Hybrid Alpha Power Marshall Olkin G Class of Distributions

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Abstract

This research presents a new class of probability distributions derived as a hybrid class between alpha power transformation class and Marshall Olkin G class and we call it the hybrid alpha power Marshall Olkin G class of distributions (HAPMOG). Characteristics properties of this new class were derived including moments, moments generating function, characteristic function, reliability and hazard functions, and its probability density function was presented in linear combination. Also, many generated distributions depending on this new class was presented and well-studied including HAPMOG-Exponential, HAPMOG-Weibull, HAPMOG-Freshet. This new class of distributions helps in modelling new forms of data, which has important applications in engineering, communication systems, networks modeling, etc.

Keywords: Probability Density Function; Cumulative Distribution Function; Alpha Power Transformation; Marshall Olkin G; Statistical Characteristics; Maximum Likelihood Estimation.

1. Introduction

For many fields of use, including physics, engineering, medicine, biology, economics, finance and insurance sectors, there is a clear need for new probability distributions that are more flexible for modeling data in these fields.

İn recent years, completely different classes have been proposed compared to the classes proposed before 1997. İn 1997 Marshall and Olkin introduced a new family of distributions by adding a parameter to obtain new classes of distributions which are more flexible than the original distributions. Many researchers used the Marshall-Olkin method to introduce new class of distributions and studied their properties and parameter estimation [1], [2] , [3], [4].

In 2016 Mahdavi and Kundu introduced a new class of distributions called Alpha Power Transformation class of distributions [5],depending on cumulative distribution function (cdf) of truncated uniform random variable U on $[0, 1]$, this new class leads to many new distributions like truncated exponential- exponential (TEE) distribution by Mahdavi and Silva [6], the Alpha Power Weibull distribution presented by Nassar et al [7], Alpha power transformed extended exponential distribution by Hassan et al in [8], The Alpha Power Transformation Family by Mead et al [9], The Extended Alpha Power Transformed Family of Distributions by Ahmed et al [10], Alpha-Power Pareto distribution by Ihtisham et al [11], A new alpha power transformed family of distributions by Elbatal et al in [12], A new extended alpha power transformed family of distributions by Ahmad et al in [13], Alpha Power Transformed Log-Logistic Distribution with Application to Breaking Stress Data by Aldahlan in [14], Alpha power transformed Pareto distribution by Sakthivel and chezhian in [15], Alpha-Power Exponentiated Inverse Rayleigh distribution by Ali [16], Alpha Power Lomax Distribution: Properties and Application by Bulut in [17], Alpha Power Transformed Weibull-G Family of Distributions by Elbatal in [18], Kumaraswamy Distribution Based on Alpha Power Transformation Methods by Hozaien in [19], Generalized Alpha-Power Transformation Family of Distributions by El-Sherpieny and Hwas in [20], Alpha Power Odd Generalized Exponential Family of Distributions: Model, Properties and Applications by Elbatal Et Al İn [21], The Pseudo-Lindley Alpha Power Transformed Distribution by Ngom et al in [22], Alpha Power Transformation of Lomax

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Distribution: Properties and Applications by Maruthan and Venkatachalam in [23], Alpha Power Transformed Extended power Lindley Distribution by Eissa and Sonar in [24], Alpha-Power of the Power Ailamujia Distribution: Properties and Applications by Gomaa et al in [25], A New Extended Alpha-power Transformation of Burr-Generalized Gamma with an Application to Income by Abdel-Zaher et al in [26].
In the paper, we introduce a new class of distributions by hybridizing Alpha Power Transformation class and Marshall Olkin G class and we call this new class the Hybrid Alpha Power Marshall Olkin G (HAPMOG).

2. Prelimaries

Definition 2.1

Let $G(x;\delta)$ and $G(x;\delta)$ be the probability density function (pdf) and cumulative distribution function (cdf) of any continuous random variable X respectively, then cdf of Alpha Power Transformation class of distributions is given by:

$$F(x) = \begin{cases} \dfrac{\alpha^{G(x;\delta)} - 1}{\alpha - 1} & for \quad \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ G(x;\delta) & for \quad \alpha = 1 \end{cases} \tag{1}$$

Its pdf is given by:

$$f(x) = \begin{cases} \dfrac{ln(\alpha)}{\alpha - 1}\, G(x;\delta)\, \alpha^{G(x;\delta)} & for \quad \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ G(x;\delta) & for \quad \alpha = 1 \end{cases} \tag{2}$$

Where $x \in \mathbb{R}$, $g(x;\delta) = \frac{dG(x;\delta)}{dx}$.

Definition 2.2

Let $g(x;\xi)$and $G(x;\ \xi)$ be the probability density function (pdf) and cumulative distribution function (cdf) of any continuous random variable X respectively then cdf of Marshall Olkin G class distribution is given by:

$$F(x) = \left(\frac{G(x;\ \xi)}{1 - \bar{\theta}R(x;\ \xi)}\right)^{\beta} \qquad for \quad x \in \mathbb{R} \tag{3}$$

Probability density function (pdf) for MOG class of distribution is:

$$f(x) = \frac{\beta\theta g(x;\xi)}{\left(1 - \bar{\theta}R(x;\xi)\right)^{2}} \left(\frac{G(x;\xi)}{1 - \bar{\theta}R(x;\xi)}\right)^{\beta - 1} ; x \in \mathbb{R} \tag{4}$$

Where $0 < \theta < 1$, $\beta > 0$ and $g(x;\xi) = \frac{dG(x;\xi)}{dx}$, $R(x) = \bar{G}(x;\xi) = 1 - G(x;\ \xi)$ is reliability function.

3. Hybrid Alpha Power Marshall Olkin G Class of Distributions (HAPMOG)

Let X be a continuous random variable substituting (3) in (1) yields to:

$$F(x) = \begin{cases} \dfrac{\alpha^{\left(\frac{G(x;\ \xi)}{1 - \bar{\theta}R(x;\ \xi)}\right)^{\beta}} - 1}{\alpha - 1} & for \quad \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ \left(\dfrac{G(x;\ \xi)}{1 - \bar{\theta}R(x;\ \xi)}\right)^{\beta} & for \quad \alpha = 1 \end{cases} \tag{5}$$

By differentiating both sides of (5), we get the probability density function:

$$f(x) = \begin{cases} \dfrac{ln(\alpha)}{\alpha - 1} \dfrac{\beta\theta g(x;\ \xi)}{\left(1 - \bar{\theta}R(x;\ \xi)\right)^{2}} \left(\dfrac{G(x;\ \xi)}{1 - \bar{\theta}R(x;\ \xi)}\right)^{\beta - 1} \alpha^{\left(\frac{G(x;\ \xi)}{1 - \bar{\theta}R(x;\ \xi)}\right)^{\beta}} ; \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ \dfrac{\beta\theta g(x;\xi)}{\left(1 - \bar{\theta}R(x;\xi)\right)^{2}} \left(\dfrac{G(x;\xi)}{1 - \bar{\theta}R(x;\xi)}\right)^{\beta - 1} ; \alpha = 1 \end{cases} \tag{6}$$

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Where $x \in \mathbb{R}$,$0 < \theta < 1, \beta > 0$, $G(x;\xi)$ is the cumulative distribution function (CDF) of random variable X and $g(x;\ \xi) = \frac{dG(x;\xi)}{dx}$, $R(x) = \bar{G}(x;\xi) = 1 - G(x;\xi)$is reliability function.

Theorem 3.1

The pdf (6) can written in linear combination as follows:

$$f(x) = \begin{cases} \frac{\beta\theta}{\alpha - 1}\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{p=0}^{j} w_{k,j,p}\ g(x;\xi)G(x;\xi)^{\beta(k+1)+p-1}\ for\ \ \alpha \in \mathbb{R}^+, \alpha \neq 1 \\ \beta\theta\sum_{i=0}^{+\infty}\sum_{p=0}^{i} v_{i,p} g(x;\xi)G(x;\xi)^{\beta+p-1} \quad for \quad \alpha = 1 \end{cases} \tag{7}$$

Where $x \in \mathbb{R}$.

Proof

For any $\alpha, u \in R$ we can write:

$$\alpha^u = e^{u\ln\alpha} = \sum_{k=0}^{+\infty}\frac{(u\ln(\alpha))^k}{k!}$$

So:

$$\alpha^{\left(\frac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta}} = \sum_{k=0}^{+\infty}\frac{(ln(\alpha)^k}{k!}\left(\frac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta k} \tag{8}$$

Substituting (8) in (6) yields to:

$$f(x) = \begin{cases} \frac{\beta\theta}{\alpha - 1}\sum_{k=0}^{+\infty}\frac{\ln(\alpha)^{k+1}}{k!} g(x;\xi)G(x;\xi)^{(\beta(k+1)-1)}\left(1 - \bar{\theta}R(x;\xi)\right)^{-(\beta(k+1)+1)} ; \alpha \in \mathbb{R}^+, \alpha \neq 1 \\ \beta\theta g(x;\xi)\ G(x;\xi)^{\beta-1}\left(1 - \bar{\theta}R(x;\xi)\right)^{-(\beta+1)} \quad ;\ \alpha = 1 \end{cases} \tag{9}$$

Where $x \in \mathbb{R}$.

By using the generalized binomial expansion and then binomial expansion, we get

$$\left(1 - \bar{\theta}R(x;\xi)\right)^{-(\beta(k+1)+1)} = \sum_{j=0}^{+\infty}\binom{\beta(k+1)+j}{j}\bar{\theta}^j\left(1 - G(x;\xi)\right)^j \tag{10}$$

$$\left(1 - G(x;\xi)\right)^j = \sum_{p=0}^{j}\binom{j}{p}G(x;\xi)^p \tag{11}$$

Where $|(1 - G(x;\xi))| < 1$ and $(\beta(k+1)+1) > 0$.

Also, we can write:

$$\left(1 - \bar{\theta}R(x;\xi)\right)^{-(\beta+1)} = \sum_{i=0}^{+\infty}\binom{\beta + i}{i}\bar{\theta}^i\left(1 - G(x;\xi)\right)^i \tag{12}$$

$$\left(1 - G(x;\xi)\right)^i = \sum_{p=0}^{i}\binom{i}{p}G(x;\xi)^p \tag{13}$$

Where $|(1 - G(x;\xi)| < 1$ and $(\beta + 1) > 0$.

Substituting (11) in (10) , (13) in (12), (10) and (12) in (9) we obtain

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$$f(x)=\begin{cases}\dfrac{\beta\theta}{\alpha-1}\displaystyle\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{p=0}^{j} w_{k,j,p}\, g(x;\xi)G(x;\xi)^{\beta(k+1)+p-1} & for \quad x\in\mathbb{R}, \alpha\in\mathbb{R}^{+}, \alpha\neq 1\\ \beta\theta\displaystyle\sum_{i=0}^{+\infty}\sum_{p=0}^{i} v_{i,p} g(x;\xi)G(x;\xi)^{\beta+p-1} & for \quad x\in\mathbb{R}, \alpha=1\end{cases}$$

where:

$$w_{k,j,p}=\frac{\ln(\alpha)^{k+1}}{k!}\frac{\Gamma(\beta(k+1)+j+1)}{\Gamma(\beta(k+1)+1)j!}\frac{\Gamma(j+1)}{\Gamma(j-p+1)p!}\bar{\theta}^{j}$$

And

$$v_{i,p}=\bar{\theta}^{i}\left(\frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)i!}\right)\left(\frac{\Gamma(i+1)}{\Gamma(i-p+1)p!}\right)$$

And this completes the proof.

3.1 Characteristic Properties

This section includes some statistical properties of HAPMOG class distributions including r^{th} moment, moment generating function, characteristic function, reliability and hazard functions, quantile function, skewness and kurtosis.

- ***r^{th} Moment***

$$A_r(X)=E(X^r)=\int_{-\infty}^{+\infty} x^r f(x)\,dx$$

$$A_r(X)=\begin{cases}\dfrac{\beta\theta}{\alpha-1}\displaystyle\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{p=0}^{+\infty} w_{k,j,p}\int_{-\infty}^{+\infty} x^r\, g(x;\xi)G(x;\xi)^{\beta(k+1)+p-1}\,dx & ; \alpha\in\mathbb{R}^{+}, \alpha\neq 1\\ \beta\theta\displaystyle\sum_{i=0}^{+\infty}\sum_{p=0}^{i} v_{i,p}\int_{-\infty}^{+\infty} x^r\, g(x;\xi)G(x;\xi)^{\beta+p-1}\,dx & ; \quad \alpha=1\end{cases} \tag{14}$$

Where $x\in\mathbb{R}$, By setting $r=1$ in equation (14) we obtain

$$A_1(X)=E(X)=\begin{cases}\dfrac{\beta\theta}{\alpha-1}\displaystyle\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{p=0}^{+\infty} w_{k,j,p}\int_{-\infty}^{+\infty} x\, g(x;\xi)G(x;\xi)^{\beta(k+1)+p-1}\,dx ; \alpha\in\mathbb{R}^{+}, \alpha\neq 1\\ \beta\theta\displaystyle\sum_{i=0}^{+\infty}\sum_{p=0}^{i} v_{i,p}\int_{-\infty}^{+\infty} x\, g(x;\xi)G(x;\xi)^{\beta+p-1}\,dx \quad ; \quad \alpha=1\end{cases} \tag{15}$$

By setting $r=2$ in equation (14) we obtain

$$A_2(X)=E(X^2)=\begin{cases}\dfrac{\beta\theta}{\alpha-1}\displaystyle\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{p=0}^{+\infty} w_{k,j,p}\int_{-\infty}^{+\infty} x^2\, g(x;\xi)G(x;\xi)^{\beta(k+1)+p-1}\,dx ; \alpha\in\mathbb{R}^{+}, \alpha\neq 1\\ \beta\theta\displaystyle\sum_{i=0}^{+\infty}\sum_{p=0}^{i} v_{i,p}\int_{-\infty}^{+\infty} x^2\, g(x;\xi)G(x;\xi)^{\beta+p-1}\,dx \quad ; \alpha=1\end{cases} \tag{16}$$

And based on (15), (16) variance can be calculated as follows:

$$Var(X)=A_2(X)-A_1(X)^2$$

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- ***Central moments***

r^{th} central moment of X can be calculated as follows:

$$\begin{aligned}\mu_r = E\big(X - E(X)\big)^r &= \sum_{n=0}^{r} (-1)^n \frac{r!}{(r-n)!\,n!}\, E(X)^{r-n} E(X)^r \\ &= \sum_{n=0}^{r} (-1)^n \frac{r!}{(r-n)!\,n!}\, A_{r-n}(X) A_r(X)\end{aligned} \tag{17}$$

- ***Moments Generating Function***

The moments generating function of HAPMOG class of distributions can be calculated as follows:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{+\infty} e^{tx} f(x)\, dx$$

$$M_X(t) = \begin{cases} \dfrac{\beta\theta}{\alpha-1} \displaystyle\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{p=0}^{+\infty} w_{k,j,p} \int_{-\infty}^{+\infty} e^{tx}\, g(x;\xi) G(x;\xi)^{\beta(k+1)+p-1}\, dx & ; \alpha \in \mathbb{R}^+, \alpha \neq 1 \\ \beta\theta \displaystyle\sum_{i=0}^{+\infty}\sum_{p=0}^{i} v_{i,p} \int_{-\infty}^{+\infty} e^{tx}\, g(x;\xi) G(x;\xi)^{\beta+p-1}\, dx & ; \alpha = 1 \end{cases} \tag{18}$$

- ***Characteristic function***

The characteristic function of HAPMOG class of distributions can be calculated as follows:

$$\varphi_X(t) = E(e^{itx}) = \int_{-\infty}^{+\infty} e^{itx} f(x)\, dx$$

$$\varphi_X(t) = \begin{cases} \dfrac{\beta\theta}{\alpha-1} \displaystyle\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{p=0}^{+\infty} w_{k,j,p} \int_{-\infty}^{+\infty} e^{itx}\, g(x;\xi) G(x;\xi)^{\beta(k+1)+p-1}\, dx & ; \alpha \in \mathbb{R}^+, \alpha \neq 1 \\ \beta\theta \displaystyle\sum_{i=0}^{+\infty}\sum_{p=0}^{i} v_{i,p} \int_{-\infty}^{+\infty} e^{itx}\, g(x;\xi) G(x;\xi)^{\beta+p-1}\, dx & ; \alpha = 1 \end{cases} \tag{19}$$

- ***Reliability and hazard functions***

$$R(x) = 1 - F(x) = \begin{cases} \dfrac{\alpha - \alpha^{\left(\frac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta}}}{\alpha - 1} & ; x \in \mathbb{R}, \alpha \in \mathbb{R}^+, \alpha \neq 1 \\ 1 - \left(\dfrac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta} ; & x \in \mathbb{R}, \alpha = 1 \end{cases} \tag{20}$$

And

$$h(x) = \frac{f(x)}{R(x)}$$
$$= \begin{cases} \dfrac{\ln(\alpha)\ \beta\theta g(x;\xi) \left(\dfrac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta-1} \alpha^{\left(\frac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta}}}{\left(1-\bar{\theta}R(x;\xi)\right)^2 \left(\alpha - \alpha^{\left(\frac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta}}\right)} & ; x \in \mathbb{R}, \alpha \in \mathbb{R}^+, \alpha \neq 1 \\ \dfrac{\dfrac{\beta\theta g(x;\xi)}{\left(1-\bar{\theta}R(x;\xi)\right)^2} \left(\dfrac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta-1}}{1 - \left(\dfrac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta}} & ; x \in \mathbb{R}, \alpha = 1 \end{cases} \tag{21}$$

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- ***Quantile function***

The quantile function is the real solution of $F(x) = u\ ;\ 0 \le u \le 1$ with respect to x.

We have:

$$F(x) = \begin{cases} \dfrac{\alpha^{\left(\frac{G(x;\,\xi)}{1-\bar{\theta}R(x;\,\xi)}\right)^{\beta}} - 1}{\alpha - 1} & for \quad x \in \mathbb{R}, \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ \left(\dfrac{G(x;\,\xi)}{1-\bar{\theta}R(x;\,\xi)}\right)^{\beta} & for \qquad x \in \mathbb{R}, \alpha = 1 \end{cases}$$

So solving the last equation with respect to $G(x;\ \xi)$ yields to:

$$G(x;\ \xi) = \begin{cases} \dfrac{\left(\dfrac{\ln(1+(\alpha-1)u)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}(1-\bar{\theta})}{\left(1-\bar{\theta}\left(\dfrac{\ln(1+(\alpha-1)u)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}\right)} & for \quad \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ \dfrac{(1-\bar{\theta})u^{\frac{1}{\beta}}}{1-\bar{\theta}u^{\frac{1}{\beta}}} & for \quad \alpha = 1 \end{cases} \tag{22}$$

Therefore:

$$x = \begin{cases} G^{-1}\left(\dfrac{\left(\dfrac{\ln(1+(\alpha-1)u)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}(1-\bar{\theta})}{\left(1-\bar{\theta}\left(\dfrac{\ln(1+(\alpha-1)u)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}\right)}\right) & for \quad \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ G^{-1}\left(\dfrac{(1-\bar{\theta})u^{\frac{1}{\beta}}}{1-\bar{\theta}u^{\frac{1}{\beta}}}\right) & for \qquad \alpha = 1 \end{cases} \tag{23}$$

From Equation (23) we can get the median (or the second quartile) of HAPMOG class of distributions when $u = 0.5$ as follows:

$$x_2 = \begin{cases} G^{-1}\left(\dfrac{\left(\dfrac{\ln(1+(\alpha-1)0.5)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}(1-\bar{\theta})}{\left(1-\bar{\theta}\left(\dfrac{\ln(1+(\alpha-1)0.5)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}\right)}\right) & for \ \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ G^{-1}\left(\dfrac{(1-\bar{\theta})(0.5)^{\frac{1}{\beta}}}{1-\bar{\theta}(0.5)^{\frac{1}{\beta}}}\right) & for \qquad \alpha = 1 \end{cases} \tag{24}$$

We can obtain the first and third quartiles of HAPMOG class of distributions when $u = 0.25$ and $u = 0.75$ respectively, as follows:

$$x_1 = \begin{cases} G^{-1}\left(\dfrac{\left(\dfrac{\ln(1+(\alpha-1)0.25)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}(1-\bar{\theta})}{\left(1-\bar{\theta}\left(\dfrac{\ln(1+(\alpha-1)0.25)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}\right)}\right) & for \ \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ G^{-1}\left(\dfrac{(1-\bar{\theta})(0.25)^{\frac{1}{\beta}}}{1-\bar{\theta}(0.25)^{\frac{1}{\beta}}}\right) & for \qquad \alpha = 1 \end{cases} \tag{25}$$

And

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$$x_3 = \begin{cases} G^{-1}\left(\dfrac{\left(\dfrac{\ln(1+(\alpha-1)0.75)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}(1-\bar{\theta})}{\left(1-\bar{\theta}\left(\dfrac{\ln(1+(\alpha-1)0.75)}{\ln(\alpha)}\right)^{\frac{1}{\beta}}\right)} \right) & for\ \ \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ G^{-1}\left(\dfrac{(1-\bar{\theta})(0.75)^{\frac{1}{\beta}}}{1-\bar{\theta}(0.75)^{\frac{1}{\beta}}} \right) & for \qquad \alpha = 1 \end{cases} \tag{26}$$

- ***Skewness and kurtosis***

Skewness of random variable X is defined as follows:

$$SK(X) = E\left[\frac{X-E(X)}{\sigma}\right]^3$$

Using equation (17) we can obtain the skewness (SK) as follows:

$$SK(X) = E\left[\frac{X-E(X)}{\sigma}\right]^3 = \frac{1}{\sigma^3}E[X-E(X)]^3 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \tag{27}$$

Kurtosis of random variable X is defined as follows:

$$K(X) = E\left[\frac{X-E(X)}{\sigma}\right]^4$$

Using equation (17) we can obtain the kurtosis (K) as follows:

$$K(X) = E\left[\frac{X-E(X)}{\sigma}\right]^4 = \frac{1}{\sigma^4}E[X-E(X)]^4 = \frac{\mu_4}{\mu_2^2} \tag{28}$$

Where $\sigma^2 = \mu_2 = Var(X)$

- ***Shannon entropy***

The entropy of a random variable X is defined by Shannon as follows:

$$H(X) = -E[\ln f(x)] = -\int_{-\infty}^{+\infty} f(x)\ln f(x)\,dx \tag{29}$$

$$\begin{aligned} H(X) = &-\frac{\beta\theta\, ln(\alpha)}{\alpha-1}\ln\left(\frac{ln(\alpha)}{\alpha-1}\right)\int_{-\infty}^{+\infty}\frac{g(x;\,\xi)G(x;\,\xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\,\xi)\right)^{\beta+1}}\,\alpha^{\left(\frac{G(x;\,\xi)}{1-\bar{\theta}R(x;\,\xi)}\right)^{\beta}}\,dx \\ &-\frac{\beta\theta\ln\beta\ \ ln(\alpha)}{\alpha-1}\int_{-\infty}^{+\infty}\frac{g(x;\,\xi)G(x;\,\xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\,\xi)\right)^{\beta+1}}\,\alpha^{\left(\frac{G(x;\,\xi)}{1-\bar{\theta}R(x;\,\xi)}\right)^{\beta}}\,dx \\ &-\frac{\beta\theta\ln\theta\ \ ln(\alpha)}{\alpha-1}\int_{-\infty}^{+\infty}\frac{g(x;\,\xi)G(x;\,\xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\,\xi)\right)^{\beta+1}}\,\alpha^{\left(\frac{G(x;\,\xi)}{1-\bar{\theta}R(x;\,\xi)}\right)^{\beta}}\,dx \\ &-\frac{\beta\theta\, ln(\alpha)}{\alpha-1}\int_{-\infty}^{+\infty}\frac{g(x;\,\xi)G(x;\,\xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\,\xi)\right)^{\beta+1}}\,\alpha^{\left(\frac{G(x;\,\xi)}{1-\bar{\theta}R(x;\,\xi)}\right)^{\beta}}\ln g(x;\,\xi)\,dx \end{aligned}$$

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$$-\frac{\beta(\beta-1)\theta\, ln(\alpha)}{\alpha-1}\int_{-\infty}^{+\infty}\frac{g(x;\ \xi)G(x;\ \xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\ \xi)\right)^{\beta+1}}\alpha^{\left(\frac{G(x;\ \xi)}{1-\bar{\theta}R(x;\ \xi)}\right)^{\beta}}\ln G(x;\ \xi)\ dx$$
$$+\frac{\beta(\beta+1)\theta ln(\alpha)}{\alpha-1}\int_{-\infty}^{+\infty}\frac{g(x;\ \xi)G(x;\ \xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\ \xi)\right)^{\beta+1}}\alpha^{\left(\frac{G(x;\ \xi)}{1-\bar{\theta}R(x;\ \xi)}\right)^{\beta}}\ln\left(1-\bar{\theta}R(x;\ \xi)\right)\ dx$$
$$-\frac{\beta\theta(ln(\alpha))^2}{\alpha-1}\int_{-\infty}^{+\infty}\frac{g(x;\ \xi)G(x;\ \xi)^{2\beta-1}}{\left(1-\bar{\theta}R(x;\ \xi)\right)^{2\beta+1}}\alpha^{\left(\frac{G(x;\ \xi)}{1-\bar{\theta}R(x;\ \xi)}\right)^{\beta}}\ dx$$

Where $\alpha \in \mathbb{R}^+, \alpha \neq 1$.

And.

$$H(X) = -\beta\theta\ln\beta\int_{-\infty}^{+\infty}\frac{g(x;\ \xi)G(x;\ \xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\ \xi)\right)^{\beta+1}}dx-\beta\theta\ln\theta\int_{-\infty}^{+\infty}\frac{g(x;\ \xi)G(x;\ \xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\ \xi)\right)^{\beta+1}}dx$$
$$-\beta\theta\int_{-\infty}^{+\infty}\frac{g(x;\ \xi)G(x;\ \xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\ \xi)\right)^{\beta+1}}\ln g(x;\ \xi)\,dx$$
$$-\beta(\beta-1)\theta\int_{-\infty}^{+\infty}\frac{g(x;\ \xi)G(x;\ \xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\ \xi)\right)^{\beta+1}}\ln G(x;\ \xi)\,dx$$
$$+\beta\theta\int_{-\infty}^{+\infty}\frac{g(x;\ \xi)G(x;\ \xi)^{\beta-1}}{\left(1-\bar{\theta}R(x;\ \xi)\right)^{\beta+1}}\ln\left(1-\bar{\theta}R(x;\ \xi)\right)^{\beta+1}dx$$

Where $\alpha = 1$.

- ***Parametesr estimation***

Let $\mathbb{X} = (X_1, X_2, \dots X_n)$ be a random sample with size n of HAPMOG then the likelihood function is:

$$L(\mathbb{X}) = \prod_{i=0}^{n} f(X_i)$$

$$L(\mathbb{X}) = \begin{cases}\prod_{i=0}^{n}\frac{ln(\alpha)}{\alpha-1}\frac{\beta\theta g(X_i;\ \xi)}{\left(1-\bar{\theta}R(X_i;\ \xi)\right)^2}\left(\frac{G(X_i;\ \xi)}{1-\bar{\theta}R(X_i;\ \xi)}\right)^{\beta-1}\alpha^{\left(\frac{G(X_i;\ \xi)}{1-\bar{\theta}R(X_i;\ \xi)}\right)^{\beta}} & ;\alpha \in \mathbb{R}^+, \alpha \neq 1\\ \prod_{i=0}^{n}\frac{\beta\theta g(X_i;\xi)}{\left(1-\bar{\theta}R(X_i;\xi)\right)^2}\left(\frac{G(X_i;\xi)}{1-\bar{\theta}R(X_i;\xi)}\right)^{\beta-1} & ;\quad \alpha = 1\end{cases} \tag{30}$$

Substituting (6) into (30) and taking the logarithm we find

$$\ln L(\mathbb{X}) = n\ln(ln(\alpha)) - n\ln(\alpha-1) + n\ln(\beta) + n\ln(\theta) + \sum_{i=1}^{n}\ln g(X_i;\ \xi)$$
$$+(\beta-1)\sum_{i=1}^{n}\ln\left(G(X_i;\ \xi)\right) - (\beta+1)\sum_{i=1}^{n}\ln\left(1-\bar{\theta}R(X_i;\ \xi)\right) \tag{31}$$
$$+\ln(\alpha)\sum_{i=1}^{n}\left(\frac{G(X_i;\ \xi)}{1-\bar{\theta}R(X_i;\ \xi)}\right)^{\beta}$$
$$;\ \alpha \in \mathbb{R}^+, \alpha \neq 1$$

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The maximum likelihood (ML) estimators of β, θ, α and ξ can be obtained by differentiating (31) with respect to λ, θ, α and ξ then equating the equations to zero we get:

$$\frac{\partial\, ln\left(L(\mathbb{X})\right)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln\left(G(X_i;\, \xi)\right) - \sum_{i=1}^{n} ln\left(1-\bar{\theta}R(X_i;\, \xi)\right) + \beta\, ln(\alpha) \sum_{i=1}^{n} \left(\frac{G(X_i;\, \xi)}{1-\bar{\theta}R(X_i;\, \xi)}\right)^{\beta-1} \tag{32}$$

$$\frac{\partial \ln\left(L(\mathbb{X})\right)}{\partial \alpha} = \frac{\mathrm{n}}{\alpha\, ln(\alpha)} - \frac{n}{\alpha-1} + \frac{1}{\alpha} \sum_{i=1}^{n} \left(\frac{G(X_i;\, \xi)}{1-\bar{\theta}R(X_i;\, \xi)}\right)^{\beta} \tag{33}$$

$$\frac{\partial\, ln\left(L(\mathbb{X})\right)}{\partial \theta} = \frac{n}{\theta} - (\beta-1) \sum_{i=1}^{n} \frac{R(X_i;\, \xi)}{\left(1-\bar{\theta}R(X_i;\, \xi)\right)} - \beta\, ln(\alpha) \sum_{i=1}^{n} \frac{R(X_i;\, \xi) G(X_i;\, \xi)^{\beta}}{\left(1-\bar{\theta}R(X_i;\, \xi)\right)^{\beta+1}} \tag{34}$$

$$\begin{aligned} \frac{\partial \ln\left(L(\mathbb{X})\right)}{\partial\, \xi} &= \sum_{i=1}^{n} \frac{[g(X_i;\, \xi)]_{\xi}}{g(X_i;\, \xi)} + (\beta-1) \sum_{i=1}^{n} \frac{[G(X_i;\, \xi)]_{\xi}}{G(X_i;\, \xi)} + (\beta-1) \sum_{i=1}^{n} \frac{\bar{\theta}[R(X_i;\, \xi)]_{\xi}}{\left(1-\bar{\theta}R(X_i;\, \xi)\right)} \\ &+ \beta \ln(\alpha) \sum_{i=1}^{n} \frac{\left([G(X_i;\, \xi)]_{\xi}\left(1-\bar{\theta}R(X_i;\, \xi)\right) + G(X_i;\, \xi)\left(\bar{\theta}[R(X_i;\, \xi)]_{\xi}\right)\right) G(X_i;\, \xi)^{\beta-1}}{\left(1-\bar{\theta}R(X_i;\, \xi)\right)^{\beta+1}} \end{aligned} \tag{35}$$

Where$\alpha \in \mathbb{R}^{+}, \alpha \neq 1$. And

$$\ln L(\mathbb{X}) = n \ln(\beta) + n \ln(\theta) + \sum_{i=1}^{n} \ln g(X_i;\, \xi) + (\beta-1) \sum_{i=1}^{n} \ln\left(G(X_i;\, \xi)\right) - (\beta+1) \sum_{i=1}^{n} \ln\left(1-\bar{\theta}R(X_i;\, \xi)\right) \quad ;\ \alpha = 1 \tag{36}$$

The maximum likelihood (ML) estimators of β, θ and ξ can be obtained by differentiating (36) with respect to $\lambda, \theta$and ξ then equating the equations to zero we get:

$$\frac{\partial\, ln\left(L(\mathbb{X})\right)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln\left(G(X_i;\, \xi)\right) - \sum_{i=1}^{n} ln\left(1-\bar{\theta}R(X_i;\, \xi)\right) \tag{37}$$

$$\frac{\partial\, ln\left(L(\mathbb{X})\right)}{\partial \theta} = \frac{n}{\theta} - (\beta+1) \sum_{i=1}^{n} \frac{R(X_i;\, \xi)}{\left(1-\bar{\theta}R(X_i;\, \xi)\right)} \tag{38}$$

$$\frac{\partial \ln\left(L(\mathbb{X})\right)}{\partial\, \xi} = \sum_{i=1}^{n} \frac{[g(X_i;\, \xi)]_{\xi}}{g(X_i;\, \xi)} + (\beta-1) \sum_{i=1}^{n} \frac{[G(X_i;\, \xi)]_{\xi}}{G(X_i;\, \xi)} + (\beta+1) \sum_{i=1}^{n} \frac{\bar{\theta}[R(X_i;\, \xi)]_{\xi}}{1-\bar{\theta}R(X_i;\, \xi)} \tag{39}$$

Where $R(X_i;\, \xi) = 1 - G(X_i;\, \xi), \alpha = 1, [g(x_i; \xi)]_{\xi_j} = \frac{\partial g(x_i;\xi)}{\partial \xi_j}, [G(x_i; \xi)]_{\xi_j} = \frac{\partial G(x_i;\xi)}{\partial \xi_j}$

$$;\, j = 1,2, \dots p$$

4. Some special classes of HAPMOG class of distributions

1. If $\beta = 1$ we obtain another new class of distribution can be called alpha power Marshall Olkin:

$$f(x) = \begin{cases} \dfrac{ln(\alpha)}{\alpha-1} \dfrac{\theta g(x;\, \xi)}{\left(1-\bar{\theta}R(x;\, \xi)\right)^2} \alpha^{\left(\frac{G(x;\, \xi)}{1-\bar{\theta}R(x;\, \xi)}\right)} & for \quad x \in \mathbb{R}, \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ \dfrac{\theta g(x;\, \xi)}{\left(1-\bar{\theta}R(x;\, \xi)\right)^2} & for \quad x \in \mathbb{R}, \alpha = 1 \end{cases} \tag{40}$$

2. If $\theta = 1$ we obtain exponentiated alpha power:

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$$f(x)=\begin{cases}\dfrac{\ln(\alpha)}{\alpha-1}\beta g(x;\ \xi)\big(G(x;\ \xi)\big)^{\beta-1}\alpha^{(G(x;\ \xi))^{\beta}} & for \quad x\in\mathbb{R},\alpha\in\mathbb{R}^{+},\alpha\neq 1\\ \beta g(x;\ \xi)G(x;\ \xi)^{\beta-1} & for \quad x\in\mathbb{R},\alpha=1\end{cases} \tag{41}$$

3. If $\beta=1$ and $\theta=1$ we return to alpha power class of distribution:

$$f(x)=\begin{cases}\dfrac{\ln(\alpha)}{\alpha-1}g(x;\ \xi)\,\alpha^{G(x;\ \xi)} & for \quad x\in\mathbb{R},\alpha\in\mathbb{R}^{+},\alpha\neq 1\\ g(x;\ \xi) & for \quad x\in\mathbb{R},\alpha=1\end{cases} \tag{42}$$

4. If $\alpha=1$ we return to Marshall Olkin G class of distribution:

$$f(x)=\frac{\beta\theta g(x;\xi)}{\left(1-\bar{\theta}R(x;\xi)\right)^{2}}\left(\frac{G(x;\xi)}{1-\bar{\theta}R(x;\xi)}\right)^{\beta-1} \quad for \quad x\in\mathbb{R} \tag{43}$$

5. *Some special distributions*

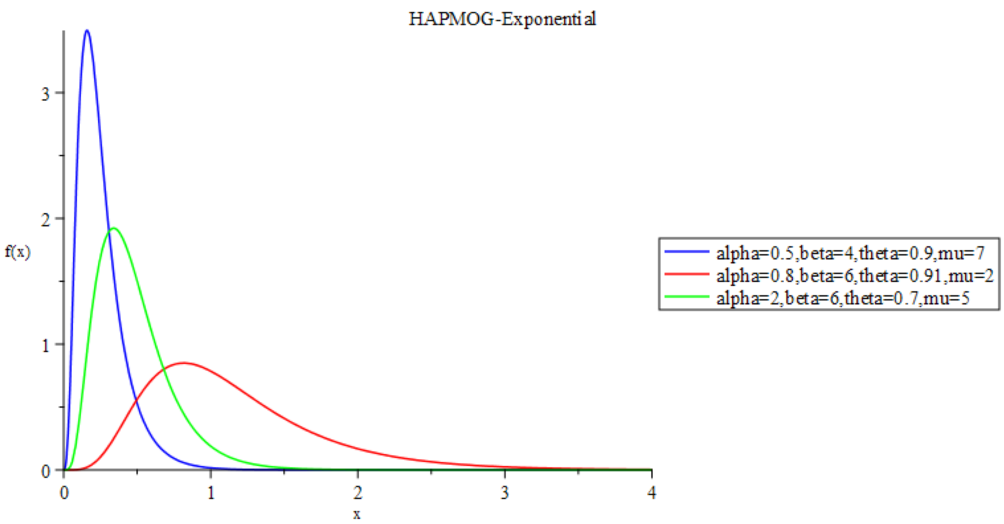
In this section, we provide three special distributions of the HAPMOG.

5.1 The HAPMOG-Exponential distribution

Considering the Exponential distribution with parameters $\mu>0$ having pdf and cdf, $g(x;\mu)=\mu e^{-\mu x}$ and $G(x;\mu)=1-e^{-\mu x}$ where $x>0$, substituting in (6), we get on the pdf of HAPMOG- Exponential distribution as:

$$f(x)=\begin{cases}\dfrac{\ln(\alpha)}{\alpha-1}\dfrac{\beta\theta\mu e^{-\mu x}}{(1-\bar{\theta}e^{-\mu x})^{2}}\left(\dfrac{1-e^{-\mu x}}{1-\bar{\theta}e^{-\mu x}}\right)^{\beta-1}\alpha^{\left(\frac{1-e^{-\mu x}}{1-\bar{\theta}e^{-\mu x}}\right)^{\beta}} & ;\ \alpha\in\mathbb{R}^{+},\alpha\neq 1\\ \dfrac{\beta\theta\mu e^{-\mu x}}{(1-\bar{\theta}e^{-\mu x})^{2}}\left(\dfrac{e^{-\mu x}}{1-\bar{\theta}e^{-\mu x}}\right)^{\beta-1} & ;\ \alpha=1\end{cases} \tag{44}$$

Where $\theta\in(0,1),\beta>0$.



5.2 The HAPMOG- Fréchet distribution

Let X be a Fréchet random variable with pdf $g(x;\sigma,\lambda)=\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}$ and $G(x;\sigma,\lambda)=e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}$ where $\sigma,\lambda>0$ and $x>0$. Then, substituting in (6), we obtain on the pdf of HAPMOG- Fréchet distribution as:

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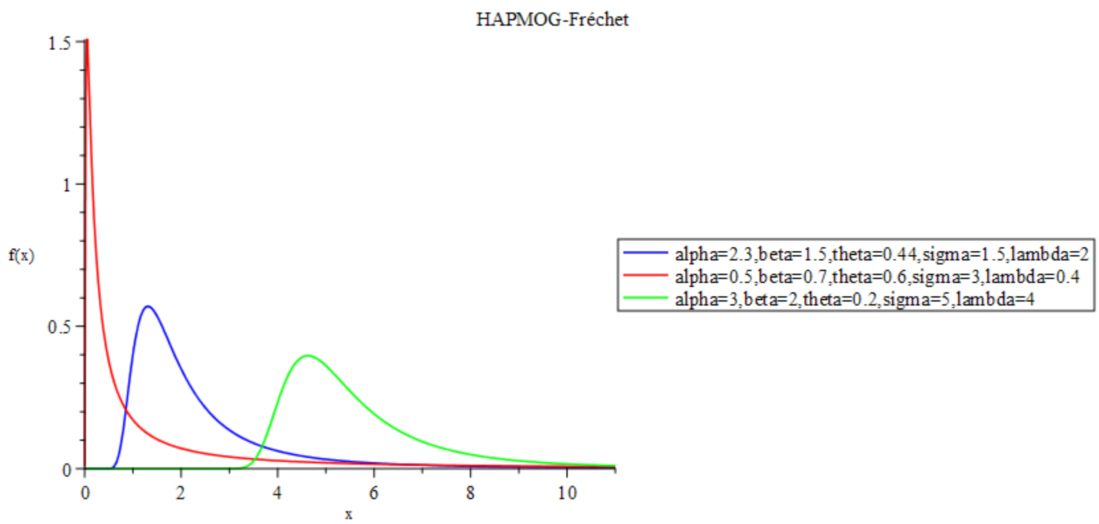
$$f(x) = \begin{cases} \dfrac{ln(\alpha)}{\alpha-1} \dfrac{\beta\theta\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{\left(1-\bar{\theta}\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)\right)^2} \left(\dfrac{e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{1-\bar{\theta}\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)}\right)^{\beta-1} \alpha^{\left(\frac{e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{1-\bar{\theta}\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)}\right)^{\beta}} & ; \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ \dfrac{\beta\theta\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{\left(1-\bar{\theta}\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)\right)^2} \left(\dfrac{e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{1-\bar{\theta}\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)}\right)^{\beta-1} & ; \alpha = 1 \end{cases} \quad (45)$$

Where $\theta \in (0,1), \beta > 0$.



5.3 The HAPMOG- Weibull distribution

Let X be a Weibull random variable with pdf $g(x;a,b) = abx^{b-1}e^{-ax^b}$ and $G(x;a,b) = 1 - e^{-ax^b}$. where $a,b > 0$ and $x > 0$. Then, substituting in (6), we obtain the pdf of HAPMOG- Weibull distribution as:

$$f(x) = \begin{cases} \dfrac{ln(\alpha)}{\alpha-1} \dfrac{\beta\theta bax^{b-1}e^{-ax^b}}{\left(1-\bar{\theta}e^{-ax^b}\right)^2} \left(\dfrac{1-e^{-ax^b}}{1-\bar{\theta}e^{-ax^b}}\right)^{\beta-1} \alpha^{\left(\frac{1-e^{-ax^b}}{1-\bar{\theta}e^{-ax^b}}\right)^{\beta}} & ; \quad \alpha \in \mathbb{R}^{+}, \alpha \neq 1 \\ \dfrac{\beta\theta abx^{b-1}e^{-ax^b}}{\left(1-\bar{\theta}e^{-ax^b}\right)^2} \left(\dfrac{1-e^{-ax^b}}{1-\bar{\theta}e^{-ax^b}}\right)^{\beta-1} & ; \alpha = 1 \end{cases} \quad (46)$$

Where $\theta \in (0,1), \beta > 0$.

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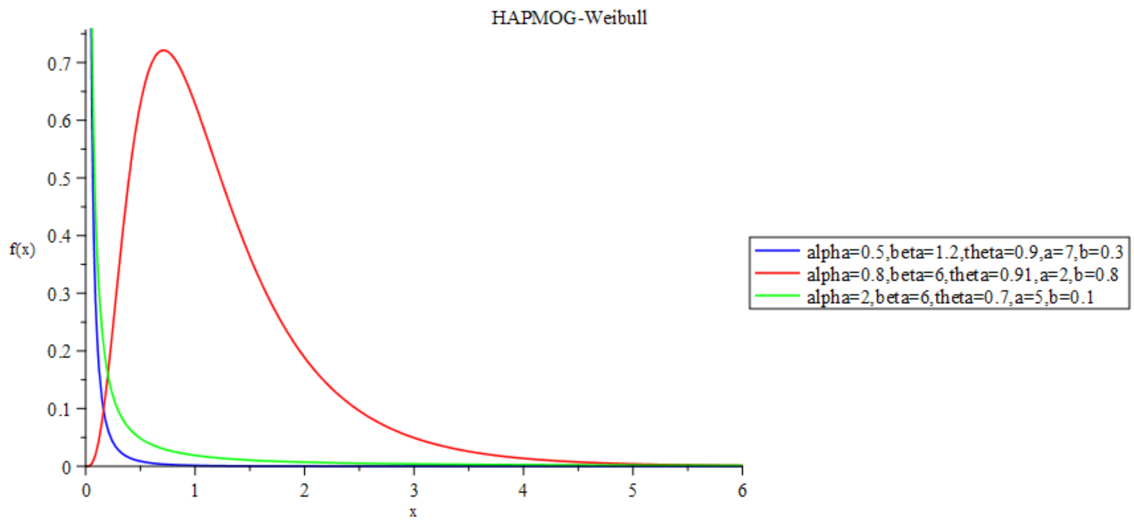

6. Conclusions and future research directions:

We defined a new class of distribution called Hybrid Alpha Power Marshall Olkin G class of distributions and found the cumulative distribution function, probability density function, r^{th} Moments, central moments, moments generating function, characteristic function, reliability and hazard functions and quantile function of this new class. We also showed special classes of the new generalized class of distributions and generated many distributions with respect to this called that are HAPMOG-Exponential distribution, HAPMOG-Freshet distribution, HAPMOG-Weibull distribution. These last three distributions may be very useful in modelling survival data and queueing systems.

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