



Introduction to Symbolic 2-Plithogenic Probability Theory

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Abstract

In this paper we present for the first time the concept of symbolic plithogenic random variables and study its properties including expected value and variance. We build the plithogenic formal form of two important distributions that are exponential and uniform distributions. We find its probability density function and cumulative distribution function in its plithogenic form. We also derived its expected values and variance and the formulas of its random numbers generating. We finally present the fundamental form of plithogenic probability density and cumulative distribution functions. All the theorems were proved depending on algebraic approach using isomorphisms. This paper can be considered the base of symbolic plithogenic probability theory.

Keywords: Plithogenic; Probability Density Function; Cumulative Distribution Function; Random Numbers Generation; Exponential Distribution; Uniform Distribution.

1. Introduction

As a generalization of fuzzy and intuitionistic fuzzy sets Smarandache presented neutrosophic sets which is more powerful definition of uncertain sets and which has been applied in many fields of science including machine learning, artificial intelligence, statistics, engineering, etc.[1]–[15].

Presenting new sets and studying its algebraic structures is very interesting subject in mathematics. As a generalization of neutrosophic sets Smarandache presented plithogenic sets which is defined depending on split indeterminacy and on absorbance law.[16]–[31].

Symbolic probability theory was first presented in [32] by M.B. Zeina and A. Hatip where symbolic (literal) neutrosophic random variable was defined in the form $X_N = X + I; I^2 = I$ and this definition was applied in many other fields of probability theory (for example see [33], [34]). This definition was generalized by M.B. Zeina and M. Abobala in [35] to the form $X_N = X + IY; I^2 = I$ which is the most general form of a neutrosophic random variable and applied in many other branches of neutrosophic probability theory [36]–[39].

In this work we are going to present for the first time the main definitions related to plithogenic probability theory defined on 2-S plithogenic sets which may be applied in many fields of probability theory and its

related branches of science including queueing theory, reliability theory, stochastic processes, survival analysis, etc. [1]–[3], [5], [10], [13], [36], [40]–[55].

2. Preliminaries

Definition 2.1

Set of symbolic 2-plithogenic real numbers is defined as follows:

$$2 - SP_R = \{a + bP_1 + cP_2; a, b, c \in R\}$$

Where:

$$P_1^2 = P_1, P_2^2 = P_2, P_1P_2 = P_2P_1 = P_{\max(1,2)} = P_2$$

Definition 2.2

Let $a_0 + a_1P_1 + a_2P_2, b_0 + b_1P_1 + b_2P_2 \in 2 - SP_R$, then addition and multiplication are defined as follows:

$$(a_0 + a_1P_1 + a_2P_2) + (b_0 + b_1P_1 + b_2P_2) = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2$$

$$(a_0 + a_1P_1 + a_2P_2) \cdot (b_0 + b_1P_1 + b_2P_2) = a_0b_0 + P_1(a_0b_1 + a_1b_0 + a_1b_1) + P_2(a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)$$

Definition 2.3

AH-Isometry on symbolic $2 - SP_R$ sets and its inverse can be defined as follows:

$$T: 2 - SP_R \rightarrow R \times R \times R;$$

$$T(a_0 + a_1P_1 + a_2P_2) = (a_0, a_0 + a_1, a_0 + a_1 + a_2)$$

$$T^{-1}: R \times R \times R \rightarrow 2 - SP_R;$$

$$T^{-1}(a_0, a_1, a_2) = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2$$

Definition 2.4

We say that $a_p \geq_p b_p$ if $a_0 \geq b_0$ and $a_0 + a_1 \geq b_0 + b_1$ and $a_0 + a_1 + a_2 \geq b_0 + b_1 + b_2$.

Definition 2.5

A plithogenic number a_p is said to be nonnegative if $a_0 \geq 0$ and $a_0 + a_1 \geq 0$ and $a_0 + a_1 + a_2 \geq 0$.

3. Plithogenic Random Variables

Definition 3.1

We define 2-SP random variable as follows:

$$X_p: \Omega_p \rightarrow 2 - SP_R; \Omega_p = \Omega_0 \times \Omega_1(P_1) \times \Omega_2(P_2)$$

$$X_p = X_0 + X_1P_1 + X_2P_2; P_1^2 = P_1, P_2^2 = P_2, P_1P_2 = P_2P_1 = P_2$$

Where X_0, X_1, X_2 are classical random variables defined on $\Omega_0, \Omega_1, \Omega_2$ respectively and each random variable takes its in R .

Remark

Taking AH-Isometry transfers X_p to three classical random variables $(X_0, X_0 + X_1, X_0 + X_1 + X_2)$.

Theorem 3.1

Let X_p be a plithogenic random variable then expectation and variance of it are:

1. $E(X_p) = E(X_0) + E(X_1)P_1 + E(X_2)P_2$
2. $V(X_p) = V(X_0) + [V(X_0 + X_1) - V(X_0)]P_1 + [V(X_0 + X_1 + X_2) - V(X_0 + X_1)]P_2$

Proof

We will prove the theorem assuming that X_p is discrete random variable and same proof can be done when X_p is continuous.

$$1. E(X_p) = E(X_0 + X_1P_1 + X_2P_2) = \sum(x_0 + x_1P_1 + x_2P_2)f(x_0 + x_1P_1 + x_2P_2)$$

Taking AH-Isometry:

$$\begin{aligned} T[E(X_p)] &= T \left[\sum (x_0 + x_1P_1 + x_2P_2)f(x_0 + x_1P_1 + x_2P_2) \right] \\ &= \sum T[(x_0 + x_1P_1 + x_2P_2)f(x_0 + x_1P_1 + x_2P_2)] \\ &= \left(\sum x_0f(x_0), \sum (x_0 + x_1)f(x_0 + x_1), \sum (x_0 + x_1 + x_2)f(x_0 + x_1 + x_2) \right) \\ &= (E(X_0), E(X_0 + X_1), E(X_0 + X_1 + X_2)) \end{aligned}$$

Taking the inverse isometry:

$$\begin{aligned} E(X_p) &= T^{-1}(E(X_0), E(X_0 + X_1), E(X_0 + X_1 + X_2)) \\ &= E(X_0) + [E(X_0 + X_1) - E(X_0)]P_1 + [E(X_0 + X_1 + X_2) - E(X_0 + X_1)]P_2 \\ &= E(X_0) + E(X_1)P_1 + E(X_2)P_2 \end{aligned}$$

$$2. E(X_p^2) = E(X_0 + X_1P_1 + X_2P_2)^2 = \sum(x_0 + x_1P_1 + x_2P_2)^2f(x_0 + x_1P_1 + x_2P_2)$$

Taking AH-Isometry:

$$\begin{aligned} T[E(X_p^2)] &= T \left[\sum (x_0 + x_1P_1 + x_2P_2)^2f(x_0 + x_1P_1 + x_2P_2) \right] \\ &= \sum T[(x_0 + x_1P_1 + x_2P_2)^2f(x_0 + x_1P_1 + x_2P_2)] \\ &= \left(\sum x_0^2f(x_0), \sum (x_0 + x_1)^2f(x_0 + x_1), \sum (x_0 + x_1 + x_2)^2f(x_0 + x_1 + x_2) \right) \\ &= (E(X_0^2), E(X_0 + X_1)^2, E(X_0 + X_1 + X_2)^2) \end{aligned}$$

Taking the inverse isometry:

$$\begin{aligned} E(X_p^2) &= T^{-1}(E(X_0^2), E(X_0 + X_1)^2, E(X_0 + X_1 + X_2)^2) \\ &= E(X_0^2) + [E(X_0 + X_1)^2 - E(X_0^2)]P_1 + [E(X_0 + X_1 + X_2)^2 - E(X_0 + X_1)^2]P_2 \end{aligned}$$

We also have:

$$\begin{aligned} [E(X_p)]^2 &= [E(X_0) + E(X_1)P_1 + E(X_2)P_2]^2 = T^{-1}T[E(X_0) + E(X_1)P_1 + E(X_2)P_2]^2 \\ &= T^{-1}[[E(X_0)]^2, [E(X_0 + X_1)]^2, [E(X_0 + X_1 + X_2)]^2] \\ &= [E(X_0)]^2 + [[E(X_0 + X_1)]^2 - [E(X_0)]^2]P_1 + [[E(X_0 + X_1 + X_2)]^2 - [E(X_0 + X_1)]^2]P_2 \end{aligned}$$

So, we can write:

$$\begin{aligned} V(X_p) &= E(X_p^2) - [E(X_p)]^2 \\ &= E(X_0^2) + [E(X_0 + X_1)^2 - E(X_0^2)]P_1 + [E(X_0 + X_1 + X_2)^2 - E(X_0 + X_1)^2]P_2 - [E(X_0)]^2 \\ &\quad - [[E(X_0 + X_1)]^2 - [E(X_0)]^2]P_1 - [[E(X_0 + X_1 + X_2)]^2 - [E(X_0 + X_1)]^2]P_2 \\ &= V(X_0) + [V(X_0 + X_1) - V(X_0)]P_1 + [V(X_0 + X_1 + X_2) - V(X_0 + X_1)]P_2 \end{aligned}$$

Definition 3.2

A function $f(x_p)$ defined on $R(P_1, P_2)$ is called a plithogenic probability density function if it satisfies the following conditions:

1. $f(x_p) \geq 0$.
2. $\int_{-\infty}^{+\infty} f(x_p)dx_p = 1$.

Example 3.1

Let $f(x_p) = 2x_p; 0 \leq_p x_p \leq_p 1$, then $f(x_p)$ is a plithogenic probability density function because:

$$T[f(x_p)] = T[2x_p] = T[2x_0 + 2x_1P_1 + 2x_2P_2] = (2x_0, 2x_0 + 2x_1, 2x_0 + 2x_1 + 2x_2)$$

Which are all nonnegative functions, we also have:

$$T \left[\int_0^1 f(x_p) dx_p \right] = \left(\int_0^1 2x_0 dx_0, \int_0^1 (2x_0 + 2x_1) d(x_0 + x_1), \int_0^1 (2x_0 + 2x_1 + 2x_2) d(x_0 + x_1 + x_2) \right) = (1,1,1)$$

Which means that:

$$\int_0^1 f(x_p) dx_p = T^{-1}(1,1,1) = 1$$

Theorem 3.2

Let $X_p = X_0 + X_1P_1 + X_2P_2$ be a plithogenic random variable, the moments generating function of it takes the form:

$$M_{X_p}(t) = M_{X_0}(t) + [M_{X_0+X_1}(t) - M_{X_0}(t)]P_1 + [M_{X_0+X_1+X_2}(t) - M_{X_0+X_1}(t)]P_2$$

Proof

$$\begin{aligned} M_{X_p}(t) &= E(e^{tX_p}) = \int_{-\infty}^{+\infty} e^{tx_p} f(x_p) dx_p = T^{-1}T \left[\int_{-\infty}^{+\infty} e^{tx_p} f(x_p) dx_p \right] \\ &= T^{-1} \left(\int_{-\infty}^{+\infty} e^{tx_0} f(x_0) dx_0, \int_{-\infty}^{+\infty} e^{t(x_0+x_1)} f(x_0 + x_1) d(x_0 + x_1), \int_{-\infty}^{+\infty} e^{t(x_0+x_1+x_2)} f(x_0 + x_1 + x_2) d(x_0 + x_1 + x_2) \right) \\ &= T^{-1} (M_{X_0}(t), M_{X_0+X_1}(t), M_{X_0+X_1+X_2}(t)) \\ &= M_{X_0}(t) + [M_{X_0+X_1}(t) - M_{X_0}(t)]P_1 + [M_{X_0+X_1+X_2}(t) - M_{X_0+X_1}(t)]P_2 \end{aligned}$$

Theorem 3.3

Let $X_p = X_0 + X_1P_1 + X_2P_2$ be a plithogenic random variable with a moments generating function $M_{X_p}(t)$ then:

$$\frac{d^k}{dt^k} M_{X_p}(t)|_{t=0} = E(X_p^k)$$

Proof

Since:

$$M_{X_p}(t) = M_{X_0}(t) + [M_{X_0+X_1}(t) - M_{X_0}(t)]P_1 + [M_{X_0+X_1+X_2}(t) - M_{X_0+X_1}(t)]P_2$$

Taking k^{th} derivative of both sides and substituting $t = 0$ yields to:

$$\begin{aligned} \frac{d^k}{dt^k} M_{X_p}(t)|_{t=0} &= \frac{d^k}{dt^k} (M_{X_0}(t) + [M_{X_0+X_1}(t) - M_{X_0}(t)]P_1 + [M_{X_0+X_1+X_2}(t) - M_{X_0+X_1}(t)]P_2)|_{t=0} \\ &= \frac{d^k}{dt^k} M_{X_0}(t)|_{t=0} + \left[\frac{d^k}{dt^k} M_{X_0+X_1}(t)|_{t=0} - \frac{d^k}{dt^k} M_{X_0}(t)|_{t=0} \right] P_1 \\ &\quad + \left[\frac{d^k}{dt^k} M_{X_0+X_1+X_2}(t)|_{t=0} - \frac{d^k}{dt^k} M_{X_0+X_1}(t)|_{t=0} \right] P_2 \\ &= E(X_0^k) + [E(X_0 + X_1)^k - E(X_0^k)]P_1 + [E(X_0 + X_1 + X_2)^k - E(X_0 + X_1)^k]P_2 = E(X_p^k) \end{aligned}$$

4. Some Continuous Plithogenic Probability Distributions

Definition 4.1

A plithogenic random variable X_p is said to be exponentially distributed with the plithogenic parameter λ_p if its probability density function takes the form:

$$f(x_p) = \lambda_p e^{-\lambda_p x_p}; x_p, \lambda_p >_P 0$$

Theorem 4.1

The formal plithogenic form of probability density function of exponential distribution is:

$$f(x_p) = \lambda_0 e^{-\lambda_0 x_0} + [(\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} - \lambda_0 e^{-\lambda_0 x_0}] P_1 + [(\lambda_0 + \lambda_1 + \lambda_2) e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)} - (\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)}] P_2$$

Proof

$$\begin{aligned} T[f(x_p)] &= T[\lambda_p e^{-\lambda_p x_p}] = T[(\lambda_0 + \lambda_1 P_1 + \lambda_2 P_2) e^{-(\lambda_0 + \lambda_1 P_1 + \lambda_2 P_2)(x_0 + x_1 P_1 + x_2 P_2)}] \\ &= T[\lambda_0 + \lambda_1 P_1 + \lambda_2 P_2] T[e^{-(\lambda_0 + \lambda_1 P_1 + \lambda_2 P_2)(x_0 + x_1 P_1 + x_2 P_2)}] \\ &= T[\lambda_0 + \lambda_1 P_1 + \lambda_2 P_2] [e^{-T(\lambda_0 + \lambda_1 P_1 + \lambda_2 P_2) T(x_0 + x_1 P_1 + x_2 P_2)}] \\ &= (\lambda_0, \lambda_0 + \lambda_1, \lambda_0 + \lambda_1 + \lambda_2) e^{-(\lambda_0, \lambda_0 + \lambda_1, \lambda_0 + \lambda_1 + \lambda_2)(x_0, x_0 + x_1, x_0 + x_1 + x_2)} \\ &= (\lambda_0 e^{-\lambda_0 x_0}, (\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)}, (\lambda_0 + \lambda_1 + \lambda_2) e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)}) \end{aligned}$$

So:

$$\begin{aligned} f(x_p) &= T^{-1}(\lambda_0 e^{-\lambda_0 x_0}, (\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)}, (\lambda_0 + \lambda_1 + \lambda_2) e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)}) \\ &= \lambda_0 e^{-\lambda_0 x_0} + [(\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} - \lambda_0 e^{-\lambda_0 x_0}] P_1 \\ &\quad + [(\lambda_0 + \lambda_1 + \lambda_2) e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)} - (\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)}] P_2 \end{aligned}$$

Example 4.1

$$f(x_p) = e^{-x_0} + [2e^{-2(x_0 + x_1)} - e^{-x_0}] P_1 + [4e^{-4(x_0 + x_1 + x_2)} - 2e^{-2(x_0 + x_1)}] P_2$$

Is a plithogenic probability density function of an exponentially distributed random variable with parameter $\lambda_p = 1 + P_1 + 2P_2$.

Theorem 4.2

The plithogenic cumulative distribution function of plithogenic exponential distribution is:

$$F(x_p) = 1 - e^{-\lambda_0 x_0} + [e^{-\lambda_0 x_0} - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)}] P_1 + [e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} - e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)}] P_2$$

Proof

$$F(x_p) = \int_0^{x_p} f(t_p) dt_p$$

$$\begin{aligned} T(F(x_p)) &= T\left(\int_0^{x_p} f(t_p) dt_p\right) \\ &= T\left(\int_0^{(x_0 + x_1 P_1 + x_2 P_2)} ((\lambda_0 + \lambda_1 P_1 + \lambda_2 P_2) e^{-(\lambda_0 + \lambda_1 P_1 + \lambda_2 P_2)(t_0 + t_1 P_1 + t_2 P_2)}) d(t_0 + t_1 P_1 + t_2 P_2)\right) \end{aligned}$$

$$\begin{aligned} &= \left(\int_0^{x_0} \lambda_0 e^{-\lambda_0 t_0} dt_0, \int_0^{x_0 + x_1} (\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1)(t_0 + t_1)} d(t_0 + t_1), \int_0^{x_0 + x_1 + x_2} (\lambda_0 + \lambda_1 + \lambda_2) e^{-(\lambda_0 + \lambda_1 + \lambda_2)(t_0 + t_1 + t_2)} d(t_0 + t_1 + t_2)\right) \end{aligned}$$

$$= (1 - e^{-\lambda_0 x_0}, 1 - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)}, 1 - e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)})$$

Which yields by taking T^{-1} to:

$$F(x_p) = 1 - e^{-\lambda_0 x_0} + [1 - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} - (1 - e^{-\lambda_0 x_0})]P_1 + [1 - e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)} - (1 - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)})]P_2$$

Or:

$$F(x_p) = 1 - e^{-\lambda_0 x_0} + [e^{-\lambda_0 x_0} - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)}]P_1 + [e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} - e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)}]P_2$$

Theorem 4.3

Let X_p be a plithogenic exponential random variable with plithogenic parameter $\lambda_p > 0$ then:

1. $E(X_p) = \frac{1}{\lambda_0} + \left(\frac{1}{\lambda_0 + \lambda_1} - \frac{1}{\lambda_0}\right)P_1 + \left(\frac{1}{\lambda_0 + \lambda_1 + \lambda_2} - \frac{1}{\lambda_0 + \lambda_1}\right)P_2$
2. $Var(X_p) = \frac{1}{\lambda_0^2} + \left[\frac{1}{(\lambda_0 + \lambda_1)^2} - \frac{1}{\lambda_0^2}\right]P_1 + \left[\frac{1}{(\lambda_0 + \lambda_1 + \lambda_2)^2} - \frac{1}{(\lambda_0 + \lambda_1)^2}\right]P_2$

Proof

1. $E(X_p) = T^{-1}T\left(\int_0^\infty x_p \lambda_p e^{-\lambda_p x_p} dx_p\right)$
 $= T^{-1}\left(\int_0^\infty x_0 \lambda_0 e^{-\lambda_0 x_0} dx_0, \int_0^\infty (x_0 + x_1)(\lambda_0 + \lambda_1)e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} d(x_0 + x_1), \int_0^\infty (x_0 + x_1 + x_2)(\lambda_0 + \lambda_1 + \lambda_2)e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)} d(x_0 + x_1 + x_2)\right) = T^{-1}\left(\frac{1}{\lambda_0}, \frac{1}{\lambda_0 + \lambda_1}, \frac{1}{\lambda_0 + \lambda_1 + \lambda_2}\right)$
 $= \frac{1}{\lambda_0} + \left(\frac{1}{\lambda_0 + \lambda_1} - \frac{1}{\lambda_0}\right)P_1 + \left(\frac{1}{\lambda_0 + \lambda_1 + \lambda_2} - \frac{1}{\lambda_0 + \lambda_1}\right)P_2$
2. $Var(X_p) = T^{-1}T\left(\int_0^\infty [x_p - E(X_p)]^2 \lambda_p e^{-\lambda_p x_p} dx_p\right) = T^{-1}\left(\int_0^\infty \left(x_0 - \frac{1}{\lambda_0}\right)^2 \lambda_0 e^{-\lambda_0 x_0} dx_0, \int_0^\infty \left(x_0 + x_1 - \frac{1}{\lambda_0 + \lambda_1}\right)^2 (\lambda_0 + \lambda_1)e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} d(x_0 + x_1), \int_0^\infty \left(x_0 + x_1 + x_2 - \frac{1}{\lambda_0 + \lambda_1 + \lambda_2}\right)^2 (\lambda_0 + \lambda_1 + \lambda_2)e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)} d(x_0 + x_1 + x_2)\right) = T^{-1}\left(\frac{1}{\lambda_0^2}, \frac{1}{(\lambda_0 + \lambda_1)^2}, \frac{1}{(\lambda_0 + \lambda_1 + \lambda_2)^2}\right) = \frac{1}{\lambda_0^2} + \left[\frac{1}{(\lambda_0 + \lambda_1)^2} - \frac{1}{\lambda_0^2}\right]P_1 + \left[\frac{1}{(\lambda_0 + \lambda_1 + \lambda_2)^2} - \frac{1}{(\lambda_0 + \lambda_1)^2}\right]P_2$

Definition 4.2

A plithogenic random variable X_p is said to be uniformly distributed with the plithogenic parameters a_p, b_p if its probability density function takes the form:

$$f(x_p) = \frac{1}{b_p - a_p}; a_p < x_p < b_p$$

Theorem 4.4

The formal plithogenic form of probability density function of uniform distribution is:

$$f(x_p) = \frac{1}{b_0 - a_0} + \left[\frac{1}{b_0 + b_1 - a_0 - a_1} - \frac{1}{b_0 - a_0}\right]P_1 + \left[\frac{1}{b_0 + b_1 + b_2 - a_0 - a_1 - a_2} - \frac{1}{b_0 + b_1 - a_0 - a_1}\right]P_2$$

Proof

$$\begin{aligned}
 f(x_p) &= T^{-1}T[f(x_p)] = T^{-1}T\left[\frac{1}{b_p - a_p}\right] = T^{-1}T\left[\frac{1}{b_0 + b_1P_1 + b_2P_2 - a_0 - a_1P_1 - a_2P_2}\right] \\
 &= T^{-1}T\left[\frac{1}{b_0 - a_0 + (b_1 - a_1)P_1 + (b_2 - a_2)P_2}\right] \\
 &= T^{-1}\left(\frac{1}{b_0 - a_0}, \frac{1}{b_0 + b_1 - a_0 - a_1}, \frac{1}{b_0 + b_1 + b_2 - a_0 - a_1 - a_2}\right) \\
 &= \frac{1}{b_0 - a_0} + \left[\frac{1}{b_0 + b_1 - a_0 - a_1} - \frac{1}{b_0 - a_0}\right]P_1 \\
 &\quad + \left[\frac{1}{b_0 + b_1 + b_2 - a_0 - a_1 - a_2} - \frac{1}{b_0 + b_1 - a_0 - a_1}\right]P_2
 \end{aligned}$$

Theorem 4.5

Plithogenic cumulative distribution function of plithogenic uniform distribution is:

$$\begin{aligned}
 F(x_p) &= \frac{x_0 - a_0}{b_0 - a_0} + \left[\frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1} - \frac{x_0 - a_0}{b_0 - a_0}\right]P_1 \\
 &\quad + \left[\frac{x_0 + x_1 + x_2 - a_0 - a_1 - a_2}{b_0 - a_0 + b_1 - a_1 + b_2 - a_2} - \frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1}\right]P_2
 \end{aligned}$$

Proof

$$F(x_p) = \int_{a_p}^{x_p} f(t_p) dt_p$$

$$\begin{aligned}
 T(F(x_p)) &= T\left(\int_{a_p}^{x_p} f(t_p) dt_p\right) \\
 &= T\left(\int_{(a_0+a_1P_1+a_2P_2)}^{(x_0+x_1P_1+x_2P_2)} \left(\frac{1}{b_0 + b_1P_1 + b_2P_2 - a_0 - a_1P_1 - a_2P_2}\right) d(t_0 + t_1P_1 + t_2P_2)\right) \\
 &= \left(\int_{a_0}^{x_0} \frac{1}{b_0 - a_0} dt_0, \int_{a_0+a_1}^{x_0+x_1} \frac{1}{b_0 - a_0 + b_1 - a_1} d(t_0 + t_1), \int_{a_0+a_1+a_2}^{x_0+x_1+x_2} \frac{1}{b_0 - a_0 + b_1 - a_1 + b_2 - a_2} d(t_0 + t_1 + t_2)\right) \\
 &= \left(\frac{x_0 - a_0}{b_0 - a_0}, \frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1}, \frac{x_0 + x_1 + x_2 - a_0 - a_1 - a_2}{b_0 - a_0 + b_1 - a_1 + b_2 - a_2}\right)
 \end{aligned}$$

Taking T^{-1} yields to:

$$\begin{aligned}
 F(x_p) &= \frac{x_0 - a_0}{b_0 - a_0} + \left[\frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1} - \frac{x_0 - a_0}{b_0 - a_0}\right]P_1 \\
 &\quad + \left[\frac{x_0 + x_1 + x_2 - a_0 - a_1 - a_2}{b_0 - a_0 + b_1 - a_1 + b_2 - a_2} - \frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1}\right]P_2
 \end{aligned}$$

Theorem 4.6

Let X_p be a plithogenic uniform random variable with plithogenic parameters a_p, b_p then:

1. $E(X_p) = \frac{a_0+b_0}{2} + \left[\frac{a_0+a_1+b_0+b_1}{2} - \frac{a_0+b_0}{2}\right]P_1 + \left[\frac{a_0+a_1+a_2+b_0+b_1+b_2}{2} - \frac{a_0+a_1+b_0+b_1}{2}\right]P_2$
2. $Var(X_p) = \frac{(b_0-a_0)^2}{12} + \left[\frac{(b_0+b_1-a_0-a_1)^2}{12} - \frac{(b_0-a_0)^2}{12}\right]P_1 + \left[\frac{(b_0+b_1+b_2-a_0-a_1-a_2)^2}{12} - \frac{(b_0+b_1-a_0-a_1)^2}{12}\right]P_2$

Proof

1. $E(X_p) = T^{-1}T\left(\int_{a_p}^{b_p} \frac{x_p}{b_p - a_p} dx_p\right)$

$$\begin{aligned}
 &= T^{-1} \left(\int_{a_0}^{b_0} \frac{x_0}{b_0 - a_0} dx_0, \int_{a_0+a_1}^{b_0+b_1} \frac{x_0 + x_1}{b_0 + b_1 - a_0 - a_1} d(x_0 + x_1), \int_{a_0+a_1+a_2}^{b_0+b_1+b_2} \frac{x_0 + x_1 + x_2}{b_0 + b_1 + b_2 - a_0 - a_1 - a_2} d(x_0 + x_1 + x_2) \right) \\
 &= T^{-1} \left(\frac{a_0 + b_0}{2}, \frac{a_0 + a_1 + b_0 + b_1}{2}, \frac{a_0 + a_1 + a_2 + b_0 + b_1 + b_2}{2} \right) \\
 &= \frac{a_0 + b_0}{2} + \left[\frac{a_0 + a_1 + b_0 + b_1}{2} - \frac{a_0 + b_0}{2} \right] P_1 + \left[\frac{a_0 + a_1 + a_2 + b_0 + b_1 + b_2}{2} - \frac{a_0 + a_1 + b_0 + b_1}{2} \right] P_2
 \end{aligned}$$

2.

$$\begin{aligned}
 \text{Var}(X_P) &= T^{-1} T \left(\int_{a_P}^{b_P} \frac{[x_P - E(X_P)]^2}{b_P - a_P} dx_P \right) \\
 &= T^{-1} \left(\int_{a_0}^{b_0} \left(x_0 - \frac{a_0 + b_0}{2} \right)^2 \frac{1}{b_0 - a_0} dx_0, \int_{a_0+a_1}^{b_0+b_1} \left(x_0 + x_1 - \frac{a_0 + a_1 + b_0 + b_1}{2} \right)^2 \frac{1}{b_0 + b_1 - a_0 - a_1} d(x_0 + x_1), \int_0^\infty \left(x_0 + x_1 + x_2 - \frac{a_0 + a_1 + a_2 + b_0 + b_1 + b_2}{2} \right)^2 \frac{1}{b_0 + b_1 + b_2 - a_0 - a_1 - a_2} d(x_0 + x_1 + x_2) \right) \\
 &= T^{-1} \left(\frac{(b_0 - a_0)^2}{12}, \frac{(b_0 + b_1 - a_0 - a_1)^2}{12}, \frac{(b_0 + b_1 + b_2 - a_0 - a_1 - a_2)^2}{12} \right) \\
 &= \frac{(b_0 - a_0)^2}{12} + \left[\frac{(b_0 + b_1 - a_0 - a_1)^2}{12} - \frac{(b_0 - a_0)^2}{12} \right] P_1 \\
 &+ \left[\frac{(b_0 + b_1 + b_2 - a_0 - a_1 - a_2)^2}{12} - \frac{(b_0 + b_1 - a_0 - a_1)^2}{12} \right] P_2
 \end{aligned}$$

5. Fundamental Form of Continuous Plithogenic Probability Densities and Plithogenic Cumulative Distribution Functions

Let X_P be a plithogenic random variable that has a probability density function $f(x_P; \Theta_P)$ and cumulative distribution function $F(x_P; \Theta_P)$ with $\Theta_P = (\theta_{1P}, \dots, \theta_{kP})$; $\theta_{iP} = \theta_{i0} + \theta_{i1}P_1 + \theta_{i2}P_2$; $i = 1, 2, \dots, k$ a vector of parameters, in this section we provide two fundamental theorems in plithogenic probability theory:

Theorem 5.1

Formal form of $f(x_P; \Theta_P)$ is:

$$f(x_P; \Theta_P) = f(x_0; \Theta_0) + [f(x_0 + x_0; \Theta_0 + \Theta_1) - f(x_0; \Theta_0)] P_1 + [f(x_0 + x_1 + x_2; \Theta_0 + \Theta_1 + \Theta_2) - f(x_0 + x_1; \Theta_0 + \Theta_1)] P_2$$

Where: $\Theta_0 = (\theta_{10}, \dots, \theta_{k0})$, $\Theta_0 + \Theta_1 = (\theta_{10} + \theta_{11}, \dots, \theta_{k0} + \theta_{k1})$, $\Theta_0 + \Theta_1 + \Theta_2 = (\theta_{10} + \theta_{11} + \theta_{12}, \dots, \theta_{k0} + \theta_{k1} + \theta_{k2})$

Proof

Using one dimensional AH-Isometry:

$$\begin{aligned}
 T(f(x_P; \Theta_P)) &= T(f(x_0 + x_1P_1 + x_2P_2; \Theta_0 + \Theta_1P_1 + \Theta_2P_2)) \\
 &= f(T(x_0 + x_1P_1 + x_2P_2; \Theta_0 + \Theta_1P_1 + \Theta_2P_2)) \\
 &= f((x_0, x_0 + x_1, x_0 + x_1 + x_2); (\Theta_0, \Theta_0 + \Theta_1, \Theta_0 + \Theta_1 + \Theta_2)) \\
 &= (f(x_0; \Theta_0), f(x_0 + x_1; \Theta_0 + \Theta_1), f(x_0 + x_1 + x_2; \Theta_0 + \Theta_1 + \Theta_2))
 \end{aligned}$$

Taking T^{-1} of both sides:

$$\begin{aligned}
 \Rightarrow f(x_P; \Theta_P) &= T^{-1}((f(x_0; \Theta_0), f(x_0 + x_1; \Theta_0 + \Theta_1), f(x_0 + x_1 + x_2; \Theta_0 + \Theta_1 + \Theta_2))) \\
 &= f(x_0; \Theta_0) + [f(x_0 + x_1; \Theta_0 + \Theta_1) - f(x_0; \Theta_0)] P_1 + [f(x_0 + x_1 + x_2; \Theta_0 + \Theta_1 + \Theta_2) - f(x_0 + x_1; \Theta_0 + \Theta_1)] P_2
 \end{aligned}$$

Theorem 5.2

Formal form of continuous plithogenic cumulative distribution function is:

$$F(x_p; \Theta_p) = F(x_0; \Theta_0) + [F(x_0 + x_1; \Theta_0 + \Theta_1) - F(x_0; \Theta_0)]P_1 + [F(x_0 + x_1 + x_2; \Theta_0 + \Theta_1 + \Theta_2) - F(x_0 + x_1; \Theta_0 + \Theta_1)]P_2$$

Proof

Straightforward by integration of results in theorem 5.1.

6. Plithogenic Random Numbers Generation

Let X_p be a random variable that has a plithogenic pdf $f(x_p)$ and plithogenic cdf $F(x_p)$, if we assumed that $U_p = F(x_p)$, then U_p is a plithogenic random variable uniformly distributed on $[0,1]$, i.e., $U_p \sim Unif(0,1)$.

We can write X_p as a function of U_p to generate random plithogenic numbers as we do in the classical case, let's work with the already presented distributions in section 4.

6.1 Random numbers generation according to plithogenic uniform distribution

We have:

$$F(x_p) = \frac{x_0 - a_0}{b_0 - a_0} + \left[\frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1} - \frac{x_0 - a_0}{b_0 - a_0} \right] P_1 + \left[\frac{x_0 + x_1 + x_2 - a_0 - a_1 - a_2}{b_0 - a_0 + b_1 - a_1 + b_2 - a_2} - \frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1} \right] P_2$$

By setting $U_p = F(x_p) \Rightarrow T(U_p) = T(F(x_p))$, then:

$$(u_0, u_0 + u_1, u_0 + u_1 + u_2) = \left(\frac{x_0 - a_0}{b_0 - a_0}, \frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1}, \frac{x_0 + x_1 + x_2 - a_0 - a_1 - a_2}{b_0 - a_0 + b_1 - a_1 + b_2 - a_2} \right)$$

$$u_0 = \frac{x_0 - a_0}{b_0 - a_0}$$

$$\Rightarrow x_0 = u_0(b_0 - a_0) + a_0$$

$$u_0 + u_1 = \frac{x_0 + x_1 - a_0 - a_1}{b_0 - a_0 + b_1 - a_1}$$

$$\Rightarrow x_0 + x_1 = (u_0 + u_1)(b_0 - a_0 + b_1 - a_1) + (a_0 + a_1)$$

$$u_0 + u_1 + u_2 = \frac{x_0 + x_1 + x_2 - a_0 - a_1 - a_2}{b_0 - a_0 + b_1 - a_1 + b_2 - a_2}$$

$$\Rightarrow x_0 + x_1 + x_2 = (u_0 + u_1 + u_2)(b_0 - a_0 + b_1 - a_1 + b_2 - a_2) + (a_0 + a_1 + a_2)$$

Hence:

$$(x_0, x_0 + x_1, x_0 + x_1 + x_2) = (u_0(b_0 - a_0) + a_0, (u_0 + u_1)(b_0 - a_0 + b_1 - a_1) + (a_0 + a_1), (u_0 + u_1 + u_2)(b_0 - a_0 + b_1 - a_1 + b_2 - a_2) + (a_0 + a_1 + a_2))$$

Now we take T^{-1} of both sides and get:

$$x_p = u_0(b_0 - a_0) + a_0 + [(u_0 + u_1)(b_0 - a_0 + b_1 - a_1) + (a_0 + a_1) - u_0(b_0 - a_0) + a_0]P_1 + [(u_0 + u_1 + u_2)(b_0 - a_0 + b_1 - a_1 + b_2 - a_2) + (a_0 + a_1 + a_2) - (u_0 + u_1)(b_0 - a_0 + b_1 - a_1) + (a_0 + a_1)]P_2$$

6.2 Random numbers generation according to plithogenic exponential distribution

We proved that:

$$F(x_p) = 1 - e^{-\lambda_0 x_0} + [1 - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} - (1 - e^{-\lambda_0 x_0})]P_1 + [1 - e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)} - (1 - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)})]P_2$$

$$\Rightarrow (u_0, u_0 + u_1, u_0 + u_1 + u_2) = (1 - e^{-\lambda_0 x_0}, 1 - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)}, 1 - e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)})$$

$$\begin{aligned}
u_0 &= 1 - e^{-\lambda_0 x_0} \\
\Rightarrow x_0 &= -\frac{\ln(1 - u_0)}{\lambda_0} \\
u_0 + u_1 &= 1 - e^{-(\lambda_0 + \lambda_1)(x_0 + x_1)} \\
\Rightarrow x_0 + x_1 &= -\frac{\ln(1 - (u_0 + u_1))}{\lambda_0 + \lambda_1} \\
u_0 + u_1 + u_2 &= 1 - e^{-(\lambda_0 + \lambda_1 + \lambda_2)(x_0 + x_1 + x_2)} \\
\Rightarrow x_0 + x_1 + x_2 &= -\frac{\ln(1 - (u_0 + u_1 + u_2))}{\lambda_0 + \lambda_1 + \lambda_2}
\end{aligned}$$

Hence:

$$(x_0, x_0 + x_1, x_0 + x_1 + x_2) = \left(-\frac{\ln(1 - u_0)}{\lambda_0}, -\frac{\ln(1 - (u_0 + u_1))}{\lambda_0 + \lambda_1}, -\frac{\ln(1 - (u_0 + u_1 + u_2))}{\lambda_0 + \lambda_1 + \lambda_2} \right)$$

Now we take T^{-1} of both sides and get:

$$\begin{aligned}
x_p &= -\frac{\ln(1 - u_0)}{\lambda_0} + \left[-\frac{\ln(1 - (u_0 + u_1))}{\lambda_0 + \lambda_1} - \left(-\frac{\ln(1 - u_0)}{\lambda_0} \right) \right] P_1 \\
&\quad + \left[-\frac{\ln(1 - (u_0 + u_1 + u_2))}{\lambda_0 + \lambda_1 + \lambda_2} - \left(-\frac{\ln(1 - (u_0 + u_1))}{\lambda_0 + \lambda_1} \right) \right] P_2 \\
\Rightarrow x_p &= -\frac{\ln(1 - u_0)}{\lambda_0} + \left[\frac{\ln(1 - u_0)}{\lambda_0} - \frac{\ln(1 - (u_0 + u_1))}{\lambda_0 + \lambda_1} \right] P_1 \\
&\quad + \left[\frac{\ln(1 - (u_0 + u_1))}{\lambda_0 + \lambda_1} - \frac{\ln(1 - (u_0 + u_1 + u_2))}{\lambda_0 + \lambda_1 + \lambda_2} \right] P_2
\end{aligned}$$

7. Conclusion

We have presented the formal form of plithogenic random variable and plithogenic probability density functions and studied its important properties including plithogenic expectation, plithogenic variance, plithogenic moments generating function, some plithogenic probability distributions and its probabilistic properties, plithogenic random numbers generation. Many theorems were presented and proved with an algebraic approach depending on algebraic isomorphisms. In future researches we are looking forward to study applications of this new definition in many branches of probability theory like reliability theory, queueing theory, dynamic systems, stochastic processes, etc.

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