right)^{-1} - \left(\sum_{i=0}^{7} U_i\right)^{-1} \\ U_9{}^t = \left(\sum_{i=0}^{9} U_i\right)^{-1} - \left(\sum_{i=0}^{8} U_i\right)^{-1} \\ U_{10}{}^t = \left(\sum_{i=0}^{10} U_i\right)^{-1} - \left(\sum_{i=0}^{9} U_i\right)^{-1} \\ U_{11}{}^t = \left(\sum_{i=0}^{11} U_i\right)^{-1} - \left(\sum_{i=0}^{10} U_i\right)^{-1} \\ U_{12}{}^t = \left(\sum_{i=0}^{12} U_i\right)^{-1} - \left(\sum_{i=0}^{11} U_i\right)^{-1} \end{cases}$$

This implies that:

$$\begin{cases} U_0{}^t = U_0{}^{-1} \\ \sum_{i=0}^{1} U_i{}^t = (\sum_{i=0}^{1} U_i)^{-1} \\ \sum_{i=0}^{2} U_i{}^t = (\sum_{i=0}^{2} U_i)^{-1} \\ \sum_{i=0}^{3} U_i{}^t = (\sum_{i=0}^{3} U_i)^{-1} \\ \sum_{i=0}^{4} U_i{}^t = (\sum_{i=0}^{4} U_i)^{-1} \\ \sum_{i=0}^{5} U_i{}^t = (\sum_{i=0}^{5} U_i)^{-1} \\ \sum_{i=0}^{6} U_i{}^t = (\sum_{i=0}^{6} U_i)^{-1}, \\ \sum_{i=0}^{7} U_i{}^t = (\sum_{i=0}^{7} U_i)^{-1} \\ \sum_{i=0}^{8} U_i{}^t = (\sum_{i=0}^{8} U_i)^{-1} \\ \sum_{i=0}^{9} U_i{}^t = (\sum_{i=0}^{9} U_i)^{-1} \\ \sum_{i=0}^{10} U_i{}^t = (\sum_{i=0}^{10} U_i)^{-1} \\ \sum_{i=0}^{11} U_i{}^t = (\sum_{i=0}^{11} U_i)^{-1} \\ \sum_{i=0}^{12} U_i{}^t = (\sum_{i=0}^{12} U_i)^{-1} \end{cases}$$

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3. Conclusion

In this research work, we studied symbolic plithogenic matrices for the special values n=9, and n=10, where we presented many new results related to their invertibility, and the calculation of eigenvalues and eigenvectors corresponding to them, by studying the algebraic relations between their classical symmetric components.
In future studies, we aim that the diagonalization and representation of these matrices will be handled and presented.

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