



On Finding Legendre Polynomials by Algebraic Methods Based on Matrix Theory

Lee Xu

University of Chinese Academy of Sciences, CAS, Mathematics Department, Beijing, China

Email: Leexu1244@yahoo.com

Abstract

This paper is dedicated to find Legendre polynomials by using novel linear algebraic methods based on matrices based on Liouville-Sturm theorem, where we find the matrix of the differential operator for Legendre polynomials, with their eigenvalues and their eigenvectors. Also, we illustrate many examples to clarify the validity of our work.

Keywords: Legendre polynomials; differential operator; eigenvalue; eigenvector.

1. Introduction

As we know that the Legendre polynomial is known to many who work in the field of mathematics due to its multiple uses, and it can be found by well-known linear algebraic methods based on the Liouville-Sturm theorem [7,8], and we will try to find the matrix of the differential exponent of these polynomials and the corresponding eigenvalues, and this method differs from other standard methods that depend on solving the Legendre differential equation using power sequences, and the generating function.

Legendre polynomials are solutions of an ordinary differential equation, such as the hyper-geometric equation that appears when solving LaPlace's equation in spherical coordinates and solving the Schrodinger equation in three dimensions [1].

The hyper-geometric equation can be written in the form:

$$s(x)F''(x) + t(x)F'(x) + \lambda F(x) = 0 \dots \dots \dots (1).$$

Where F is a real function $F: U \rightarrow \mathcal{R}$, and $U \subset \mathcal{R}$, $\mathcal{R} \in \lambda$ is an eigenvalue. Also, s(x), t(x) are real polynomials with degree one or two. Many special cases can be obtained according to s(x), for example if s(x) is constant, then (1) becomes $F''(x) - 2\alpha x F'(x) + \lambda F(x) = 0$.

If s(x) is of degree one, then (1) becomes $x F''(x) + (-\alpha x + \beta + 1)F'(x) + \lambda F(x) = 0$.

There is a close correlation between Liouville-sturm theorem and other linear algebraic concepts, for example, functions take the role of vectors while effects take the role of Matrices and diagonalization of a real symmetric matrix corresponding to the solution of an ordinary differential equation, this matrix is known through the effect of the self-adjoint where its eigenfunctions are continuous and corresponding to eigenvectors [2,3].

Legendre functions are studied in most Sciences as we have already mentioned, which focus on differential equations and special functions, and here we will get the Legendre differential operator matrix in addition to eigenvalues and eigenvectors and from the form of vectors we get Legendre polynomials and we can apply this method to other polynomials such as, Hermite and Chebyshev [3,4,5,6].

Legendre Polynomials and matrices

The general formula of a polynomial is $\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots \dots \dots + \alpha_n x^n$ (2)

Where $\alpha_0, \alpha_1, \dots \dots \dots, \alpha_n \in \mathcal{R}$ can be represented as a vector

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} \tag{3}$$

The derivatives of (2) are:

$$\frac{d}{dx} \rightarrow \frac{d}{dx} [\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n] = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \dots + n\alpha_n x^{n-1} \tag{4}$$

$$\frac{d^2}{dx^2} \rightarrow \frac{d}{dx} [\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \dots + n\alpha_n x^{n-1}] = 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 + \dots + n(n-1)\alpha_n x^{n-2} \tag{5}$$

The matrix representations of first derivative are:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & n \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ 3\alpha_3 \\ \dots \\ n\alpha_n \\ 0 \end{bmatrix} \tag{6}$$

or

$$\frac{d}{dx} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & n \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{7}$$

The matrix representations of the second derivative are:

$$\begin{bmatrix} 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 6 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & n(n-1) \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 2\alpha_2 \\ 6\alpha_3 \\ 12\alpha_4 \\ \dots \\ n(n-1)\alpha_n \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

or

$$\frac{d^2}{dx^2} \rightarrow \begin{bmatrix} 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 6 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & n(n-1) \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{9}$$

We change (7), (9) in Legendre differential operator to get:

$$(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx}$$

where

$$(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} = (1-x^2)[2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 + \dots + n(n-1)\alpha_n x^{n-2}] - 2x[\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \dots + n\alpha_n x^n] = 2\alpha_2 + (-2\alpha_1 + 6\alpha_3)x + (-6\alpha_2 + 12\alpha_4)x^2 + \dots + (-n(n-1) - 2n)\alpha_n x^n.$$

Thus, we get the matrix representation of Legendre differential operator:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -2 & 0 & 6 & 0 & 0 & \dots & 0 \\ 0 & 0 & -6 & 0 & 12 & 0 & \dots & 0 \\ 0 & 0 & 0 & -12 & 0 & 20 & \dots & 0 \\ 0 & 0 & 0 & 0 & -20 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & n(n-1) \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & -n(n-1)-2n \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \dots \\ \alpha_{n-2} \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 2\alpha_2 \\ -2\alpha_1 + 6\alpha_3 \\ -6\alpha_2 + 12\alpha_4 \\ -12\alpha_3 + 20\alpha_5 \\ \dots \\ n(n-1)\alpha_n \\ 0 \\ [-n(n-1)-2n]\alpha_n \end{bmatrix}$$

(10)

or

$$(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -2 & 0 & 6 & 0 & 0 & \dots & 0 \\ 0 & 0 & -6 & 0 & 12 & 0 & \dots & 0 \\ 0 & 0 & 0 & -12 & 0 & 20 & \dots & 0 \\ 0 & 0 & 0 & 0 & -20 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & n(n-1) \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & -n(n-1)-2n \end{bmatrix}$$

Example:

Consider the Legendre differential operator of size 4*4:

$$M = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 & 6 \\ 0 & 0 & -6 & -6 & 0 \\ 0 & 0 & 0 & 0 & -12 \end{bmatrix} \quad (12)$$

The eigenvalues are:

$$\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = -6, \lambda_4 = -12$$

The eigenvectors are:

$$v = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \begin{bmatrix} 0 - \lambda_i & 0 & 2 & 0 \\ 0 & -2 - \lambda_i & 0 & 6 \\ 0 & 0 & -6 - \lambda_i & 0 \\ 0 & 0 & 0 & -12 - \lambda_i \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (13).$$

For $\lambda_1 = 0$, we get:

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is the corresponding vector to the polynomial:

$$P_0(x) = 1.$$

For $\lambda_2 = -2$, we get:

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus,

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

And $P_1(x) = x$.

For $\lambda_3 = -6$, we get:

$$\begin{bmatrix} 6 & 0 & 2 & 0 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

And

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus $P_2(x) = \frac{(3x^2-1)}{2}$.

For $\lambda_4 = -12$, hence

$$\begin{bmatrix} 12 & 0 & 2 & 0 \\ 0 & 10 & 0 & 6 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

And

$$v_4 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -5 \end{bmatrix}$$

so that, $P_3(x) = \frac{(3x^3-3x)}{2}$.

Finally, here the Legendre polynomials were obtained in a direct way using basic linear algebra concepts such as the eigenvalue and eigenvector of a matrix, once the Matrix corresponding to the Legendre differential exponent is obtained, the eigenvalues of this matrix and the eigenvectors corresponding to these values corresponding to the coefficients of the Legendre polynomials are found and the above was obtained in a way that does not use classical methods and which is based on solving the Legendre equation using power sequences, the Rodrigue formula and the generating function.

Suggestions and recommendations

We recommend using this method that was followed in this research on other polynomials such as orthogonal polynomials and especially the classical chebchev – Laguerre – Hermite – Jacobi polynomials.

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