On Fully Right Prime Ternary Rings and Their Algebraic Properties

Zahraa Hasan

University Of Applied Science, Faculty of Literature and Science, Manama, Bahrain

Email: Zahraamathscience1243@gmail.com

Abstract

The objective of this paper to study some algebraic properties of ternary rings in which all of their proper right ideals are prime. Also, we characterize the relation between this class of ternary rings and the rings that all of their right ideals are weakly prime.

Keywords: Prime right ideal; Fully right prime; Ternary ring.

1. Introduction and basic definitions

The algebraic properties of rings have attracted many authors to contribute to this field [13-20], where ideals, elements, unit, and homomorphism are the basis of these studies.
For example, we mention the rings that all of their ideals are prime or semi-prime [1-3], and the rings that all of their right ideals are weakly prime or generalized to the right [4-8].
Ternary rings were defined by Lister, with many advances about them, see [9-11].
In this work, we try to find necessary and sufficient conditions for right ideals in ternary ring to be prime.
First we recall some basic definitions:

Definition:

Let $T \neq \emptyset$ be a set with two binary and ternary operations:
$(+): T \times T \to T$ and $(.): T \times T \times T \to T$.
$(T, +, .)$ Is called ternary ring if:
1). $(T, +)$ is abelian group.

2). $\begin{cases} (abc)de = a(bcd)e = ab(cde) \\ (a+b)cd = acd + bcd \\ a(b+c)d = abd + acd \\ ab(c+d) = abc + abd \end{cases}$

For all $a, b, c, d, e \in T$.

Definition.

Let T is a ternary ring, then the zero of T (0) is defined with the following properties.
$0 + a = a\ , 0ab = a0b = ab0 = 0;\ \forall a, b \in T.$

Definition.

Let T be a ternary ring, if there exists $1 \in T$ such that $1.1.a = 1.a.1 = a.1.1 = a$ for all $a \in T$, then 1 is called the unity of T, and T is called ternary ring of unity.

Definition.

Let T be a ternary ring , and $(I, +)$ is a subgroup of $(T, +)$, we say:
1). I is left ideal of T if:
$x.y.i \in I\ ;\ \forall i \in I\ , x, y \in T.$
2). I is right ideal of T if:
$i.x.y \in I\ ;\ \forall i \in I\ , x, y \in T.$
3). I is called side ideal of T if:

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$x.i.y \in I$; $\forall i \in I, x, y \in T$.
4). I is called two-side ideal of T if I is right and left of T.
5). I is called ideal of T if it is right, left, and side ideal.
Definition.
Let P be a proper right ideal of T, then P is called semi-prime right ideal if:
$I \subseteq P$ implies $P \subseteq I$.
T is called ternary right semi-prime if $\{0\}$ is semi-prime right ideal.
T is called ternary fully right semi-prime if all proper right ideals are semi prime.
Definition.
Let P be a proper right ideal of T, P is called right prime ideal if it has the following property:
If I, J, K are right ideals of T with $I.J.K \subseteq P$, then $I \subseteq P, J \subseteq P, K \subseteq P$.
T is called ternary right prime ring if $\{0\}$ is right prime ideal of T.
T is called fully right prime ring if all of its proper right ideals are prime.
Main results.
Theorem.
Let P be a proper right ideal of T, then:
P is semi-prime nilpotent right ideal if and only if P is prime eight ideal.
Proof.
The first implementation is easy.
For the converse, we assume that I, J, K are right ideals of T with $I.J.K \subseteq P$, then there exists an odd natural number n such that $I^n \subseteq P$ or $J^n \subseteq P$ or $K^n \subseteq P$.
Since P is right semi-prime ideal, we get that:
$I \subseteq P$ or $J \subseteq P$ or $K \subseteq P$, thus P is prime.
Remark.
1). If T is fully prime right, then it weakly prime from the right.
2). T is nilpotent and fully semi prime from the right equivalents to that T is fully prime from the right.
Theorem.
Let T be a prime right ternary ring, and P is right ideal of T, then P is weakly prime if and only if P is prime.
Proof.
The first implementation holds directly from the previous remark.
For the converse, we suppose that I, J, K are right ideals of T with $I.J.K \subseteq P$.
If $I.J.K \neq \{0\}$, then $I \subseteq P$ or $J \subseteq P$ or $K \subseteq P$ a a result of the weakly primary of P, thus P is prime.
If $I.J.K = \{0\}$, since T is prime from the right, we get that $I = \{0\}$ or $J = \{0\}$ or $K = \{0\}$, thus $I \subseteq P$ or $J \subseteq P$ or $K \subseteq P$, thus P is prime.
Theorem.
Let T be an idempotent fully right ternary ring, and its set of right ideals is totally ordered, then T T is fully prime from the right.
Proof.
Let P be a proper right ideal of T, and let that I, J, K be right ideals of T with $I.J.K \subseteq P$, from the totally ordering property, we get:
$I \subseteq J$ or $J \subseteq I$, hence $I \subseteq K$ or $K \subseteq I$.
For $I \subseteq J$ with $I \subseteq K$, we can write:
$I = I.I.I \subseteq I.J.K$, thus $I \subseteq I.J.K \subseteq P$ and $I \subseteq P$.
For $J \subseteq I$ with $I \subseteq K$, then:
$J = J.J.J \subseteq I.J.I \subseteq I.J.K \subseteq P$.
For $I \subseteq J$ with $K \subseteq I$, then $K \subseteq I \subseteq J$, thus $K = K.K.K \subseteq I.J.K \subseteq P$.
For $K \subseteq J \subseteq I$, we have:
$K = K.K.K \subseteq I.J.K \subseteq P$
This means that P is prime right ideal of T, and the proof is complete.
Theorem.
If T primitive ternary ring fully generalized to the ring, then its set of semi-prime ideal I totally ordered.
Proof.
Let P_1, P_2 be two semi-prime ideals of T, then $P_1 \cap P_2$is semi-prime ideal of T.
We have $P_1.P_2.P_2 \subseteq P_1 \cap P_2$.
And since $P_1 \cap P_2$ is generalized to the right, we get:
$P_1 \subseteq P_2$ or $P_2 \subseteq P_1$, hence the set of semi-prime ideals is totally ordered.
Theorem.
Let P be a proper right ideals of T such that $I.J.K \subseteq P$.
We put $H = I \cap J \cap K$, then H is right ideal of T with $H^3 \subseteq P$, that is because:

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$H^3 = h.H.H = (I \cap J \cap K)(I \cap J \cap K)(I \cap J \cap K) \subseteq I.J.K \subseteq P$.
According to the assumption, P is semi-prime right ideal, then $H \subseteq P$ and $I \subseteq P$ or $J \subseteq P$ or $K \subseteq P$ from the irreducibility of P, hence P is prime right ideal.

Theorem.

Let T be a sub-commutative ternary ring, then:
1). T is fully prime right if and only if T is fully idempotent from the right and its set of right ideals is totally ordered.
2). T I fully prime right if and only if T is fully right non-irreducible and fully semi-prime.

Proof.

1). The first implementations is clear.
Assume that T is fully prime right ternary ring, then T I fully semi-prime from the right, which implies that T is fully idempotent from the right, and primitive fully generalized to the right, hence if P_1, P_2 are two semi-prime right ideals of P_1, P_2, this means that:
$P_1.P_2.P_3 \subseteq TTP_2 \subseteq P_2TT \subseteq P_2$, and $P_1.P_2.P_3 \subseteq P_1TT \subseteq P_1$, then $P_1.P_2.P_3 \subseteq P_1 \cap P_2$, hence $P_1 \subseteq P_1 \cap P_2 \subseteq P_2$ or there exists an odd natural number n such that ${P_2}^n \subseteq P_1 \cap P_2 \subseteq P_1 \Longrightarrow P_2 \subseteq P_1$.
The previous argument gives the following result.
Semi-prime ideals of T is totally ordered.
2). The first implementation is clear.
For the converse, assume that P is right proper ideal of T.
According to the assumption, we get that P is prime right ideal.
Now, let I, J, K be right ideals of T such that $I \cap J \cap K \subseteq P$.
Since P is fully prime right of T, then its set of right ideals is totally ordered, which implies that:
$I \subseteq J$ or $J \subseteq I$, $I \subseteq K$ or $K \subseteq I$,.
If $I \subseteq K$ and $I \subseteq J$, then $I \subseteq I \cap J \cap K \subseteq P$.
If $I \subseteq K$ and $J \subseteq I$, then $J \subseteq I \subseteq K$, thus $J \subseteq I \cap J \cap K \subseteq P$.
If $I \subseteq J$ and $K \subseteq I$, then $K \subseteq I \subseteq J$, thus $K \subseteq I \cap J \cap K \subseteq P$.
Now, for $J \subseteq I$ and $K \subseteq I$, we have two cases:
If $J \subseteq K$, then $J \subseteq I \cap J \cap K \subseteq P$.
If $K \subseteq J$, then $K \subseteq I \cap J \cap K \subseteq P$.
Hence $I \subseteq P$ or $J \subseteq P$ or $K \subseteq P$, and P is non-irreducible right ideal, which means that T is fully non-irreducible from the right.

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