Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices

M. Anandhkumar [1,*], T. Harikrishnan[2], S. M. Chithra[3], V. Kamalakannan[4], B. Kanimozhi[5], Broumi Said[6]

[1]Department of Mathematics, IFET College of Engineering (Autonomous), Villupuram, Tamilnadu, India
[2]Department of Mathematics, Faculty of Science and Humanities, SRM Institute of Science and Technology, Ramapuram, Tamilnadu, India
[3]Department of Mathematics, R.M.K College of Engineering and Technology Chennai, Tamilnadu, India
[4]Department of Mathematics, Panimalar Engineering College, Chennai, Tamilnadu, India
[5]Department of Mathematics, Sri Manakula Vinayagar Engineering College (Autonomous), Madagadipet, Puducherry, India
[6]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco
Emails: anandhkumarmm@gmail.com; mokshihari2009@gmail.com; chithra.sm@rmkcet.ac.in; vkamalakannan@panimalar.ac.in; kanigopi.a20@gmail.com; broumisaid78@gmail.com

Abstract

In this paper, we introduce the concept of reverse sharp ordering on Neutrosophic Fuzzy matrix (NFM) as a special case of minus ordering. We also introduce the concept of reverse left-T and right-T orderings for NFM as an analogue of left-star and right-star partial orderings for complex matrices. Several properties of these ordering are derived. We show that these ordering preserve its Moore-penrose inverse property. Finally, we show that these ordering are identical for certain class of NFM.

Keywords: Neutrosophic fuzzy matrices; Reverse sharp ordering; Reverse left-T and right-T ordering; g-inverse; Moore-penrose inverses**.**

1. Introduction

A significant challenge in today's practical world is the complexity of problems in Economics, Engineering, Environmental Sciences, and Social Sciences that cannot be resolved using the well-known techniques of classical Mathematics. To handle this type of situation Zadeh [16] first introduced the notion of fuzzy set to investigate both theoretical and practical applications of our daily activities. At times, it may be exceedingly challenging to assign the membership value for standard fuzzy sets. In the current scenario Neutrosophic set—a generalization of the intuitionistic fuzzy set by Florentin Smarandache [17] is appropriate for such a situation.

Thomson [15] has studied Convergence of Power of a Fuzzy Matrix. The partial orderings on fuzzy matrices, which are equivalent to the star orderings on complex matrices, were started by Jian Miao Chen [4] has discussed Fuzzy matrix partial orderings and generalized inverses. Meenachi [5] has characterized the minus ordering on matrices in terms of their generalized inverses. Meenachi [6] has studied Fuzzy matrix – Theory and its applications. Punithavalli and Anandhkumar [11] have studied partial ordering on K - Idempotent intuitionistic fuzzy matrices. Punithavalli [10] has studied the Partial Orderings of m-Symmetric Fuzzy Matrices. Sriram and Murugadas [14] have discussed the Moore-Penrose Inverse of Intuitionistic Fuzzy Matrices. Muthu Guru Packiam and Krishna Mohan [7] have studied Partial orderings on k−idempotent fuzzy matrices. Atanassov [2][3] has introduced Intuitionistic fuzzy implications and modus ponens and on some types of fuzzy negations. Padder and Murugadas have discussed Algorithm for controllable and nilpotent intuitionistic fuzzy matrices and determinant theory for intuitionistic fuzzy matrices. Pradhan and Pal

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

[12] have studied the generalized inverse of Atanassov's intuitionistic fuzzy matrices. Shyamal and Pal [13] have characterized distance between intuitionistic fuzzy matrices. This work is to compute the undetermined equation by using generalized inverses. Atanassov [1] has studied Intuitionistic Fuzzy Sets. Wang, Smarandache, Zhang YQ, Sunderraman [18] have discussed Single valued neutrosophic sets. Anandh kumar [19] has introduced Pseudo Similarity of Neutrosophic Fuzzy matrices. Anandhkumar [20] has studied On various Inverse of Neutrosophic Fuzzy Matrices.

Partial ordering is a reflexive, anti-symmetric, transitive crisp binary relation R(X,X). The properties of this class of relations are denoted by the common symbol ≤. Therefore, <x,y> represents <x,y> ∈ R and indicates that x comes before y. The symbol ≥ denotes the reverse partial ordering R^{-1} (X, X). We say that y succeeds x if y ≤ x implies that <x,y> ∈ R^{-1}. The symbols $\leq^{P}$, $\leq^{Q}$ and $\leq^{R}$ are used to denote the various partial orderings P, Q, and R respectively.

In section 2, we introduce the concept of reverse sharp ordering on NFM as a special case of minus ordering. We established that for commuting pairs of matrices, sharp ordering and minus ordering are identical. We prove that under certain conditions sharp ordering reduces to the T-ordering on NFM. We establish a set of necessary condition for NFM with specified row and column spaces to be under sharp order. We derive some properties of NFM under sharp ordering. Let $(\mathrm{NF})^{\#}{}_{n}$ denote the set of all NFM $N \in F_n$ for which is group inverse $N^{\#}$.

In section 3, we introduce the concept of reverse left-T and right-T orderings for NFM as an analogue of left-star and right-star partial orderings for complex matrices. Several properties of these ordering are derived. We discuss the relation between these ordering with the T-ordering and minus ordering. We show that these ordering preserve its Moore - penrose inverse property. By using various generalized inverses, the new type of minus orderings are discussed. Finally, we show that these ordering are identical for certain class of NFM.

2. Research gaps

As mentioned in the above introduction section, Meenakshi introduced the concept of Left T Right T and minus ordering on fuzzy matrices and Jian Miao Chen introduced Fuzzy matrix partial orderings and generalized inverses. Here, we have applied the concept of Reverse Sharp and Left-T Right-T partial ordering on Neutrosophic Fuzzy Matrices. Both these concepts plays a significant role in hybrid fuzzy structure and we have applied Reverse Sharp and Left-T Right-T Partial ordering on Neutrosophic Fuzzy Matrices and studied some of the results in detail. First, we present equivalent characterizations of a Reverse Sharp and Left-T Right-T Partial ordering on Neutrosophic Fuzzy Matrices and then, derive equivalent conditions for an Neutrosophic fuzzy matrices. Also, using the g- inverses, we discuss some Theorems and examples for the reverse Sharp and Left-T Right-T Partial ordering on NFM.

2.1 Notations: For NFM of M ∈ $(NF)_n$,

M^T : Transpose of M,

R(M) : Row space of M,

C(M) : Column space of M,

M^+: Moore-Penrose inverse of M ,

$(NF)_n$ = Square Neutrosophic Fuzzy matrices of order n,

$(NF)_n^{\#}$ = Neutrosophic Fuzzy group inverse of order n,

$M \leq^{T} N$ = T-ordering,

$M \overset{T}{\geq} N$ = Reverse T-ordering,

$M \overset{\#}{>} N$ = Reverse Sharp ordering,

$M, N \in (NF)^{-}_{m\times n}$ = Minus ordering,

$M \leq^{*} N$ = Partial ordering.

3. Definitions

Definition :3.1 A neutrosophic set P on the universe of discourse X is defined as

$$P = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}, \text{where } T, I, F : X \to]^{-}0, 1^{+}[\text{ and}$$

$$^{-}0 \leq T_P(x) + I_P(x) + F_P(x) \leq 3^{+}$$

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

Definition:3.2 [5] For $P \in (\mathrm{NF})^{\#}{}_{n}$ and $Q \in \left(NF\right)_n$ the reverse order sharp ordering denoted as $\overset{\#}{>}$ is defined as $P \overset{\#}{>} Q \Leftrightarrow Q^{\#}Q = Q^{\#}P$ and $QQ^{\#} = PQ^{\#}$. Since $Q^{\#} \in Q\{1\}$, $P \overset{\#}{>} Q \Leftrightarrow P \geq Q$ with respect to $Q^{\#}$. Thus sharp ordering is the special case of minus ordering. In general, minus order need not imply sharp order need not imply T-order.
This is illustrated in the following examples

Example 3.2 Let $M = \begin{bmatrix} <1,0,0> & <0,0,1> \\ <0,0,1> & <1,0,0> \end{bmatrix}, \mathrm{N} = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix}$

$$\text{Now, } \mathrm{N}\{1\} = \left\{ X / X = \begin{bmatrix} <0,0,1> & <1,0,0> \\ <1,0,0> & <\alpha,0,0> \end{bmatrix}, \alpha \in \left(NF\right) \right\}$$

Clearly, $M \geq N$ with respect to $N^{-} = \begin{bmatrix} <0,0,1> & <1,0,0> \\ <1,0,0> & <1,0,0> \end{bmatrix}$

Here $NN^{-} \neq N^{-}N$ for all $N^{-} \in N\{1\} \Rightarrow N^{\#}$ does not exist and hence M and N are not sharp ordering. Thus minus order need not imply sharp order.

Definition:3.4[6] For M, N belongs to $\left(NF\right)_{m\times n}$ the T−ordering $\mathrm{M} \leq^{T} \mathrm{N}$ is well-defined as $\mathrm{M} \leq^{T} \mathrm{N} \Leftrightarrow \mathrm{M}^{t}\mathrm{M} = \mathrm{M}^{t}\mathrm{N}$ and $\mathrm{MM}^{t} = \mathrm{NM}^{t}$.

Definition:3.5. [7] For M, N belongs to $\left(NF\right)_{m\times n}$ the T− Reverse ordering $\mathrm{M} \overset{T}{\geq} \mathrm{N}$ is defined as $\mathrm{M} \overset{T}{\geq} \mathrm{N} \Leftrightarrow \mathrm{N}^{t}\mathrm{N} = \mathrm{N}^{t}\mathrm{M}$ and $\mathrm{NN}^{t} = \mathrm{MN}^{t}$.

Definition 3.6. [6] Let $\mathrm{M}, \mathrm{N} \in \left(NF\right)^{-}_{m\times n}$ the minus ordering denoted as $\leq$ is defined as

$$M \leq N \Leftrightarrow M^{-}M = M^{-}N \text{ and } MM^{-} = NM^{-} \text{ for some } M^{-} \in M\{1\}$$

Definition:3.7 [4] For M, N belongs to $\left(NF\right)_{m\times n}$ partial ordering $\mathrm{M} \leq^{*} \mathrm{N}$ is well-defined as $\mathrm{M} \leq^{*} \mathrm{N} \Leftrightarrow \mathrm{M}^{+}\mathrm{M} = \mathrm{M}^{+}\mathrm{N}$ and $\mathrm{MM}^{+} = \mathrm{NM}^{+}$.

4. Main Results

Theorem 4.1 For $\mathrm{M}, \mathrm{N} \in (\mathrm{N}F)^{\#}{}_{n}$, if $M \overset{\#}{>} N$ then we have

(i) $N = NN^{\#}M = MN^{\#}N = MN^{\#}M$

(ii) $NM^{\#}N = MM^{\#}N = NM^{\#}M = N$

Proof: $M \overset{\#}{>} N \Rightarrow M \geq N$ with respect to $N^{\#}$.

$$M \geq N \Leftrightarrow N^{\#}N = N^{\#}M \text{ and } NN^{\#} = MN^{\#} \text{ for some } N^{\#} \in N\{1\}$$

Now, $N = N\left(N^{\#}N\right) = NN^{\#}M$

$$N = \left(NN^{\#}\right)N = MN^{\#}N$$

$$N = M\left(N^{\#}N\right) = MN^{\#}M$$

(ii) $M \geq N \Rightarrow N = NN^{\#}M = MN^{\#}N$

For, $M^{\#} \in M\{1\}$

$$NM^{\#}N = (\mathrm{NN}^{\#}\mathrm{M})\,\mathrm{M}^{\#}(\mathrm{MN}^{\#}\mathrm{N})$$

$$NM^{\#}N = NN^{\#}(\mathrm{MM}^{\#}\mathrm{M})\,\mathrm{N}^{\#}N$$

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

$= (NN^{\#}M)\, N^{\#} N = NN^{\#}N = N$

Hence, $NM^{\#}N = N$ for each $M^{\#} \in M\{1\}$

Similarly, $MM^{\#}N = NM^{\#}M = N$

Theorem 4.2 For $M, N \in (NF)^{\#}_{n}$, $M \overset{\#}{>} N \Leftrightarrow M^{\#} \overset{\#}{>} N^{\#}$

Proof: Let $M \overset{\#}{>} N$

Now, $\left(N^{\#}\right)^{\#} N^{\#} = NN^{\#} = N^{\#}N$

$= N^{\#}\left(NM^{\#}M\right)$

$= \left(N^{\#}N\right)\left(M^{\#}M\right)$

$= \left(N^{\#}N\right)\left(MM^{\#}\right)$

$= \left(N^{\#}NM\right)M^{\#}$

$= NM^{\#}$

$\left(N^{\#}\right)^{\#} N^{\#} = \left(N^{\#}\right)^{\#} M^{\#}$

Similarly, $N^{\#}\left(N^{\#}\right)^{\#} = M^{\#}\left(N^{\#}\right)^{\#}$

Thus, $M^{\#} \overset{\#}{>} N^{\#}$ converse follows from the fact $\left(N^{\#}\right)^{\#} = N$

Theorem 4.3 For $M \in (NF)^{\#}_{n}$, and $N \in (NF)_{n}$ the conditions are equivalent

(i) $M \overset{\#}{>} N$

(ii) $MN = N^2 = NM$.

Proof: Since $N^{\#}$ exists, $N^{\#}N^2 = (N^{\#}N)N = NN^{\#}N = N^2N^{\#} = N$

(i) $\Rightarrow$ (ii) $NM = N^2N^{\#}M = N(NN^{\#}M) = NN = N^2$

Similarly, $MN = N^2$

(ii) $\Rightarrow$ (i) $NN^{\#} = (N^2N^{\#})N^{\#}$

$= (MNN^{\#})\, N^{\#}$

$= M(N^{\#}NN^{\#})$

$= MN^{\#}$

Similarly, $N^{\#}N = N^{\#}M$. Hence $M \overset{\#}{>} N$

Theorem 4.4 For $N \in (IF)^{\#}_{n}$, and $M \in (IF)_{n}$ then $M \overset{\#}{>} N \Leftrightarrow NM = MN$ and $M \geq N$.

Proof: $M \overset{\#}{>} N \Rightarrow M \geq N, NM = MN = N^2$ (By Theorem 4.3)

Conversely, $M \geq N \Rightarrow N = NM^{-}M = MN^{-}M$ (By Theorem 4.3)

$NM = \left(NM^{-}N\right)M$

$= NM^{-}\left(NM\right) = \left(NM^{-}\right)\left(MN\right)$

$= \left(NM^{-}N\right)N$

$= NN = N^2$

Similarly, $MN = N^2$

Hence $NM = MN = N^2 \Rightarrow M \overset{\#}{>} N$ (By Theorem 4.3)

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

Theorem 4.5 For $N \in (IF)^{-}_{mn}$, and $M \in (IF)_{mn}$ we have the following.

$R(M) \subseteq R(N) \Leftrightarrow C(M^t) \subseteq C(N^t)$

Proof:

$R(M) \subseteq R(N) \Leftrightarrow M = MMN^{-}N$ (Taking Transpose on both sides)

$\Leftrightarrow M^t = N^t (N^{-})^t M^t$

$\Leftrightarrow M^t = N^t (N^t)^{-} M^t$

$\Leftrightarrow C(M^t) \subseteq C(N)^t$

Theorem 4.6 Let $M, N \in (NF)^{\#}_{n}$, If $M \overset{\#}{>} N$ then $M \geq N$ and $MN^{\#}M = N$. conversely $M \geq N$, $C(MN^{\#}M) \subseteq C(N)$ and $R(MN^{\#}M) \subseteq R(N)$ imply $N \overset{\#}{>} M$.

Proof: Clearly, $M \overset{\#}{>} N \Rightarrow M > N$ with respect to $N^{\#}$ and $MN^{\#}M = N$.

Now assume $N > M$ and $C(MN^{\#}M) \subseteq C(N)$ hold. By Theorem 4.5 .Since $N > M$ and $M^{\#} \in M\{1\}$, we have , $NM^{\#}N = N, NM^{\#} M = MM^{\#}N = N$

$M(MN^{\#}M) \subseteq C(N) \Rightarrow NN^{\#}(MN^{\#}M) = MN^{\#}M$

$\Rightarrow NM^{\#}(NN^{\#}MN^{\#}M) = NM^{\#}(MN^{\#}M)$ (Premultiply by $NM^{\#}$)

$\Rightarrow NN^{\#}MN^{\#}M = NN^{\#}M = MN^{\#}M$

$\Rightarrow NN^{\#}M(M^{\#}N) = MN^{\#}M(M^{\#}N)$ (Premultiply by $M^{\#}$ N)

$\Rightarrow N = NN^{\#}N = MN^{\#}N$

$\Rightarrow NN^{\#} = MN^{\#}$ (Premultiply by $N^{\#}$)

Thus $C(MN^{\#}M) \subseteq C(N) \Rightarrow NN^{\#} = MN^{\#}$................(1)

Similarly, $R\left[(MN^{\#}M)^t\right] \subseteq R(N)$ *and* $N \geq M$

$\Rightarrow C\left[(NM^{\#}M)^t\right] \subseteq C(N)^t$ *and* $N^t \geq M^t$

$\Rightarrow N^t (N^t)^{\#} = M^t (N^t)^{\#}$ (By 1)

$\Rightarrow N^t (N^{\#})^t = M^t (N^{\#})^t$

$\Rightarrow N^{\#}N = N^{\#}M$

Hence $N \geq M$

In general $(NM)^{\#} \neq M^{\#}N^{\#}$. This is illustrated in the given example.

Example 4.1 Let us consider

$M = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <0,0,1> & <0,0,1> \end{bmatrix}$, $N = \begin{bmatrix} <1,0,0> & <0,0,1> \\ <1,0,0> & <0,0,1> \end{bmatrix}$. Here $NM = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$

Since N,M and NM are idempotent , $N^{\#} =$ N, $M^{\#} =$ M and $(NM)^{\#}$ = NM. But

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

$$M^{\#}N^{\#}=\begin{bmatrix}<1,0,0> & <1,0,0>\\ <0,0,1> & <0,0,1>\end{bmatrix}\begin{bmatrix}<1,0,0> & <0,0,1>\\ <1,0,0> & <0,0,1>\end{bmatrix}$$
$$=\begin{bmatrix}<1,1,0> & <0,1,1>\\ <0,1,1> & <0,1,1>\end{bmatrix}\neq\begin{bmatrix}<1,1,0> & <1,1,0>\\ <1,1,0> & <1,1,0>\end{bmatrix}=(QP)^{\#}$$

Hence $(\mathrm{NM})^{\#}\neq M^{\#}N^{\#}$ .Since $(\mathrm{NM})^{\#}\neq M^{\#}N^{\#}$ Theorem 4.6 can be restated involving group inverses in the following.

Theorem 4.7 Let $\mathrm{M,N}\in(\mathrm{NF})^{\#}{}_{n}$, If $M\overset{\#}{>}N$ then $M\geq N$ and $M^{\#}NM^{\#}=N^{\#}$. conversely $M\geq N$,

$C\left(M^{\#}NM^{\#}\right)\subseteq C\left(N^{\#}\right)$ *and* $R\left(M^{\#}NM^{\#}\right)\subseteq R\left(N^{\#}\right)$, *then* $N\overset{\#}{>}M$.

Proof: Clearly, $M\overset{\#}{>}N\Rightarrow M>N$ with respect to $N^{\#}$ and $M^{\#}\overset{\#}{>}N^{\#}$ follows by the definition 3.2 Theorem 4.2

Now $M^{\#}\overset{\#}{>}N^{\#}\Rightarrow N^{\#}=N^{\#}NN^{\#}=M^{\#}NM^{\#}$.

Conversely,

$M\geq N\Rightarrow NM^{\#}N=N,\mathrm{NM}^{\#}M=M^{\#}MN=N$ (By theorem 4.1)

$C\left(M^{\#}NM^{\#}\right)\subseteq C\left(M^{\#}\right)\Rightarrow N^{\#}NM^{\#}NM^{\#}=M^{\#}NM^{\#}$

$\Rightarrow N^{\#}\left(NM^{\#}N\right)M^{\#}N=M^{\#}\left(NM^{\#}N\right)$ (Post multiply by N)

$\Rightarrow N^{\#}\left(NM^{\#}N\right)=M^{\#}N$

$\Rightarrow N^{\#}M=M^{\#}N$

Silarly, $R\left(M^{\#}NM^{\#}\right)\subseteq R\left(M^{\#}\right)\Rightarrow C\left[\left(M^{\#}NM^{\#}\right)^{t}\right]\subseteq C\left[\left(M^{\#}\right)^{t}\right]$

$\Rightarrow\left(N^{t}\right)^{\#}N^{t}=\left(M^{t}\right)^{\#}N^{t}$

$\Rightarrow NN^{\#}=NM^{\#}$

Therefore $M^{\#}\overset{\#}{>}N^{\#}$ which implies $M\overset{\#}{>}N$ (By Theorem 4.2)

Hence, $M\overset{\#}{>}N$

Theorem 4.8 Let $N\in(\mathrm{NF})^{\#}{}_{n}$, and $M\in(\mathrm{NF})_{n}$, if both N and M are symmetric NFM then

$M\geq N=M^{2}\Rightarrow N^{2}=N\overset{\#}{>}M$

Proof: $M\geq N=N^{2}\Rightarrow NM^{-}M=MM^{-}N=NM^{-}N$

$M\geq N$ with M idempotent which implies N is idempotent

Now $MN=M(\mathrm{MM^{-}N})$

$MN=M^{2}M^{-}N=MN^{-}N=N=Q^{t}=(\mathrm{MN})^{t}=N^{t}M^{t}=NM$

Hence, $NM=MN=N=N^{2}$

$\Rightarrow M\overset{\#}{>}N$ (By Theorem 4.3)

Theorem 4.9 For $\mathrm{M,N}\in(\mathrm{NF})^{\#}{}_{n}$, if N is symmetric with N^{+} exists then $N\overset{\#}{>}M\Leftrightarrow N\overset{T}{>}M$.

Proof: N is symmetric which implies N is range symmetric .We know that N is range symmetric and N^{+} exists imply $N^{\#}$ exists and $N^{\#}=N^{t}$

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

Thus, $M \overset{\#}{>} N \Leftrightarrow N \overset{T}{>} M$ holds

5. Reverse Left –T and Right-T Partial ordering

Definition:5.1[7] Let $M, N \in (NF)_{mn}$ We say that M and N with respect to the left-T ordering if $N^t N = N^t M \ and \ C(N) \subseteq C(M)$ and is denoted as $Nt > M$. We say that N is below M with respect to the right –T ordering if $NN^t = MN^t \ and \ R(N) \subseteq R(M)$ and is denoted as $N > tM$.

Example:5.1 Let us consider $M = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix}, N = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <0,0,1> & <0,0,1> \end{bmatrix}$

$$N^T N = \begin{bmatrix} <1,0,0> & <0,0,1> \\ <1,0,0> & <0,0,1> \end{bmatrix} \begin{bmatrix} <1,0,0> & <1,0,0> \\ <0,0,1> & <0,0,1> \end{bmatrix} = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$$

$$N^T M = \begin{bmatrix} <1,0,0> & <0,0,1> \\ <1,1,0> & <0,0,1> \end{bmatrix} \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix} = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$$

$$N = My = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix} \begin{bmatrix} <0,1,1> & <0,1,1> \\ <1,1,0> & <1,1,0> \end{bmatrix} = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <0,0,1> & <0,0,1> \end{bmatrix}$$

$$N^T N = N^T M \ and \ C(N) \subseteq C(M)$$

$$MN^T = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix} \begin{bmatrix} <1,0,0> & <0,0,1> \\ <1,0,0> & <0,0,1> \end{bmatrix} = \begin{bmatrix} <1,1,0> & <0,1,1> \\ <1,1,0> & <0,1,1> \end{bmatrix}$$

$N^T N \neq N^T M$ and $R(N) \not\subseteq R(M)$

Example:5.2 Let us consider , $M = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix}, N = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <1,0,0> \end{bmatrix}$

$$N^t N = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$$

$$N^t M = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$$

$$y = \begin{bmatrix} <1,0> & <0,0,1> \\ <1,0> & <1,0> \end{bmatrix}$$

$$N = My = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix} \begin{bmatrix} <1,0,0> & <0,0,1> \\ <1,0,0> & <1,0,0> \end{bmatrix} = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$$

$$N = yM = \begin{bmatrix} <1,0> & <0,0> \\ <1,0> & <1,0> \end{bmatrix} \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix} = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$$

In particular $M, N \in (NF)_{mn}$ since by $M^+ = M^T$. Definition is equal to the following

$$N^+ N = N^+ M \text{ and } C(N) \subseteq C(M)$$

$$NN^+ = MN^+ \text{ and } R(N) \subseteq R(M)$$

Example:5.3 Let us consider

$$M = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix}, N = \begin{bmatrix} <0,0,1> & <1,0,0> \\ <1,0,0> & <1,0,0> \end{bmatrix}, y = \begin{bmatrix} <0,0,1> & <0,0,1> \\ <1,0,0> & <1,0,0> \end{bmatrix}$$

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

$$N^tN = \begin{bmatrix} <0,0,1> & <1,0,0> \\ <1,0,0> & <1,0,0> \end{bmatrix}\begin{bmatrix} <0,0,1> & <1,0,0> \\ <1,0,0> & <1,0,0> \end{bmatrix} = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$$

$$N^tM = \begin{bmatrix} <0,0,1> & <1,0,0> \\ <1,0,0> & <1,0,0> \end{bmatrix}\begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix} = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$$

$N^tN = N^tM$ but $C(N) \nsubseteq C(M)$

$$MN^t = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <0,0,1> \end{bmatrix}\begin{bmatrix} <0,0,1> & <1,0,0> \\ <1,0,0> & <1,0,0> \end{bmatrix} = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <1,0,0> & <1,0,0> \end{bmatrix}$$

$N^tN = MN^t$ *but* $R(N) \nsubseteq R(M)$

Hence T-ordering need not imply both left-T and right-T orderings. In the following we discuss relationship between the left –T and right-T orderings with T- ordering.

Theorem: 5.1 Let $N \in (\mathrm{NF})^+_{mn}, \mathrm{M} \in (\mathrm{NF})_{mn}, \mathrm{N}t > M$ and $N > tM \Leftrightarrow N \overset{T}{>} M$

Proof: Given, $Nt > M$ and $N > tM$ implies $N^tN = N^tM$ *and* $NN^t = MN^t \Rightarrow N \overset{T}{>} M$

Conversely, $N \overset{T}{>} M \Rightarrow N^tN = N^tM$ *and* $NN^t = MN^t$

$N^+N = N^+M$ *and* $NN^+ = MN^+$

$N \overset{T}{>} M \Rightarrow N^+N = N^+M \Rightarrow NN^+N = NN^+M$

$\Rightarrow N = XM$ where $X = NN^+$

$R(N) \subseteq R(M)$

$NN^+ = MN^+ \Rightarrow N = MN^+N \Rightarrow N = MY$ where $Y = N^+N$

$C(N) \subseteq C(M)$

Thus, $\mathrm{N} \overset{T}{>} M \Rightarrow Nt > M$ and $Mt > N$

Hence, $N \in (\mathrm{NF})^+_{mn}, \mathrm{M} \in (\mathrm{NF})_{mn}, \mathrm{N}t > M$ and $N > tM \Leftrightarrow N \overset{T}{>} M$

Theorem:5.2 Let $N \in (\mathrm{NF})^+_{mn}, \mathrm{M} \in (\mathrm{NF})^-_{mn}$, *if either* $Nt > M$ *or* $N > tM$ *then* $N \geq M$

Proof: $Nt > M \Rightarrow N^tN = N^tM$

$\Rightarrow NN^+N = NN^+M$ (Pre multiply by N and replace N^t by N^+)

$\Rightarrow N = (NN^+)M \quad \Rightarrow R(\mathrm{N}) \subseteq R(\mathrm{M})$

Now, $\mathrm{N}^tN = N^tM \Rightarrow N^+NM^-N = N^+MM^-N$ (Post multiply by MN and replace N^t by N^+)

$\Rightarrow (NN^+N)M^-N = NN^+(MM^-N)$ (Pre multiply by N)

$\Rightarrow NM^-N = N$

Hence, $Nt > M \Rightarrow C(N) \subseteq C(M), R(N) \subseteq R(M)$ *and* $NM^-N = N$

$\Rightarrow N \geq M$

Proof of $N > tM \Rightarrow N \geq M$ can be proved in the same manner.

Theorem:5.3 For $\mathrm{M}, \mathrm{N} \in (\mathrm{NF})^+_{mn}$ we have

i. $Nt > M \Leftrightarrow N^+t > M^+$

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

ii. $N > tM \Leftrightarrow N^+ > tM^+$

Proof: $Nt > M \Rightarrow N^tN = N^tM$ and $C(N) \subseteq C(M)$

Now, $C(N) \subseteq C(M) \Rightarrow N = MM^+N$

$\Rightarrow N = MM^tN$

$\Rightarrow N^t = N^tMM^t$

$\Rightarrow N^t = \left(M^tN\right)M^t$ (N^{t}M is symmetric)

$\Rightarrow N^t = M^t\left(NM^t\right)$

$\Rightarrow C\left(N^t\right) \subseteq C\left(M^t\right)$

$NN^t = N^tM \Rightarrow C\left(N^tN\right)M^t = N\left(N^tM\right)M^t$

$\Rightarrow NM^t = N\left(MM^tN\right)^t$

$\Rightarrow NM^t = NN^t \quad \left(By\ N = MM^tN\right)$

$\Rightarrow NM^+ = NN^+$

Thus $\left(N^+\right)^t M^+ = \left(N^+\right)^t N^+$ and $C\left(N^+\right) \subseteq C\left(M^+\right) \Rightarrow N^+t > M^+$

Converse follows from above part by using $\left(N^+\right)^+ = N$

Proof of (ii) is similar.

Theorem: 5.4 For $\mathrm{M}, \mathrm{N} \in (\mathrm{NF})^+_{mn}$ we have

i. $Nt > M \Leftrightarrow N^+N = NM^+ \ and \ R(N) \subseteq R(M) \Leftrightarrow N > tM$

ii. $N > tM \Leftrightarrow N^+N = M^+N$ and $C(N) \subseteq C(M) \Leftrightarrow Nt > M$.

Proof: (i) $Nt > M \Leftrightarrow \ \Leftrightarrow N^+t > M^+$

$\Leftrightarrow \left(N^+\right)^t N^+ = \left(N^+\right)^t M^+$ and $C\left(N^+\right) \subseteq C\left(M^+\right)$

$\Leftrightarrow NN^t = NM^t$ and $R(N) \subseteq R(M)$

$\Leftrightarrow NN^t = MN^t \ and \ R(N) \subseteq R(M)$ (NM^t is symmetric)

$\Leftrightarrow N > tM$

Proof of (ii) is same and hence omitted.

Theorem: 5.5 For $\mathrm{M}, \mathrm{N} \in (\mathrm{NF})^+_{mn}$ the following are equivalent

i. $Nt > M$

ii. $N > tM$

iii. $N \geq M$ and N$^+$M is symmetric.

iv. $N \geq M$ and NM$^+$ is symmetric.

Proof: $(i) \Leftrightarrow (ii)$ Follow from theorem 3.4

$(i) \Leftrightarrow (iii)$ $Nt > M \Rightarrow N \geq M$ (By Theorem 5.2)

$Nt > M \Rightarrow N^tN = N^tM$

$\Rightarrow N^tM$ symmentric

$\Rightarrow N^+M$ symmentric (By using $\mathrm{N}^+ = N^t$)

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

$(iii) \Leftrightarrow (i)$ If N^+M is symmetric then $N^+M = M^+N$ follows by replacing M^t by M^+

Since $N \geq M$ from theorem (5.2.) we have $MM^+N = N = NM^+N$

$N^TN = N^+N = N^+(MM^+N) = (N^+M)(M^+N)=M^t(NM^+N) =M^+N = N^tM$

Hence $N \geq M$

$(iii) \Leftrightarrow (iv)\, N \geq M$ *and* N^+M *is symmetric*

$NM^+ = (MM^+N)M^+ = (MM^tN)M^t = MM^tNM^t =M(M^tN)M^t = M(N^tMM^t)$
$=M(MM^tN)^t=MN^t=(NM^t)^t=(NM^+)^t$

$(iv) \Leftrightarrow (iii): N^t M = \left(NM^+M\right)M = M^t\left(MN^t\right)^t M = M^t\left(NM^TM\right) = M^tN$

Hence N^tM is symmetric.

In the above theorem 5.5, the condition N^+M and NM^+ are symmetric is essential.

Example: 5.4 Let us consider, $M = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <0,0,1> & <0,0,1> \end{bmatrix}, N = \begin{bmatrix} <1,0,0> & <1,0,0> \\ <0,0,1> & <0,0,1> \end{bmatrix}$. Clearly $N \geq M$

With respect to $M = M^-$ .Here $N^tM = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <1,1,0> & <1,1,0> \end{bmatrix}$ and $NM^t = \begin{bmatrix} <1,1,0> & <0,1,1> \\ <0,1,1> & <0,1,1> \end{bmatrix}$ are not symmetric. Here $Nt \ngeq M$.

Theorem: 5.6 For $N \in (\mathrm{NF})_{mn}$, if $N^{(1,4)}$ *and* $N^{(1,3)}$ then N^+ exists and $N^+ = N^{(1,4)}NN^{(1,3)}$

Proof: Let $Y = N^{(1,4)}NN^{(1,3)}$ one can easily verify that $Y \in N\{1,2\}$

$$NY = N\left(N^{(1,4)}NN^{(1,3)}\right) = \left(NN^{(1,4)}N\right)N^{(1,3)} = NN^{(1,3)}$$

$$\left(NY\right)^t = \left(NN^{(1,3)}\right)^t = NN^{(1,3)} = NY$$

$$YN = \left(NN^{(1,3)}N\right) = N^{(1,4)}\left(NN^{(1,3)}N\right) = N^{(1,4)}N$$

$$\left(YN\right)^t = \left(N^{(1,4)}N\right)^t = N^{(1,4)}N = YN$$

Thus, $Y \in N\{1,2,3,4\}$

Hence $Y = N^{(1,4)}NN^{(1,3)} = N^+$

4. Conclusion:

We established that for commuting pairs of matrices, sharp ordering and minus ordering are identical. We prove that under certain conditions sharp ordering reduces to the T-ordering on IFM. We establish a set of necessary condition for IFM with specified row and column spaces to be under sharp order. The concept of left-T and right-T orderings for IFM as an analogue of left-star and right-star partial orderings for complex matrices. We show that these ordering preserve its Moore-penrose inverse property. By using various generalized inverses, the new type of minus orderings is discussed. Finally, we show that these ordering are identical for certain class of IFM.

Acknowledgements

I render my heartful thanks to Prof. Dr. (Mrs.) AR. Meenakshi, Former AICTE - Emeritus Professor of Mathematics, Annamalai University, for her expert guidance and Dr. D. Jayashree, Assistant Professor, Department of Mathematics. Government Arts and Science College, Hosur

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023

References

[1] Atanassov, K., (1983), Intuitionistic Fuzzy Sets, Fuzzy Sets and System., 20, pp. 87- 96.
[2] Atanassov, K., (2005), Intuitionistic fuzzy implications and modus ponens, In Notes on Intuitionistic Fuzzy Sets., 11(1), pp. 1–5.
[3] Atanassov, K., (2005), On some types of fuzzy negations, In Notes on Intuitionistic Fuzzy Sets., 11(4), pp. 170–172.
[4] Jian Miao Chen., (1982), Fuzzy matrix partial orderings and generalized inverses, Fuzzy sets sys., 105, pp. 453 – 458.
[5] Meenakshi, AR., Inbam, C., (2004), The minus partial order in Fuzzy matrices, The Journal of Fuzzy matrices., 12 (3), pp. 695 - 700.
[6] Meenakshi, AR., (2008), Fuzzy matrix – Theory and its applications, MJP Publishers.
[7] Muthu Guru Packiam, K., Krishna Mohan, K.S., (2019), Partial orderings on k−idempotent fuzzy matrices., 15(5), pp. 733-741.
[8] Padder, R. A., Murugadas, P., (2022), Algorithm for controllable and nilpotent intuitionistic fuzzy matrices, Afr. Mat., 33(3), pp.84.
[9] Padder, R. A., Murugadas, P., (2019), Determinant theory for intuitionistic fuzzy matrices, Afr. Mat., 30, pp.943–955.
[10] Punithavalli, G., (2020), The Partial Orderings of m-Symmetric Fuzzy Matrices, International journal of Mathematics Trends and Technology., 66(4), pp.705-803.
[11] Punithavalli, G., Anandhkumar, M., (2022) , Partial Orderings On K−Idempotent Intuitionistic Fuzzy Matrices., 54(2), pp.2096-3246.
[12] Pradhan, R., Pal, M., (2014), The generalized inverse of Atanassov's intuitionistic fuzzy matrices, International Journal of Computational Intelligence Systems., 7(6), pp.1083-1095.
[13] Shyamal, A. K., Pal, M., (2002), Distance between intuitionistic fuzzy matrices, V.U.J. Physical Sciences., 8, pp.81-91.
[14] Sriram, S., Murugadas, P., (2010), The Moore-Penrose Inverse of Intuitionistic Fuzzy Matrices Int. Journal of Math. Analysis., 4(36), pp. 1779 – 1786.
[15] Thomson, M. G., (1977), Convergence of Power of a Fuzzy Matrix, Journal of Mathematical Analysis and Applications., 57, pp.476-480.
[16] Zadeh, L. A., (1965), Fuzzy Sets, Information and control., 8, pp.338-353.
[17] Smarandache F (2005) Neutrosophic set—a generalization of the intuitionistic fuzzy set. Int J Pure Appl Math 24(3):287–297
[18] Wang H, Smarandache F, Zhang YQ, Sunderraman R (2010) Single valued neutrosophic sets. Multispace Multistruct 4:410–413
[19] M. Anandhk umar, Pseudo Similarity of Neutrosophic Fuzzy matrices, International Journal of Neutrosophic Science, Vol. 20, No. 04, PP. 191-196, 2023
[20] M. Anandhkumar, On various Inverse of Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, Vol. 21, No. 02, PP. 20-31, 2023

Doi: https://doi.org/10.54216/IJNS.210413
Received: February 22, 2023 Revised: May 29, 2023 Accepted: July 23, 2023