Introduction to Neutrosophic Bayes Estimation Theory

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Abstract

This research presents the concept of neutrosophic Bayesian estimation defining the neutrosophic loss function, neutrosophic risk function, neutrosophic posterior risk function and neutrosophic maximum a posteriori estimator. Minimization of the neutrosophic posterior risk of the estimator is also discussed. An algebraic isomorphism is used to simplify equations solving. As an application of the presented theorems, a sample drawn from a neutrosophic gamma distribution with a conjugate prior is discussed and studied and the parameter of the formulated distribution is successfully estimated using neutrosophic quadratic loss function which results an estimator that equals the posterior mean.

Keywords: Neutrosophic; Loss Function; Risk Function; Conditional Density Function; Conditional Expectation; Posterior Risk Function; Maximum a Posteriori Estimator.

1. Introduction

Neutrosophic probability theory is considered a new extension to traditional probability theory, an extension that allows us to work on phenomena that contain ambiguity and uncertainty in its results, such indeterminacy could happen because of the behavior of the experiment, its physical environment or other reasons, and due to taking indeterminacy into consideration must improve our understandings of the phenomenon, neutrosophic was presented in probability theory.

Neutrosophic probability theory has been presented in many points of view, first one considered probability as a triplet of values, second one considering parameters of the distributions are interval valued and the last one defined by using an algebraic structure I satisfies $I^2 = I$ which makes a beautiful change in classical Euclidian geometry and this new generalization of probability theory is called literal or symbolic neutrosophic probability theory, which we will be used in our work.[1]–[11]

Many important papers were presented in neutrosophic probability, AH-Isometry and its inverse that helps to transfer problems from a neutrosophic form to classical form or vice versa was presented by Abobala and Hatip in [8], neutrosophic real analysis was presented by Abobala and Zeina in [12], Astambli, Zeina et al. presented many estimation methods of neutrosophic parameters in [13], [14] and there're lots of other researches related in different fields such as mathematics, decision making, artificial intelligence, etc. [15]–[48].

In the paper, we will work on literal neutrosophic probability theory and define the neutrosophic maximum a posteriori estimator (MAP), the maximum a posteriori estimation plays a big rule in estimation theory for its great performance and applications, it deals with the parameter that we want to estimate as a random variable which has a probability distribution, and use it with the drawn sample to find an estimator that minimizes the posterior risk, this allows to use more provided information to enhance the estimation ability to capture the true parameter, so, when we treat the indeterminacy with these distributions we provide more general and flexible estimations than the classical cases.

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2. Preliminaries:

Definition 2.1

Let $R(I) = \{a_0 + a_1 I; a_0, a_1 \in R, I^2 = I\}$ be the neutrosophic field of reals. The one-dimensional AH-Isometry is defined as the following:

$$T: R(I) \longrightarrow R^2\,; T(a_0 + a_1 I) = (a_0, a_0 + a_1)$$
$$T^{-1}: R^2 \longrightarrow R(I)\,; T^{-1}(a_0, a_1) = a_0 + (a_1 - a_0)I$$

Definition 2.2

Let $f: R(I) \longrightarrow R(I); f = f(x_N), x_N \in R(I)$, we call f a neutrosophic real function with one neutrosophic real variable.

Definition 2.3

Literal neutrosophic random variable X_N is defined as follows:

$$X_N: \Omega_N \longrightarrow R(I);\ \Omega_N = \Omega_1 \times \Omega_2(I)$$
$$X_N = X_1 + X_2 I;\ \ I^2 = I$$

Where X_1, X_2 are classical random variables defined on Ω_1, Ω_2 respectively.

Definition 2.4

Let $a_N = a_1 + a_2 I,\ b_N = b_1 + b_2 I \in R(I)$, we say that $a_N \geq_N b_N$ if:

$$a_1 \geq b_1, a_1 + a_2 \geq b_1 + b_2$$

3. Neutrosophic Conditional Density, Neutrosophic Conditional Expectation, Neutrosophic Loss, Neutrosophic Risk and Neutrosophic Posterior Risk Functions:

Theorem 3.1

Let X_N be a neutrosophic random variable that has a neutrosophic probability density function $f(x_N;\Theta_N)$ with a vector of parameters $\Theta_N = (\theta_{1N}, \dots, \theta_{kN}); \theta_{iN} = \theta_{i1} + \theta_{i2} I\,; i = 1,2, \dots, k$, then $f(x_N;\Theta_N)$ is proved to have the form:

$$f(x_N;\Theta_N) = f(x_1;\Theta_1) + [f(x_1 + x_2;\Theta_1 + \Theta_2) - f(x_1;\Theta_1)]I \qquad (1)$$

Where: $\Theta_1 = (\theta_{11}, \dots, \theta_{k1})$, $\Theta_1 + \Theta_2 = (\theta_{11} + \theta_{12}, \dots, \theta_{k1} + \theta_{k2})$.
see [15]

Theorem 3.2

Neutrosophic conditional density function of a neutrosophic random variables X_N given another neutrosophic random variables Y_N satisfies:

$$f(x_N|y_N;\Theta_N) = f(x_1|y_1;\Theta_1) + [f(x_1 + x_2|y_1 + y_2;\Theta_1 + \Theta_2) - f(x_1|y_1;\Theta_1)]I \qquad (2)$$

Where $\Theta_N = (\theta_{1N}, \dots, \theta_{kN}); \theta_{iN} = \theta_{i1} + \theta_{i2} I\,; i = 1,2, \dots, k$ is a vector of parameters.

Proof:

$$f(x_N|y_N;\Theta_N) = f(x_1 + x_2 I|y_1 + y_2 I;\Theta_1 + \Theta_2 I)$$

We take one-dimensional AH-Isometry:

$$T(f(x_N|y_N;\Theta_N)) = T(f(x_1 + x_2 I|y_1 + y_2 I;\Theta_1 + \Theta_2 I))$$
$$= f(x_1|y_1;\Theta_1, x_1 + x_2|y_1 + y_2;\Theta_1 + \Theta_2)$$
$$= (f(x_1|y_1;\Theta_1), f(x_1 + x_2|y_1 + y_2;\Theta_1 + \Theta_2))$$

Taking T^{-1} for both sides yield:

$$f(x_N|y_N;\Theta_N) = f(x_1|y_1;\Theta_1) + [f(x_1 + x_2|y_1 + y_2;\Theta_1 + \Theta_2) - f(x_1|y_1;\Theta_1)]I$$

Theorem 3.3

Let X_N, Y_N be two neutrosophic random variables and let $f(x_N|y_N)$ be the conditional probability density function of X_N given Y_N then the neutrosophic conditional expectation $E(X_N|Y_N)$ is:

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$$E(X_N|Y_N) = E(X_1|Y_1) + [E(X_1 + X_2|Y_1 + Y_2) - E(X_1|Y_1)]I \quad (3)$$

Proof:

$$E(X_N|Y_N) = E(X_1 + X_2 I|Y_1 + Y_2 I)$$
$$T(E(X_N|Y_N)) = T(E(X_1 + X_2 I|Y_1 + Y_2 I))$$
$$= E(X_1|Y_1, X_1 + X_2|Y_1 + Y_2)$$
$$= (E(X_1|Y_1), E(X_1 + X_2|Y_1 + Y_2))$$

Now we take T^{-1}:

$$E(X_N|Y_N) = E(X_1|Y_1) + [E(X_1 + X_2|Y_1 + Y_2) - E(X_1|Y_1)]I$$

Theorem 3.4

Let $\tau_N = \tau_1 + \tau_2 I$ be a neutrosophic parameter of a probability distribution, and let $\hat{\tau}_N = \hat{\tau}_1 + \hat{\tau}_2 I$ be an estimator of τ_N then neutrosophic loss, neutrosophic risk and neutrosophic posterior risk are respectively:

1.

$$Loss(\tau_N, \hat{\tau}_N) = Loss(\tau_1, \hat{\tau}_1) + [Loss(\tau_1 + \tau_2, \hat{\tau}_1 + \hat{\tau}_2) - Loss(\tau_1, \hat{\tau}_1)]I \quad (4)$$

Where $Loss$ function could be absolute: $Loss(a, \hat{a}) = |a - \hat{a}|$, quadratic: $Loss(a, \hat{a}) = (a - \hat{a})^2$, or other loss functions.

2.

$$Risk(\tau_N, \hat{\tau}_N) = E(Loss(\tau_N, \hat{\tau}_N))$$
$$= Risk(\tau_1, \hat{\tau}_1) + [Risk(\tau_1 + \tau_2, \hat{\tau}_1 + \hat{\tau}_2) - Risk(\tau_1, \hat{\tau}_1)]I \quad (5)$$

3.

$$Risk(\tau_N, \hat{\tau}_N|X_N) = E(Loss(\tau_N, \hat{\tau}_N)|X_N) \quad (6)$$
$$= Risk(\tau_1, \hat{\tau}_1|X_1) + [Risk(\tau_1 + \tau_2, \hat{\tau}_1 + \hat{\tau}_2|X_1 + X_2) - Risk(\tau_1, \hat{\tau}_1|X_1)]I$$

Proof:

Straightforward.

4. Neutrosophic Maximum a Posteriori Estimation:

Theorem 4.1

Let $\mathbb{S}_N = (S_{1N}, \dots, S_{nN})$ be an i.i.d sample of neutrosophic random variables and $\mathrm{T}_N = (\tau_{1N}, \dots, \tau_{kN}); \tau_{iN} = \tau_{i1} + \tau_{i2} I\ ; i = 1,2, \dots, k$ be a vector of parameters, suppose T_N is a random variable follows a distribution that has a probability density function of $\pi(\mathrm{T}_N)$, which we call it a prior distribution, and suppose $L_N(\mathrm{T}_N)$ is the neutrosophic maximum likelihood function of the sample [13], hence the neutrosophic posterior density function of T_N is as follows:

$$f(\mathrm{T}_N|\mathbb{S}_N) \sim L(\mathrm{T}_1)\pi(\mathrm{T}_1) + [L(\mathrm{T}_1 + \mathrm{T}_2)\pi(\mathrm{T}_1 + \mathrm{T}_2) - L(\mathrm{T}_1)\pi(\mathrm{T}_1)]I \quad (7)$$

Where: $\mathrm{T}_1 = (\tau_{11}, \dots, \tau_{k1})$, $\mathrm{T}_1 + \mathrm{T}_2 = (\tau_{11} + \tau_{12}, \dots, \tau_{k1} + \tau_{k2})$.

Proof:

$f(\mathrm{T}_N|\mathbb{d}_N)$ can be written as the following:

$$f(\mathrm{T}_N|\mathbb{S}_N) = \frac{f(\mathbb{S}_N|\mathrm{T}_N)\pi(\mathrm{T}_N)}{f(\mathbb{S}_N)}$$

The denominator can be excluded due to the fact that it doesn't affect the final shape of the distribution:

$$f(\mathrm{T}_N|\mathbb{S}_N) \sim f(\mathbb{S}_N|\mathrm{T}_N)\pi(\mathrm{T}_N)$$
$$\Rightarrow f(\mathrm{T}_N|\mathbb{S}_N) \sim (L(\mathrm{T}_1) + [L(\mathrm{T}_1 + \mathrm{T}_2) - L(\mathrm{T}_1)]I)(\pi(\mathrm{T}_1) + [\pi(\mathrm{T}_1 + \mathrm{T}_2) - \pi(\mathrm{T}_1)]I)$$
$$= L(\mathrm{T}_1)\pi(\mathrm{T}_1) + L(\mathrm{T}_1)\pi(\mathrm{T}_1 + \mathrm{T}_2)I - L(\mathrm{T}_1)\pi(\mathrm{T}_1)I + L(\mathrm{T}_1 + \mathrm{T}_2)\pi(\mathrm{T}_1)I + L(\mathrm{T}_1 + \mathrm{T}_2)\pi(\mathrm{T}_1 + \mathrm{T}_2)I$$
$$- L(\mathrm{T}_1 + \mathrm{T}_2)\pi(\mathrm{T}_1)I - L(\mathrm{T}_1)\pi(\mathrm{T}_1)I - L(\mathrm{T}_1)\,\pi(\mathrm{T}_1 + \mathrm{T}_2)I + L(\mathrm{T}_1)\pi(\mathrm{T}_1)I$$

Doing some simplifications yields to:

$$f(\mathrm{T}_N|\mathbb{S}_N) \sim L(\mathrm{T}_1)\pi(\mathrm{T}_1) + [L(\mathrm{T}_1 + \mathrm{T}_2)\pi(\mathrm{T}_1 + \mathrm{T}_2) - L(\mathrm{T}_1)\pi(\mathrm{T}_1)]I$$

Theorem 4.2

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The estimator $\widehat{T}_N$ satisfies the minimization condition of the neutrosophic posterior risk function if $\widehat{T}_N$ is the posterior mean of T_N for a neutrosophic quadratic loss function, or $\widehat{T}_N$ is the posterior median for a neutrosophic absolute loss function.

Proof:
The minimization of $Risk(T_N, \widehat{T}_N|\mathbb{S}_N)$ by equation (6) is the minimization of $Risk(T_1, \widehat{T}_1|\mathbb{S}_1)$ and $Risk(T_1 + T_2, \widehat{T}_1 + \widehat{T}_2|\mathbb{S}_1 + \mathbb{S}_2)$, which occurs when:

$$\frac{d}{dT_1} Risk(T_1, \widehat{T}_1|\mathbb{S}_1) = 0$$

$$\frac{d}{d(T_1 + T_2)} Risk(T_1 + T_2, \widehat{T}_1 + \widehat{T}_2|\mathbb{S}_1 + \mathbb{S}_2) = 0$$

We notice that we can deal with both conditions as the classical case, which means that for a quadratic loss function, $\widehat{T}_N$ must equals:

$$\widehat{T}_N = E(T_N|\mathbb{D}_N) \Leftrightarrow \begin{cases} \widehat{T}_1 = E(T_1|\mathbb{S}_1) \\ \widehat{T}_1 + \widehat{T}_2 = E(T_1 + T_2|\mathbb{S}_1 + \mathbb{S}_2) \end{cases} \quad (10)$$

Which is the posterior mean.
The proof for the absolute loss function goes also straightforward as before, where the estimator must equal the posterior median.

5. MAP estimation of neutrosophic gamma distribution's parameter with a neutrosophic gamma prior:

Let $X_{1N}, \ldots, X_{nN} \sim Gamma(\alpha_N, \beta_N)$, where α_N is known, and let β_N has a neutrosophic gamma prior, i.e. $\beta_N \sim Gamma(\alpha_{0N}, \beta_{0N})$ where α_{0N}, β_{0N} are known, $\alpha_{0N} = \alpha_{01} + \alpha_{02}I, \beta_{0N} = \beta_{01} + \beta_{02}I$, then:

$$\pi(\beta_N) = \frac{{\beta_{0N}}^{\alpha_{0N}}}{(\alpha_{0N} - 1)!} {\beta_N}^{\alpha_{0N}-1} e^{-\beta_{0N}\beta_N}$$

$$\Rightarrow \pi(\beta_N) = \frac{{\beta_{01}}^{\alpha_{01}}}{(\alpha_{01} - 1)!} {\beta_1}^{\alpha_{01}-1} e^{-\beta_{01}\beta_1} + \left[\frac{(\beta_{01} + \beta_{02})^{(\alpha_{01}+\alpha_{02})}}{(\alpha_{01} + \alpha_{02} - 1)!} (\beta_1 + \beta_2)^{\alpha_{01}+\alpha_{02}-1} e^{-(\beta_{01}+\beta_{02})(\beta_1+\beta_2)} - \frac{{\beta_{01}}^{\alpha_{01}}}{(\alpha_{01} - 1)!} {\beta_1}^{\alpha_{01}-1} e^{-\beta_{01}\beta_1}\right] I$$

$$L(\beta_N) = \prod_{i=1}^{n} \frac{{\beta_N}^{\alpha_N}}{(\alpha_N - 1)!} {x_{iN}}^{\alpha_N - 1} e^{-\beta_N x_{iN}}$$

$$\Rightarrow L(\beta_N) = \prod_{i=1}^{n} \frac{{\beta_1}^{\alpha_1}}{(\alpha_1 - 1)!} {x_{i1}}^{\alpha_1 - 1} e^{-\beta_1 x_{i1}} + \left[\prod_{i=1}^{n} \frac{(\beta_1 + \beta_2)^{(\alpha_1+\alpha_2)}}{(\alpha_1 + \alpha_2 - 1)!} (x_{i1} + x_{i2})^{\alpha_1+\alpha_2-1} e^{-(\beta_1+\beta_2)(x_{i1}+x_{i2})} - \prod_{i=1}^{n} \frac{{\beta_1}^{\alpha_1}}{(\alpha_1 - 1)!} {x_{i1}}^{\alpha_1 - 1} e^{-\beta_1 x_{i1}}\right] I$$

By equation (7) we write:

$$f(\beta_N|\mathbb{x}_N) \sim \frac{{\beta_{01}}^{\alpha_{01}}}{(\alpha_{01} - 1)!} {\beta_1}^{\alpha_{01}-1} e^{-\beta_{01}\beta_1} \frac{{\beta_1}^{n\alpha_1}}{((\alpha_1 - 1)!)^n} e^{-\beta_1 \Sigma x_{i1}} \prod_{i=1}^{n} {x_{i1}}^{\alpha_1 - 1} + \left[\frac{(\beta_{01} + \beta_{02})^{(\alpha_{01}+\alpha_{02})}}{(\alpha_{01} + \alpha_{02} - 1)!} (\beta_1 + \beta_2)^{\alpha_{01}+\alpha_{02}-1} e^{-(\beta_{01}+\beta_{02})(\beta_1+\beta_2)} \frac{(\beta_1 + \beta_2)^{n(\alpha_1+\alpha_2)}}{((\alpha_1 + \alpha_2 - 1)!)^n} e^{-(\beta_1+\beta_2)\Sigma(x_{i1}+x_{i2})} \prod_{i=1}^{n} (x_{i1} + x_{i2})^{\alpha_1+\alpha_2-1} - \frac{{\beta_{01}}^{\alpha_{01}}}{(\alpha_{01} - 1)!} {\beta_1}^{\alpha_{01}-1} e^{-\beta_{01}\beta_1} \frac{{\beta_1}^{n\alpha_1}}{((\alpha_1 - 1)!)^n} e^{-\beta_1 \Sigma x_{i1}} \prod_{i=1}^{n} {x_{i1}}^{\alpha_1 - 1}\right] I$$

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$$= \frac{{\beta_{01}}^{\alpha_{01}}.\prod_{i=1}^{n} {x_{i1}}^{\alpha_1 - 1}}{(\alpha_{01}-1)!.\left((\alpha_1-1)!\right)^n} {\beta_1}^{n\alpha_1+\alpha_{01}-1} e^{-\beta_1(\beta_{01}+\sum x_{i1})} + \left[\frac{(\beta_{01}+\beta_{02})^{(\alpha_{01}+\alpha_{02})}\prod_{i=1}^{n}(x_{i1}+x_{i2})^{\alpha_1+\alpha_2-1}}{(\alpha_{01}+\alpha_{02}-1)!.\left((\alpha_1+\alpha_2-1)!\right)^n}(\beta_1 + \beta_2)^{n(\alpha_1+\alpha_2)+\alpha_{01}+\alpha_{02}-1} e^{-(\beta_1+\beta_2)(\beta_{01}+\beta_{02}+\sum(x_{i1}+x_{i2}))} - \frac{{\beta_{01}}^{\alpha_{01}}.\prod_{i=1}^{n} {x_{i1}}^{\alpha_1 - 1}}{(\alpha_{01}-1)!.\left((\alpha_1-1)!\right)^n} {\beta_1}^{n\alpha_1+\alpha_{01}-1} e^{-\beta_1(\beta_{01}+\sum x_{i1})}\right] I$$

We exclude constants because it doesn't change the resulting distribution by taking T, we do the exclusion and then we take T^{-1}:

$$\Rightarrow f(\beta_N|\mathbb{x}_N) \sim {\beta_1}^{n\alpha_1+\alpha_{01}-1} e^{-\beta_1(\beta_{01}+\sum x_{i1})} + \left[(\beta_1+\beta_2)^{n(\alpha_1+\alpha_2)+\alpha_{01}+\alpha_{02}-1} e^{-(\beta_1+\beta_2)(\beta_{01}+\beta_{02}+\sum(x_{i1}+x_{i2}))} - {\beta_1}^{n\alpha_1+\alpha_{01}-1} e^{-\beta_1(\beta_{01}+\sum x_{i1})}\right] I$$

We notice that this is the neutrosophic probability density function of gamma distribution with parameter $n\alpha_N + \alpha_{0N}$ and $\beta_{0N} + \sum x_{iN}$, which means that $\beta_N \sim Gamma(n\alpha_N + \alpha_{0N}, \beta_{0N} + \sum x_{iN})$

For a quadratic loss:

$$\hat{\beta}_N = \frac{n\alpha_N + \alpha_{0N}}{\beta_{0N} + \sum x_{iN}}$$

$$\Rightarrow \hat{\beta}_N = \frac{n\alpha_1 + \alpha_{01}}{\beta_{01} + \sum x_{i1}} + \left[\frac{n(\alpha_1+\alpha_2) + \alpha_{01} + \alpha_{02}}{\beta_{01} + \beta_{02} + \sum(x_{i1}+x_{i2})} - \frac{n\alpha_1 + \alpha_{01}}{\beta_{01} + \sum x_{i1}}\right] I$$

Or simply we write:

$$\hat{\beta}_N = \hat{\beta}_1 + \hat{\beta}_2 I$$

Where:

$$\hat{\beta}_1 = \frac{n\alpha_1 + \alpha_{01}}{\beta_{01} + \sum x_{i1}}$$

$$\hat{\beta}_1 + \hat{\beta}_2 = \frac{n(\alpha_1+\alpha_2) + \alpha_{01} + \alpha_{02}}{\beta_{01} + \beta_{02} + \sum(x_{i1}+x_{i2})}$$

5. Conclusions and future research directions:

We defined the neutrosophic MAP and found the formal estimators that minimizes the neutrosophic posterior risk function which made its application easy and useful to work on for any given sample distribution and prior distribution, the resulted definitions are similar to the classical case, but it takes the indeterminacy into account. We considered a case of a gamma distributed neutrosophic sample with a neutrosophic parameter that has a gamma prior also and found that the posterior density is also neutrosophic gamma distribution. Non-informative priors should be studied to check its indeterminacy effectiveness, which is the goal of future researches.

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