



The Integrals on Spherical Regions in \mathfrak{R}^n

Murat Ozcek

Gaziantep University, Gaziantep, Turkey

Email: muratozcek.12@gmail.com

Abstract

This paper is dedicated to find the values of the integrals in the spherical region of \mathfrak{R}^n depending on the generative Kernel method by finding the integral formula that we use in the orthogonal and regular operations to find Ortho-normal polynomials on the sphere with radius r . Also, we illustrate many examples to clarify the validity of our work.

Keywords: Generative kernel; spherical region; Euclidean space; integral formula

1. Introduction

The generative kernel method was discovered for the first time in 1968 by the scientist misovskikh in order to calculate the multiplicative integrals in regions that have a center of symmetry and do not have a center of symmetry [5,6], but the important development of the generating kernel method was obtained by Mueller in 1973, where he was able to prove the axioms, and found the cubic relation with $2k + 1$ algebraic accuracy in the case of symmetric regions, and $2K$ in the case of asymmetric regions [2,3,4], and in 1975 the ongoing research in this field is about studying the properties of the generated nucleus and applying this method in wider integral areas, obtaining New cubic relations can be used in order to calculate the approximate value of multiplicative integrals.

Some important axioms and definitions:

Before presenting the method of generating nuclei for the formation of cubic relations, it is necessary to clarify some basic concepts, the cubic relation is an approximate equality of the form:

$$\int_{\Omega} \omega(x)f(x)dx \cong \sum_{j=1}^N C_j f(x_j) \quad (1)$$

Since x_j are different points of a bent triangle and are called Integral points (integral points) or nodes of the cubic relationship, N the number of Integral points and C_j constants corresponding to those points, $f(x)$ the function to be integrated and $\omega(x)$ the weight function, while Ω is the integral area.

Theorem (1): [7] assuming that Ω is a region containing Interior points, and assuming that $\omega(x)$ satisfies the condition:

$\omega(x) \geq 0$ & $\int_{\Omega} \omega(x)dx > 0$; $x \in \Omega$, and the cubic relation (1) has algebraic precision d^1 , and for $d \geq 0$, then:

$$N \geq \partial = M(n, k) = \frac{(n+k)!}{n!k!} : k = \left\lfloor \frac{d}{2} \right\rfloor$$

The symbol $k = \left\lfloor \frac{d}{2} \right\rfloor$ means the integer division of the fraction $n, \frac{d}{2}$ after the space.

Definition (2): [7] the generating kernel is a polynomial of degree k containing $2n$ transformations denoted by $k_k(u, x)$ where $x = (x_1, x_2, \dots, x_n)$ and $u = (u_1, u_2, \dots, u_n)$ is called a generator because it achieves the following special:

$$F(u) = \int_{\Omega} \omega(x) \cdot k_k(u, x) \cdot F(x) dx$$

Where $dx = dx_1 \cdot dx_2 \dots dx_n$ and k are the degree of a regular orthogonal polynomial $F(x)$ we can find the generated kernel in the following form:

$$\tilde{K}_k(u, x) = \sum_{j=1}^{n'} F_j^k(u) F_j^k(x) \quad (2)$$

Where the sign of the aperture above the sum means that the addition operation j , which corresponds to the polynomial $F_j^k(x)$, which has the same odd (even) with k in the case k is an odd (even) number, and $\tilde{K}_k(u, x)$ is a polynomial with $2n$ variables, whose degree is k , which has the same odd (even) with k , from which we can form the cubic relation with algebraic accuracy $2k$ and to form the cubic relation with odd accuracy $2k+1$, we use the formula:

$$K_k(u, x) = \tilde{K}_k(u, x) + \tilde{K}_{k-1}(u, x) \quad (3)$$

We use pairwise precision in the case when both the weight $\omega(x)$ and the region Ω possess a special central symmetry, that is, that:

$$x \in \Omega \Rightarrow -x \in \Omega, \omega(x) = \omega(-x) \quad (4)$$

Theorem 1:

Suppose that $\int_{\Omega} \omega(x) dx > 0$ & $\omega(x) \geq 0$, and Ω has internal points, so if the cubic relation (1) has m algebraic precision, then N the number of nodes of the cubic relation achieves the retracement:

$$N \geq \chi = M(n, k) = \frac{(n+k)!}{n!k!}; k = \left\lfloor \frac{m}{2} \right\rfloor$$

Theorem 2:

Suppose that $\int_{\Omega} \omega(x) dx > 0$ & $\omega(x) \geq 0$, and the quotient has internal points, so that $\Omega, \omega(x)$ fulfill the special analog for $\theta=(0,0,\dots,0)$ as in (4)

If the cubic relation (1) has $d = 2k+1$ algebraic precision, and if θ is not among the nodes of the cubic relation, then the number of nodes of the cubic relation achieves the regression:

$$N \geq \begin{cases} 2(\chi - \nu); k: \text{odd} \\ 2\nu \end{cases}$$

And if θ is among the nodes of the cubic relation, then the number of nodes of the cubic relation achieves the retracement:

$$N \geq \begin{cases} 2(\chi - \nu) - 1; k: \text{odd} \\ 2\nu + 1 \end{cases}$$

Where ν is the number of non-pairwise limit singularities whose degree does not exceed k by n variant. To form cubic relations we use the following two theorems:

Theorem (2): [7] assuming that the points u^1, u^2, \dots, u^n check the condition of the $K_k(u^i, u^j) = b_i \delta_{ij}$ where $b_i = K_k(u^i, u^i)$ consists $\cap_{i=1}^n H_i$ of points $s = k^n, j = 1, 2, \dots, s, x^j$, we take u^1 optional point of the ball $B_n^{(r)}$ as a result of the offset coordinates in (3) We get the coefficient of the surface H_1 get the point u^2 from the intersection $B_n^{(r)} \cap H_1$ which set the surface H_2 and $B_n^{(r)} \cap H_1 \cap H_2$ we get u^3 , and so on... Then the inverse relationship can be formed, the accuracy of which is $2k$:

$$\int_{\Omega} \omega(x) f(x) dx \cong \sum_{i=1}^n \frac{1}{b_i} f(u^i) + \sum_{j=1}^s C_j f(x^j) \quad (5)$$

Theorem (3): [7] assuming that both $\omega(x)$ and Ω achieve a special central symmetry (4), the points $a^{(i)}$ fulfill the condition $\tilde{K}_k(u^i, u^j) = b_i \delta_{ij}$ and consists $\cap_{i=1}^n H_i$ of the different points bent $s = k^n, x^{(1)}, x^{(2)}, \dots, x^{(s)}$, then the cubic relation whose accuracy is $2k+1$ can be formed:

$$\int_{\Omega} \omega(x)f(x)dx \cong \sum_{i=1}^n \frac{1}{2b_i} (f(u^i) + f(-u^i)) + \sum_{j=1}^s C_j f(x^j) \tag{6}$$

2-3. Finding the integral Constitution:

Let B_n be a sphere in the Space \mathfrak{R}^n , where $B_n^{(r)} = \{x \in \mathfrak{R}^n; x_1^2 + \dots + x_n^2 = r^2\}$ and r is the radius of the sphere, and let's find the value of the integral

$$P_{\alpha} = \int_{B_n} x^{\alpha} dx \tag{7}$$

Where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $x = (x_1, x_2, \dots, x_n)$, if one of the powers α_i is an odd number, then $P_{\alpha} = 0$, but if the powers α_i are even numbers, we can write it in the form $\alpha_i = 2q_i$ where $i=1,2,\dots,n$ we find:

$$P_{\alpha} = r^{|\alpha|+n} \frac{\prod_{i=1}^n \Gamma\left(\frac{\alpha_i+1}{2}\right)}{\Gamma\left(\frac{|\alpha|+n}{2} + 1\right)}; \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \tag{8}$$

3-3: finding orthogonal and ordered polynomials:

To find orthogonal and ordered polynomials, we will rely on the gram-Schmidt principle of orthogonality and order:

$$F_0^0 = \frac{1}{\sqrt{\mu(B_n^{(r)})}} \tag{9}$$

$$F_i^1 = \frac{\sqrt{n+2}}{r\sqrt{\mu(B_n^{(r)})}} x_i; \quad i = 1,2,3, \dots, n \tag{10}$$

$$F_{ij}^2 = \begin{cases} F_{ii}^2 = \sqrt{\frac{(n+3-i)(n+2)(n+4)}{2r^4(n+2-i)\mu(B_n^{(r)})}} \left[x_i^2 + \frac{1}{n+3-i} \left(\sum_{j=1}^{i-1} x_j^2 - r^2 \right) \right]; & i = j \\ F_{ij}^2 = \sqrt{\frac{(n+2)(n+4)}{r^4\mu(B_n^{(r)})}} x_i x_j & ; i \neq j \end{cases} \tag{11}$$

$$F_{ijk}^3 = \begin{cases} F_{iij}^3 = \sqrt{\frac{(n+5-i)\lambda}{2(n+4-i)r^6\mu(B_r^{(n)})}} \left[x_i^2 + \frac{1}{n+5-i} \left(\sum_{s=1}^{i-1} x_s^2 - r^2 \right) \right] x_j; & i < j \\ F_{iii}^3 = \sqrt{\frac{(n+5-i)\lambda}{6r^6\mu(B_r^{(n)}) (n+2-i)}} \left[x_i^2 + \frac{3}{n+5-i} \left(\sum_{s=1}^{i-1} x_s^2 - r^2 \right) \right] x_i; & i = j = k \\ F_{ijk}^3 = \sqrt{\frac{\lambda}{r^6\mu(B_r^{(n)})}} x_i x_j x_k & ; i \neq j \neq k \\ F_{iij}^3 = \sqrt{\frac{(n+3-i)\lambda}{2(n+2-i)r^6\mu(B_n^{(r)})}} \left[x_i^2 + \frac{1}{n+3-i} \left(\sum_{t=1}^{i-1} x_t^2 - r^2 \right) \right] x_j; & i > j \end{cases} \tag{12}$$

Where $\mu(B_n^{(r)}) = \frac{r^n \Gamma\left(\frac{1}{2}\right)^n}{\Gamma\left(\frac{n}{2}+1\right)}$, $\lambda = (n+2)(n+4)(n+6)$

4-3: finding the formula of the generating kernel [9,10]:

1-4-3. The generating kernel of First-Order polynomials:

Substituting (10) in (2), we find:

$$\tilde{K}_1(u, x) = \frac{n+2}{r^2\mu(B_n^{(r)})} \sum_{j=1}^n u_j x_j \tag{13}$$

By substituting in (3), where the $\tilde{K}_0(u, x) = \frac{1}{\mu(B_n^{(r)})}$ we find:

$$K_1(u, x) = \tilde{K}_1(u, x) + \tilde{K}_0(u, x) = \frac{1}{\mu(B_n^{(r)})} + \frac{n+2}{r^2\mu(B_n^{(r)})} \sum_{j=1}^n u_j x_j \quad (14)$$

2-4-3. The generating kernel of second-order polynomials:

The generated kernel is given by the formula:

$$\tilde{K}_2 = \frac{1}{\mu(B_n^{(r)})} + \sum_{j=1}^n F_j^2(x)F_j^2(u) + \sum_{i \neq j} F_i^2(x)F_j^2(u) \quad (15)$$

Substituting the polynomials of the second degree and of the zero degree, we find:

$$\tilde{K}_2(u, x) = \frac{1}{\mu(B_n^{(r)})} + N \sum_{j=1}^n \frac{n+3-j}{n+2-j} (U_j)(X_j) + \sum_{i \neq j} F_i^2(x)F_j^2(u) \quad (16)$$

Where the: $N = \frac{(n+2)(n+4)}{2r^4\mu(B_n^{(r)})}$

$$U_j = u_j^2 + \frac{1}{n+3-j} \left(\sum_{s=1}^{j-1} u_s^2 - r^2 \right)$$

$$X_j = x_j^2 + \frac{1}{n+3-j} \left(\sum_{s=1}^{j-1} x_s^2 - r^2 \right)$$

(16) can be written as:

$$\begin{aligned} \tilde{K}_2(u, x) &= \frac{1}{\mu(B_n^{(r)})} + N \sum_{j=1}^n \frac{n+3-j}{n+2-j} [U_j] \left[x_j^2 + \frac{1}{n+3-j} \sum_{s=1}^{j-1} x_s^2 - \frac{r^2}{n+3-j} \right] + \sum_{i \neq j} F_i^2(x)F_j^2(u) \\ &= \frac{1}{\mu(B_n^{(r)})} + N \sum_{j=1}^n \frac{n+3-j}{n+2-j} [U_j] x_j^2 + N \sum_{j=1}^n \frac{[U_j]}{n+2-j} \sum_{s=1}^{j-1} x_s^2 - N \sum_{j=1}^n \frac{[U_j]}{n+2-j} r^2 + \sum_{i \neq j} F_i^2(x)F_j^2(u) \end{aligned} \quad (18)$$

$$\tilde{K}_2(u, x) = \frac{1}{\mu(B_n^{(r)})} + N[S_1 + S_2 + S_3] + \frac{(n+2)(n+4)}{r^4\mu(B_n^{(r)})} \sum_{i \neq j} x_i x_j u_i u_j$$

Let's find both S_1, S_2, S_3 : we get out in S_1 the sum with proof n , then we replace each j with s and we find:

S_1 is the sum with proof n , then we replace each j with s , and we find:

$$S_1 = \frac{3}{2} U_n x_n^2 + \sum_{s=1}^{n-1} \frac{n+3-s}{n+2-s} [U_s] x_s^2 \quad (19)$$

In S_2 we perform a permutation in the summation process, we find:

$$S_2 = \sum_{s=1}^{n-1} x_s^2 \sum_{j=s+1}^n \frac{U_j}{n+2-j} \quad (20)$$

Adding (19) and (20), we find:

$$S_1 + S_2 = \frac{3}{2} U_n x_n^2 + \sum_{s=1}^{n-1} x_s^2 \cdot S_4 \quad (21)$$

Where the:

$$S_4 = U_s + \sum_{j=s}^n \frac{U_j}{n+2-j} \quad (22)$$

Substituting (17) in (22), we find:

$$S_4 = u_s^2 + \frac{1}{n+3-s} \left(\sum_{t=1}^{s-1} u_t^2 - r^2 \right) + \sum_{j=s}^n \frac{1}{n+3-j} \left[u_j^2 + \frac{1}{n+3-j} \left(\sum_{t=1}^{s-1} u_t^2 - r^2 \right) \right]$$

$$S_4 = u_s^2 + \frac{1}{n+3-s} \sum_{t=1}^{s-1} u_t^2 - \frac{r^2}{n+3-s} + \sum_{j=s}^n \frac{1}{n+2-j} u_j^2 + \sum_{j=s}^n \frac{1}{(n+2-j)(n+3-j)} \sum_{t=1}^{j-1} u_t^2 - \sum_{j=s}^n \frac{r^2}{(n+2-j)(n+3-j)} \tag{23}$$

We perform a permutation in the order of the sum in the fifth term, which we will denote by the symbol S_5 in (23), we find:

$$S_5 = \sum_{t=1}^{s-1} u_t^2 \sum_{j=s}^n \frac{1}{(n+2-j)(n+3-j)} + \sum_{t=s}^{n-1} u_t^2 \sum_{j=t+1}^n \frac{1}{(n+2-j)(n+3-j)} \tag{24}$$

As for the sum in (24), it can be written by the formula:

$$\sum_{j=s}^n \frac{1}{(n+3-j)(n+2-j)} = \sum_{j=s}^n \frac{1}{n+2-j} - \frac{1}{n+3-j} = \frac{1}{2} - \frac{1}{n+3-s} \tag{25}$$

$$\sum_{j=t+1}^n \frac{1}{(n+2-j)(n+3-j)} = \sum_{j=t+1}^n \frac{1}{n+2-j} - \frac{1}{n+3-j} = \frac{1}{2} - \frac{1}{n+2-t}$$

Substituting (25) in (24), we find:

$$S_5 = \sum_{t=1}^{s-1} u_t^2 \left(\frac{1}{2} - \frac{1}{n+3-s} \right) + \sum_{t=s}^{n-1} u_t^2 \left(\frac{1}{2} - \frac{1}{n+2-t} \right) \tag{26}$$

Substituting (26) in (23), we find:

$$S_4 = u_s^2 + \frac{1}{n+3-s} \sum_{t=1}^{s-1} u_t^2 - \frac{r^2}{n+3-s} + \sum_{j=s}^n \frac{1}{n+2-j} u_j^2 + \frac{1}{2} \sum_{t=1}^{s-1} u_t^2 - \frac{1}{n+3-s} \sum_{t=1}^{s-1} u_t^2 + \frac{1}{2} \sum_{t=s}^{n-1} u_t^2 - \sum_{t=1}^{s-1} \frac{1}{n+2-t} u_t^2 - \frac{r^2}{2} + \frac{r^2}{n+3-s} \tag{27}$$

Summing the limits, we find:

$$S_4 = u_s^2 + \frac{1}{2} \sum_{t=1}^n u_t^2 - \frac{r^2}{2} \tag{28}$$

Substituting (28) in (21), we find:

$$\begin{aligned} S_1 + S_2 &= \frac{3}{2} U_n x_n^2 + \sum_{s=1}^{n-1} x_s^2 \left(u_s^2 + \frac{1}{2} \sum_{t=1}^n u_t^2 - \frac{r^2}{2} \right) \\ &= \frac{3}{2} x_n^2 \left(u_n^2 + \frac{1}{3} \left(\sum_{s=1}^{n-1} u_s^2 - r^2 \right) \right) + \sum_{s=1}^{n-1} x_s^2 \left(u_s^2 + \frac{1}{2} \sum_{t=1}^n u_t^2 - \frac{r^2}{2} \right) \\ &= \sum_{s=1}^n x_s^2 \left(u_s^2 + \frac{1}{2} \sum_{s=1}^n u_s^2 - \frac{r^2}{2} \right) \end{aligned}$$

$$S_1 + S_2 = \sum_{s=1}^n x_s^2 \left(u_s^2 + \frac{1}{2} \sum_{s=1}^n u_s^2 - \frac{r^2}{2} \right) \tag{29}$$

Let's find S_3 :

$$\begin{aligned} -S_3 &= \sum_{j=1}^n \frac{[U_j]}{n+2-j} r^2 = \sum_{j=1}^n \frac{r^2}{n+2-j} \left(u_j^2 + \frac{1}{n+3-j} \left(\sum_{s=1}^{j-1} u_s^2 - r^2 \right) \right) \\ &= \sum_{j=1}^n \frac{r^2}{n+2-j} u_j^2 + \sum_{j=1}^n \frac{r^2}{(n+2-j)(n+3-j)} \sum_{s=1}^{j-1} u_s^2 - \sum_{j=1}^n \frac{r^4}{(n+2-j)(n+3-j)} \end{aligned}$$

Hence:

$$-S_3 = \frac{r^2}{2} \sum_{s=1}^n u_s^2 - r^4 \left(\frac{n}{2(n+2)} \right) \tag{30}$$

Adding (29) and (30), we find:

$$S_1 + S_2 + S_3 = \frac{1}{2} \left[2 \sum_{s=1}^n x_s^2 u_s^2 + \sum_{s=1}^n x_s^2 \sum_{s=1}^n u_s^2 - r^2 \left(\sum_{s=1}^n u_s^2 + \sum_{s=1}^n x_s^2 \right) + r^4 \binom{n}{n+2} \right]$$

Substituting in (18), we find:

$$\tilde{K}_2(u, x) = \frac{1}{\mu(B_n^{(r)})} + \frac{(n+2)(n+4)}{4r^4 \mu(B_n^{(r)})} \left[\frac{2 \sum_{s=1}^n x_s^2 u_s^2 + \sum_{s=1}^n x_s^2 \sum_{s=1}^n u_s^2}{-r^2 (\sum_{s=1}^n u_s^2 + \sum_{s=1}^n x_s^2) + r^4 \binom{n}{n+2}} \right] + \frac{(n+2)(n+4)}{r^4 \mu(B_n^{(r)})} \sum_{i \neq j} x_i x_j u_i u_j$$

The last relation can be written in the form:

$$\tilde{K}_2(u, x) = \frac{(n+2)}{r^4 \mu(B_n^{(r)})} \left\{ \frac{r^4 \frac{n+2}{4} + (n+4) \sum_{i \neq j} x_i x_j u_i u_j + \binom{n+4}{4} [2 \sum_{s=1}^n x_s^2 u_s^2 + \sum_{s=1}^n x_s^2 \sum_{s=1}^n u_s^2 - r^2 (\sum_{s=1}^n u_s^2 + \sum_{s=1}^n x_s^2)]}{4} \right\} \tag{31}$$

And to find $K_2(u, x)$:

$$K_2(u, x) = \tilde{K}_2(u, x) + \tilde{K}_1(u, x) = \frac{(n+2)}{r^4 \mu(B_n^{(r)})} \left\{ \frac{r^4 \frac{n+2}{4} + (n+4) \sum_{i \neq j} x_i x_j u_i u_j + r^2 \sum_{j=1}^n u_j x_j + \binom{n+4}{4} [2 \sum_{s=1}^n x_s^2 u_s^2 + \sum_{s=1}^n x_s^2 \sum_{s=1}^n u_s^2 - r^2 (\sum_{s=1}^n u_s^2 + \sum_{s=1}^n x_s^2)]}{4} \right\} \tag{32}$$

Suppose that $(u_1, u_2, \dots, u_n) \in S_{n-1}^{(r)}$ thus the $\sum_{s=1}^n u_s^2 = r^2$, substituting in (31) and (32), we find

$$\tilde{K}_2(u, x) = \frac{(n+2)(n+4)}{2r^4 \mu(B_n^{(r)})} \left\{ \left(\sum_{s=1}^n x_s u_s \right)^2 - \frac{r^4}{n+4} \right\}$$

Thus the final formula for $\tilde{K}_2(u, x)$ is:

$$\tilde{K}_2(u, x) = \frac{(n+2)(n+4)}{2r^4 \mu(B_n^{(r)})} \left(\sum_{s=1}^n x_s u_s + \frac{r^2}{\sqrt{n+4}} \right) \left(\sum_{s=1}^n x_s u_s - \frac{r^2}{\sqrt{n+4}} \right) \tag{33}$$

In the same way as before, we find that the final formula for $K_2(u, x)$ is:

$$K_2(u, x) = \frac{(n+2)(n+4)}{2r^4 \mu(B_n^{(r)})} \left(\sum_{s=1}^n x_s u_s + \frac{1 + \sqrt{n+5}}{n+4} r^2 \right) \left(\sum_{s=1}^n x_s u_s + \frac{1 - \sqrt{n+5}}{n+4} r^2 \right) \tag{34}$$

3-4-3. The generating kernel of polynomials of the third degree:

Following the same steps in paragraph 2.4.3, where the kernel formula was deduced which is in the form:

$$\tilde{K}_3(u, x) = \frac{(n+2)(n+4)(n+6)}{6r^6 \mu(B_n^{(r)})} \left[\left(\sum_{i=1}^n u_i x_i \right)^3 - \frac{3r^4}{(n+6)} \sum_{i=1}^n u_i x_i \right] \tag{35}$$

$$K_3(u, x) = \frac{(n+2)(n+4)(n+6)}{6r^6 \mu(B_n^{(r)})} \left[\left(\sum_{i=1}^n u_i x_i \right)^3 + \frac{3r^2}{n+6} \left(\sum_{i=1}^n x_i u_i \right)^2 - \frac{3r^4}{(n+6)} \sum_{i=1}^n u_i x_i - \frac{3r^6}{(n+4)(n+6)} \right] \tag{36}$$

4. Results and their discussion [1,8,11,12]:

1-4. Formation of the cubic relation of the generating nuclei $K_1(u, x)$ and $\tilde{K}_1(u, x)$:

To find the cubic relation for $d = 2k + 1 = 3$, we choose the point $a^1 = (\alpha_1, 0, 0, \dots, 0)$ since $-r < \alpha_1 < r$ we substitute in (13) and we get the plane:

$$H_1 := \frac{n+2}{r^2} \frac{1}{\mu(B_n^{(r)})} \alpha_1 x_1 = 0 \tag{37}$$

We choose from $H_1 \cap B_n^{(r)}$ the point a^2 , from (37) $x_1 = 0$ we substitute in the retracement $x_1^2 + x_2^2 \leq r^2$ we find that $x_2 \leq r$ and suppose that $x_2 = \alpha_2$ we find that $a^2 = (0, \alpha_2, 0, \dots, 0)$ which in turn assigns the level H_2

$$H_2 := \frac{n+2}{r^2} \frac{1}{\mu(B_n^{(r)})} \alpha_2 x_2 = 0 \tag{38}$$

So we choose from $B_n^{(r)} \cap (H_1 \cap H_2)$ the point u^3 , from (38) and from (39) $x_1 = x_2 = 0$ by substituting in the gradient $x_1^2 + x_2^2 + x_3^2 \leq r^2$ we find that $a^3 = (0, 0, \alpha_3, 0, \dots, 0)$ and so on So we find that the planes intersect at one point is $x = (0, 0, \dots, 0)$

The number of Integral points is $2n+1$ and the minimum number of points is $N \geq 2\nu + 1 = 2[M(n - 1, 1)] + 1 = 2n + 1$, we substitute in the cubic relation (6) where:

$\frac{1}{2b_i} = [2\tilde{K}_1(u^i, u^i)]^{-1} = r^2 \frac{\mu(B_n^{(r)})}{2(n+2)\alpha_i^2}$, the cubic relation (6) is accurate for $f(x) = 1$, x_i, x_i^2, x_i^3 to find the constant C , we take the function $f(x) = 1$ and substitute in (6), we find

$$C f(x) = \int_{B_n^{(r)}} 1 dx - \sum_{i=1}^n \frac{1}{2b_i} [f(u^i) + f(-u^i)] \Rightarrow C = \mu(B_n^{(r)}) \left(\frac{(n+2)\alpha_i^2 - r^2 n}{(n+2)\alpha_i^2} \right)$$

To find the cubic relation for $d = 2k = 2$ we choose the point $u^1 = (r, 0, 0, \dots, 0)$ we substitute in (14) and we get the level:

$$H_1 := 1 + \frac{n+2}{r} x_1 = 0 \tag{39}$$

We choose from $H_1 \cap B_n^{(r)}$ the point u^2 , from (39) $x_1 = \frac{r}{(n+2)}$ we substitute in equality $x_1^2, x_2^2 = r^2$

$$\text{We find that } x_2 = \frac{r}{(n+2)} \sqrt{(n+1)(n+3)}$$

Thus $u^2 = \left(-\frac{r}{(n+2)}, \frac{r}{(n+2)} \sqrt{(n+1)(n+3)}, 0, \dots, 0 \right)$ which in turn assigns the level H_2

$$H_2 := 1 - \frac{1}{r} x_1 + \frac{1}{r} \sqrt{(n+1)(n+3)} x_2 = 0 \tag{40}$$

So we choose from $B_n^{(r)} \cap (H_1 \cap H_2)$ the point u^3 , from (39) and from (40) $x_2 = \frac{-r}{(n+2)} \sqrt{\frac{n+3}{n+1}}$ substituting in the Equality $x_1^2 + x_2^2 + x_3^2 \leq r^2$ we find that $x_3 = r \sqrt{\frac{n(n+3)}{(n+1)(n+2)}}$ so that

$$u^3 = \left(\frac{-r}{(n+2)}, \frac{-r}{(n+2)} \sqrt{\frac{n+3}{n+1}}, r \sqrt{\frac{n(n+3)}{(n+1)(n+2)}}, 0, 0, \dots, 0 \right) \text{ Which sets the level:}$$

$$H_3 := 1 - \frac{1}{r} x_1 - \frac{1}{r} \sqrt{\frac{(n+3)}{(n+1)}} x_2 + \frac{1}{r} \sqrt{\frac{n(n+2)(n+3)}{n+1}} x_3 = 0 \tag{41}$$

So we choose from $B_n^{(r)} \cap (H_1 \cap H_2 \cap H_3)$ the point u^4 , from (40) and from (41) $x_3 = -r \sqrt{\frac{n+3}{n(n+1)(n+2)}}$ substituting in the equality $x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2$ we find that $x_4 = r \sqrt{\frac{(n+3)(n-1)}{n(n+2)}}$ thus the

$$u^4 = \left(\frac{-r}{n+2}, \frac{-r}{n+2} \sqrt{\frac{n+3}{n+1}}, -r \sqrt{\frac{(n+3)}{(n+1)(n+2)}}, r \sqrt{\frac{(n+3)(n-1)}{n(n+2)}}, 0, \dots, 0 \right) \text{ which designates the level:}$$

$$H_3 := 1 - \frac{1}{r} x_1 - \frac{1}{r} \sqrt{\frac{(n+3)}{(n+1)}} x_2 + \frac{1}{r} \sqrt{\frac{n(n+2)(n+3)}{n+1}} x_3 = 0 \tag{42}$$

Thus generalizing, we choose from $B_n^{(r)} \cap (H_1 \cap H_2 \cap H_3 \cap \dots \cap H_i)$ the point u^i , which is given by:

$$u^i = \left(\begin{array}{c} \frac{-r}{n+2}, \frac{-r}{n+2}, \sqrt{\frac{n+3}{n+1}}, -r \sqrt{\frac{(n+3)}{n(n+1)(n+2)}}, \dots, \dots, \dots, \\ -r \sqrt{\frac{(n+3)}{(n+4-(i-1))(n+3-(i-1))(n+2)}}, r \sqrt{\frac{(n+3)(n-i+3)}{(n+4-i)(n+2)}}, 0, \dots, 0 \end{array} \right)$$

So the levels $H_i \ i=1, \dots, n$ intersecting at one point are:

$$x = \left(\begin{array}{c} \frac{-r}{n+2}, \frac{-r}{n+2}, \sqrt{\frac{n+3}{n+1}}, -r \sqrt{\frac{(n+3)}{n(n+1)(n+2)}}, \dots, \dots, -r \sqrt{\frac{(n+3)}{20(n+2)}}, r \sqrt{\frac{3(n+3)}{4(n+2)}} \end{array} \right)$$

The number of points of integration equal to $n+1$ and the minimum number of these points is $N = M(n, k) = M(n, 1) = n + 1$ we get back in the relationship Cubism (14) where: $\frac{1}{b_i} = [K_1(u^i, u^i)]^{-1} = \frac{\mu(B_n^{(r)})}{n+3}$, the relationship Cubist (5) minutes in order to $f(x) = 1 \iota, x_i, x_i^2$ and let's get the value of constant C , we take the function $f(x) = 1 \iota$, we find $C f(x) = \int_{B_n^{(r)}} 1 dx - \sum_{i=1}^n \frac{1}{b_i} [f(u^i)] \Rightarrow C = \mu(B_n^{(r)}) \frac{n+2}{n+3}$

2-4. Formation of the cubic relation of the generating nuclei $K_2(u, x)$ and $\tilde{K}_2(u, x)$:

1-2-4. Formation of cubic relations for $\tilde{K}_2(u, x)$:

First: for $n=2$:

Let's form the cubic relation for the kernel generated in (33), and for algebraic precision

$d = 2k + 1 = 5$ and for $n = 2$, we substitute in (33) and find:

$$\tilde{K}_2(u, x) = \frac{12}{r^2 \mu(B_n^{(r)})} \left(x_1 u_1 + x_2 u_2 + \frac{r^2}{\sqrt{6}} \right) \left(x_1 u_1 + x_2 u_2 - \frac{r^2}{\sqrt{6}} \right) \quad (42)$$

Let's take the point $u^1 = (r, 0)$, we substitute in (42), we get the equation of the surface:

$$H_1: \left(r x_1 + \frac{r^2}{\sqrt{6}} \right) \left(r x_1 - \frac{r^2}{\sqrt{6}} \right) = 0 \quad (43)$$

We choose from $H_1 \cap B_2^{(r)}$ the point u^2 , from (43) $x_1 = -\frac{r}{\sqrt{6}}$ we substitute in the Equality $x_1^2 + x_2^2 = r^2$ we find

that $x_2 = \sqrt{\frac{5}{6}} r$ and therefore $u^2 = \left(-\frac{r}{\sqrt{6}}, \sqrt{\frac{5}{6}} r \right)$ which set the surface:

$$H_2: \left(-\frac{r}{\sqrt{6}} x_1 + \sqrt{\frac{5}{6}} r x_2 + \frac{r^2}{\sqrt{6}} \right) \left(-\frac{r}{\sqrt{6}} x_1 + \sqrt{\frac{5}{6}} r x_2 - \frac{r^2}{\sqrt{6}} \right) = 0 \quad (44)$$

The intersecting surfaces $H_i \ i=1, 2; s = k^n = 2^2 = 4$ with four points are:

$$\begin{aligned} x^1 &= \left(-\frac{r}{\sqrt{6}}, -\frac{r}{\sqrt{5}} \left(1 + \frac{1}{\sqrt{6}} \right) \right), \quad x^2 = \left(-\frac{r}{\sqrt{6}}, \frac{r}{\sqrt{5}} \left(1 - \frac{1}{\sqrt{6}} \right) \right) \\ x^3 &= \left(\frac{r}{\sqrt{6}}, \frac{r}{\sqrt{5}} \left(-1 + \frac{1}{\sqrt{6}} \right) \right), \quad x^4 = \left(\frac{r}{\sqrt{6}}, \frac{r}{\sqrt{5}} \left(1 + \frac{1}{\sqrt{6}} \right) \right) \end{aligned} \quad (50)$$

The number of Integral points is equal to 8 and the minimum number of points is $N \geq 2[\chi - \nu] = 2[M(n, k) - M(n - 1, 1)] = 8$ and we note that all points are located inside the circle, where the two nodes x^1, x^4 are located on the surface of the Circle $x^2 + y^2 = \frac{6-\sqrt{6}}{15} r^2$ has the same constant, and the nodes x^2, x^3 are located on the surface of the Circle $x^2 + y^2 = \frac{6-\sqrt{6}}{15} r^2$ has the same constant, we substitute in the cubic relation where $\frac{1}{2b_i} = [2K_k(u^i, u^i)]^{-1} = \frac{r^2 \pi}{20}$, and to find the constants C_1, C_2, C_3, C_4 we choose the following functions:

$$f_1 = \left(x - \frac{r}{\sqrt{6}}\right) \left(y + \frac{r}{\sqrt{5}} \left(-1 + \frac{1}{\sqrt{6}}\right)\right), f_2 = \left(x - \frac{r}{\sqrt{6}}\right) \left(y + \frac{r}{\sqrt{5}} \left(+1 + \frac{1}{\sqrt{6}}\right)\right)$$

$$f_3 = \left(x + \frac{r}{\sqrt{6}}\right) \left(y - \frac{r}{\sqrt{5}} \left(+1 + \frac{1}{\sqrt{6}}\right)\right), f_4 = \left(x + \frac{r}{\sqrt{6}}\right) \left(y + \frac{r}{\sqrt{5}} \left(+1 - \frac{1}{\sqrt{6}}\right)\right)$$

We obtain the constant C_1 by substituting the function f_1 in the cubic relation (33), since the function

f_1 is absent at all points in (50) except node x_1 , and we find the rest of the constants for the rest of the functions on the sequence, so we get the constants of the cubic relationship as in the following table:

Table 1: constants of the cubic relation (6) for $\tilde{K}_2(u, x)$, $n = 2$.

Points x_i	C_i
$x^1 = \left(-\frac{r}{\sqrt{6}}, -\frac{r}{\sqrt{5}} \left(1 + \frac{1}{\sqrt{6}}\right)\right)$ $x^4 = \left(\frac{r}{\sqrt{6}}, \frac{r}{\sqrt{5}} \left(1 + \frac{1}{\sqrt{6}}\right)\right)$	$\frac{8\sqrt{6} - 3}{40\sqrt{6}} \pi r^2$
$x^2 = \left(-\frac{r}{\sqrt{6}}, \frac{r}{\sqrt{5}} \left(1 - \frac{1}{\sqrt{6}}\right)\right)$ $x^3 = \left(\frac{r}{\sqrt{6}}, \frac{r}{\sqrt{5}} \left(-1 + \frac{1}{\sqrt{6}}\right)\right)$	$\frac{8\sqrt{6} + 3}{40\sqrt{6}} \pi r^2$

Second: for n=3:

Let's form the cubic relation for the kernel generated in (33), and for algebraic precision

$$d = 2k + 1 = 5 \text{ and for } n = 3, \text{ we substitute in (33) and find: } \mu(B_n^{(r)})$$

$$\tilde{K}_3(u, x) = \frac{35}{2r^4 \mu(B_n^{(r)})} \left(\sum_{i=1}^3 x_i u_i + \frac{r^2}{\sqrt{7}}\right) \left(\sum_{i=1}^3 x_i u_i - \frac{r^2}{\sqrt{7}}\right)$$

following the same steps in the previous paragraph, we

find that: $u^1(r, 0, 0)$, $u^3 = \left(\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(1 - \frac{1}{\sqrt{7}}\right), \frac{r}{\sqrt{3}} \sqrt{2 + \frac{1}{\sqrt{7}}}\right)$ the intersection of the surfaces resulting from the compensation of u^1, u^2, u^3 in (33) are the following points:

$$x^1 = \left(\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(1 - \frac{1}{\sqrt{7}}\right), \frac{r}{\sqrt{3}} \frac{4 - \sqrt{7}}{\sqrt{14 - \sqrt{7}}}\right), x^2 = \left(\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(1 - \frac{1}{\sqrt{7}}\right), -\frac{r}{\sqrt{3}} \frac{2 + \sqrt{7}}{\sqrt{14 - \sqrt{7}}}\right)$$

$$x^3 = \left(\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(-1 - \frac{1}{\sqrt{7}}\right), \sqrt{\frac{3}{14 - \sqrt{7}}} r\right),$$

$$x^4 = \left(\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(-1 - \frac{1}{\sqrt{7}}\right), -\sqrt{\frac{3}{14 - \sqrt{7}}} r\right)$$

$$x^5 = \left(-\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(1 + \frac{1}{\sqrt{7}}\right), -\sqrt{\frac{3}{14 - \sqrt{7}}} r\right),$$

$$x^6 = \left(-\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(1 + \frac{1}{\sqrt{7}}\right), \sqrt{\frac{3}{14 - \sqrt{7}}} r\right)$$

$$x^7 = \left(-\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(-1 + \frac{1}{\sqrt{7}}\right), \frac{r}{\sqrt{3}} \frac{2 + \sqrt{7}}{\sqrt{14 - \sqrt{7}}}\right)$$

$$x^8 = \left(-\frac{r}{\sqrt{7}}, \frac{r}{\sqrt{6}} \left(-1 + \frac{1}{\sqrt{7}}\right), \frac{r}{\sqrt{3}} \frac{-4 + \sqrt{7}}{\sqrt{14 - \sqrt{7}}}\right)$$

The number of Integral points is 14, and the minimum number of points is $N \geq 2[\chi - v] = 2[M(n, k) - M(n - 1, 1)] = 14$, so the number of Integral points increases by one amount from the minimum, the points

$x^2, x^3, x^4, x^5, x^6, x^8$ are located on the surface of the ball $xx^2 + y^2 + z^2 = \frac{8 + \sqrt{7}}{14 + \sqrt{7}} r^2$ has the same constant, and

the points x^1, x^8 are located on the surface of the ball $x^2 + y^2 + z^2 = \frac{12 - 3\sqrt{7}}{14 + \sqrt{7}} r^2$ has the same constant.

Let's form the cubic relationship where $\frac{1}{2b_i} = [2K_k(u^i, u^i)]^{-1} = 2\frac{r^3\pi}{45}$ and to find the constant C_1 we choose the function $f_1 = \left(x - \frac{r}{\sqrt{7}}\right)\left(y + \frac{r}{\sqrt{6}}\left(1 + \frac{1}{\sqrt{7}}\right)\right)\left(z + \frac{r}{\sqrt{3}}\frac{2+\sqrt{7}}{\sqrt{14-\sqrt{7}}}\right)$ which is absent at all points except the node x^1 , and we find the rest of the constants in the same way, so we get the constants of the cubic relationship as in the following table:

Table 2: constants of the cubic relation (6) for $\tilde{K}_2(u, x)$, $n = 3$.

Points x_i	C_i
$x^2, x^3, x^4, x^5, x^6, x^8$	$\frac{9\sqrt{7} - 21}{90\sqrt{7}}\pi r^3$
x^1, x^8	$\frac{13\sqrt{7} + 7}{90\sqrt{7}}\pi r^3$

2-1-5-2. Formation of cubic relations for $K_2(u, x)$:

First: for n=2:

Let's form the cubic relation for the generated kernel in (34), for algebraic precision $d = 2K = 4$, and for $n = 2$, we substitute in (34), we find

$$K_2(u, x) = \frac{24}{2r^4\mu(B_n^{(r)})} \left(x_1u_1 + x_2u_2 + \frac{1 + \sqrt{7}}{6}r^2\right) \left(x_1u_1 + x_2u_2 + \frac{1 - \sqrt{7}}{6}r^2\right)$$

Where $u^1 = (r, 0)$, $u^2 = \left(\frac{-1+\sqrt{7}}{6}r, \frac{\sqrt{28+2\sqrt{7}}}{6}r\right)$ the intersection of the surfaces resulting from the compensation of u^1, u^2 in (34) are the following points:

$$x^1 = \left(\frac{-1+\sqrt{7}}{6}r, \frac{-7-2\sqrt{7}}{3\sqrt{28+2\sqrt{7}}}r\right), x^2 = \left(\frac{-1+\sqrt{7}}{6}r, \frac{-7+4\sqrt{7}}{3\sqrt{28+2\sqrt{7}}}r\right)$$

$$x^3 = \left(\frac{-1-\sqrt{7}}{6}r, \frac{-\sqrt{7}}{\sqrt{28+2\sqrt{7}}}r\right), x^4 = \left(\frac{-1-\sqrt{7}}{6}r, \frac{\sqrt{7}}{\sqrt{28+2\sqrt{7}}}r\right)$$

The number of Integral points is equal to 6 and the minimum number of points is $N \geq M(n, k) = M\left(2, \left\lfloor \frac{d}{2} \right\rfloor\right) = M(2, 2) = 6$, and we note that all points are located inside the circle, where the nodes x^1, x^3, x^4 are located on the circular surface $x^2 + y^2 = \frac{14+2\sqrt{7}}{28+2\sqrt{7}}r^2$ where $\frac{1}{b_i} = \frac{r^2\pi}{14}$ to find the constant C_1 we choose the function $f_1 = \left(x + \frac{1+\sqrt{7}}{6}r\right)\left(y + \frac{7-4\sqrt{7}}{3\sqrt{28+2\sqrt{7}}}r\right)$ we obtain the constant C_1 by substituting the function f_1 in the cubic relationship (5), since the function f_1 is absent at all points except node x_1 , and we find the rest of the constants for the rest of the functions on the sequence, we obtain the constants of the cubic relationship as in the following table:

Table 3: constants of the cubic relation (5) for $K_2(u, x)$, $n = 2$.

Points x_i	C_i
$x^2, x^3, x^4, x^5, x^6, x^8$	$\frac{13 - \sqrt{7}}{56}\pi r^2$
x^2	$\frac{9 + 3\sqrt{7}}{56}\pi r^2$

Let's form the cubic relation for the generated kernel in (34) and for algebraic precision $d = 2K = 4$, and for $n = 3$, we substitute in (34) and find:

$$K_2(u, x) = \frac{35}{2r^4\mu(B_3^{(r)})} \left(\sum_{i=1}^3 x_iu_i + \frac{1 + \sqrt{8}}{7}r^2\right) \left(\sum_{i=1}^3 x_iu_i + \frac{1 - \sqrt{8}}{7}r^2\right)$$

Where $u^1 = (r, 0, 0)$, $u^2 = \left(\frac{\sqrt{8}-1}{7}r, \frac{\sqrt{40+2\sqrt{8}}}{7}r, 0\right)$, $u^3 = \left(\frac{\sqrt{8}-1}{7}r, \frac{9\sqrt{8}-16}{7\sqrt{40+2\sqrt{8}}}r, \sqrt{\frac{104+64\sqrt{8}}{7(40+2\sqrt{8})}}r\right)$ the intersection of the surfaces resulting from the compensation of u^1, u^2, u^3 in (34) are the following points:

$$\begin{aligned}
 x^1 &= \left(\frac{\sqrt{8}-1}{7}r, \frac{-8+9\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r, \frac{-50+22\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right) \\
 x^2 &= \left(\frac{\sqrt{8}-1}{7}r, \frac{-8+9\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r, \frac{-58+2\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right) \\
 x^3 &= \left(\frac{\sqrt{8}-1}{7}r, \frac{-8-5\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r, \frac{14(\sqrt{8}-1)}{\sqrt{2268+2422\sqrt{8}}}r \right) \\
 x^4 &= \left(\frac{\sqrt{8}-1}{7}r, \frac{-8-5\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r, \frac{-22-6\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right) \\
 x^5 &= \left(\frac{-1-\sqrt{8}}{7}r, \frac{\sqrt{2}}{\sqrt{10+\sqrt{2}}}r, \frac{14(\sqrt{8}-1)}{\sqrt{2268+2422\sqrt{8}}}r \right) \\
 x^6 &= \left(\frac{-1-\sqrt{8}}{7}r, \frac{\sqrt{2}}{\sqrt{10+\sqrt{2}}}r, \frac{-22-6\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right) \\
 x^7 &= \left(\frac{-1-\sqrt{8}}{7}r, \frac{-\sqrt{2}}{\sqrt{10+\sqrt{2}}}r, \frac{22+6\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right) \\
 x^8 &= \left(\frac{-1-\sqrt{8}}{7}r, \frac{-\sqrt{2}}{\sqrt{10+\sqrt{2}}}r, \frac{14(1-\sqrt{8})}{\sqrt{2268+2422\sqrt{8}}}r \right)
 \end{aligned}$$

The number of Integral points is 11 and the minimum number of points is $N = M(n, k) = M\left(3, \left[\frac{d}{2}\right]\right) = M(3,2) = 10$, so the number of Integral points is one more than the minimum and let's form the cubic relationship, where $\frac{1}{b_i} = [K_k(u^i, u^i)]^{-1} = \frac{r^2\pi}{15}$ and to find the constant C_1 we choose the function $f_1 = \left(x + \frac{\sqrt{8}+1}{7}\right)\left(y + \frac{8+5\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r\right)\left(z + \frac{58-2\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r\right)$ which is absent at all points except node x_1 , and we find the rest of the constants in the same way, so we get the constants of the cubic relationship as in the following table :

x_i points	C_i
$x^1 = \left(\frac{\sqrt{8}-1}{7}r, \frac{-8+9\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r, \frac{-50+22\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right)$	$\frac{175\sqrt{2} + 231}{240 + 1200\sqrt{2}}\pi r^3$
$x^2 = \left(\frac{\sqrt{8}-1}{7}r, \frac{-8+9\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r, \frac{-58+2\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right)$ $x^3 = \left(\frac{\sqrt{8}-1}{7}r, \frac{-8-5\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r, \frac{14(\sqrt{8}-1)}{\sqrt{2268+2422\sqrt{8}}}r \right)$ $x^5 = \left(\frac{-1-\sqrt{8}}{7}r, \frac{\sqrt{2}}{\sqrt{10+\sqrt{2}}}r, \frac{14(\sqrt{8}-1)}{\sqrt{2268+2422\sqrt{8}}}r \right)$ $x^8 = \left(\frac{-1-\sqrt{8}}{7}r, \frac{-\sqrt{2}}{\sqrt{10+\sqrt{2}}}r, \frac{14(1-\sqrt{8})}{\sqrt{2268+2422\sqrt{8}}}r \right)$	$\frac{81\sqrt{2} + 173}{240 + 1200\sqrt{2}}\pi r^3$
$x^4 = \left(\frac{\sqrt{8}-1}{7}r, \frac{-8-5\sqrt{2}}{7\sqrt{10+\sqrt{2}}}r, \frac{-22-6\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right)$ $x^6 = \left(\frac{-1-\sqrt{8}}{7}r, \frac{\sqrt{2}}{\sqrt{10+\sqrt{2}}}r, \frac{-22-6\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right)$ $x^7 = \left(\frac{-1-\sqrt{8}}{7}r, \frac{-\sqrt{2}}{\sqrt{10+\sqrt{2}}}r, \frac{22+6\sqrt{8}}{\sqrt{2268+2422\sqrt{8}}}r \right)$	$\frac{81\sqrt{2} + 173}{240 + 1200\sqrt{2}}\pi r^3$

Table (4): constants of the cubic relation (5) for $\tilde{K}_2(u, x)$ $n = 3$.

3-4. The formation of the cubic relation of the generating nuclei $K_3(u, x)$ and $\tilde{K}_3(u, x)$:

First: for $n=2$:

Let's form the cubic relation for the generated kernel in (35), and for the algebraic precision $d = 2k + 1 = 7$ for $n=2$, we substitute in (35), we find:

$\tilde{K}_3(u, x) = \frac{32}{r^6 \mu(B_n^{(r)})} (x_1 u_1 + x_2 u_2) \left(x_1 u_1 + x_2 u_2 + \frac{\sqrt{3} r^2}{\sqrt{8}} \right) \left(x_1 u_1 + x_2 u_2 - \frac{\sqrt{3} r^2}{\sqrt{8}} \right)$ Where $u^1 = \left(\frac{r}{2}, \frac{\sqrt{3}}{2} r \right)$ $u^2 = \left(-\frac{\sqrt{3}}{2} r, \frac{1}{2} r \right)$ the surfaces resulting from the offset of u^1, u^2 intersect in (35) by $s = k^n = 3^2 = 9$ nine points, are:

$$\begin{aligned} x^1 &= (0,0), x^2 = \left(\frac{-3}{4\sqrt{2}} r, \frac{\sqrt{3}}{4\sqrt{2}} r \right), x^3 = \left(\frac{3}{4\sqrt{2}} r, \frac{-\sqrt{3}}{4\sqrt{2}} r \right) \\ x^4 &= \left(\frac{-\sqrt{3}}{4\sqrt{2}} r, \frac{-3}{4\sqrt{2}} r \right), x^5 = \left(\frac{-\sqrt{3}-3}{4\sqrt{2}} r, \frac{\sqrt{3}-3}{4\sqrt{2}} r \right), x^6 = \left(\frac{-\sqrt{3}+3}{4\sqrt{2}} r, \frac{-\sqrt{3}-3}{4\sqrt{2}} r \right) \\ x^7 &= \left(\frac{\sqrt{3}}{4\sqrt{2}} r, \frac{3}{4\sqrt{2}} r \right), x^8 = \left(\frac{\sqrt{3}-3}{4\sqrt{2}} r, \frac{\sqrt{3}+3}{4\sqrt{2}} r \right), x^9 = \left(\frac{\sqrt{3}+3}{4\sqrt{2}} r, \frac{-\sqrt{3}+3}{4\sqrt{2}} r \right) \end{aligned}$$

The number of Integral points is equal to 13 and the minimum number of points, $N \geq 2v + 1 = 2[M(1,3) + M(1,1)] + 1 = 13$ and we note that all points are located inside the circle, where the points x^2, x^3, x^4, x^7 are located on the circular $x^2 + y^2 = \frac{9}{16} r^2$ have the same constant, and the points x^5, x^6, x^8, x^9 are located on the circular surface $x^2 + y^2 = \frac{3}{4} r^2$ has the same constant, and $\frac{1}{2b_i} = [2K_k(u^i, u^i)]^{-1} = \frac{r^2 \pi}{40}$ since the cubic relation (6) is true for functions up to the seventh degree, to find C_2 we choose the function $x^2 \left(y^2 - \frac{12-6\sqrt{3}}{32} \right) \left(y^2 - \frac{12+6\sqrt{3}}{32} \right)$ to find C_5 , we choose the function $x^2 \left(y^2 - \frac{3}{32} \right) \left(y^2 - \frac{9}{32} \right)$, so we find:

Table 5: constants of the cubic relation (6) for $\tilde{K}_3(u, x)$, $n = 2$.

x_i points	Constants
x^2, x^3, x^4, x^7	$\frac{16}{135} \pi r^2$
x^5, x^6, x^8, x^9	$\frac{2}{27} \pi r^2$
x^2	$\frac{7}{54} \pi r^2$

Second: for $n=3$:

We substitute in (35), we find:

$$\tilde{K}_3(u, x) = \frac{105}{2r^6 \mu(B_2^{(r)})} \left(\sum_{i=1}^3 x_i u_i \right) \left(\sum_{i=1}^3 x_i u_i + \frac{r^2}{\sqrt{3}} \right) \left(\sum_{i=1}^3 x_i u_i - \frac{r^2}{\sqrt{3}} \right)$$

Where $u^1 = (r, 0, 0)$, $u^2 = (0, r, 0)$, $u^3 = (0, 0, r)$ the surfaces resulting from the offset of u^1, u^2, u^3 intersect at (35) $s = k^n = 3^3 = 27$ points, are:

$$\begin{aligned} x^1 &= (0,0,0), x^{2,3} = \left(\mp \frac{1}{\sqrt{3}} r, 0, 0 \right), x^{4,5} = \left(0, \mp \frac{1}{\sqrt{3}} r, 0 \right), \\ x^{6,7} &= \left(0, 0, \mp \frac{1}{\sqrt{3}} r \right), x^{8,9} = \left(\mp \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r, 0 \right), \\ x^{10,11} &= \left(\mp \frac{1}{\sqrt{3}} r, 0, \mp \frac{1}{\sqrt{3}} r \right), x^{12,13} = \left(0, \mp \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r \right) \\ x^{14,15} &= \left(\pm \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r, 0 \right), x^{16,17} = \left(\pm \frac{1}{\sqrt{3}} r, 0, \mp \frac{1}{\sqrt{3}} r \right), \\ x^{18,19} &= \left(0, \pm \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r \right), x^{20,21} = \left(\mp \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r \right), x^{22,23} = \left(\pm \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r \right), x^{24,25} = \\ &\left(\mp \frac{1}{\sqrt{3}} r, \pm \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r \right), x^{26,27} = \left(\mp \frac{1}{\sqrt{3}} r, \mp \frac{1}{\sqrt{3}} r, \pm \frac{1}{\sqrt{3}} r \right) \end{aligned}$$

The number of Integral points is equal to 33, and the minimum number of points is:

$$N \geq 2v + 1 = 2[M(2,3) + M(2,1)] + 1 = 27$$

The points $x^{2,3,4,5,6,7}$ are located on the spherical surface $x^2 + y^2 + z^2 = \frac{r^2}{3}$ have the same constant, the points $x^{8,9,10,11, \dots, 19}$ on the spherical surface $x^2 + y^2 + z^2 = \frac{2r^2}{3}$ has the same constant, as for the points

$x^{20,21,\dots,27}$ located on the spherical surface $x^2 + y^2 + z^2 = r^2$ it has the same constants $u^{1,2,3}$ we substitute in the cubic relation (6) where $\frac{1}{2b_i} = [2K_k(u^i, u^i)]^{-1} = \frac{2r^3\pi}{105}$ since the cubic relation (6) is true for functions up to the seventh degree, to find C_1 we choose the function

$(x^2 - \frac{1}{3})(y^2 - \frac{1}{3})(z^2 - \frac{1}{3})$ to find C_2 we choose the function $x^2(y^2 - \frac{1}{3})(z^2 - \frac{1}{3})$, to find C_8 we choose the function $x^2y^2(z^2 - \frac{1}{3})$, and to find C_{20} we choose the function $x^2y^2z^2$, we find:

Table 7: constants of the cubic relation (6) for $K_3(u, x)$ and $n=3$.

x_i points	constants
x^1	$\frac{8}{105}\pi r^3$
$x^{2,3,4,5,6,7}$	$\frac{2}{35}\pi r^3$
$x^{8,9,10,11,\dots,19}$	$\frac{2}{35}\pi r^3$
$x^{20,21,\dots,27}$	$\frac{1}{70}\pi r^3$

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