Neutrosophic set theory applied to Hilbert algebras

Aiyared Iampan[1*], **N. Rajesh**[2], **B. Brundha**[3]
[1]Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics,
School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand
[2]Department of Mathematics, Rajah Serfoji Government College (affiliated to Bharathidasan University),
Thanjavur-613005, Tamilnadu, India
[3]Department of Mathematics, Government Arts College for Women, Orathanadu-614625, Tamilnadu, India
Emails: aiyared.ia@up.ac.th; nrajesh_topology@yahoo.co.in; brindamithunraj@gmail.com

Abstract

In this paper, the notions of neutrosophic subalgebras, neutrosophic ideals, and neutrosophic deductive systems of Hilbert algebras are introduced, and some related properties are investigated. Relations between the notions are given. Finally, we study the properties of homomorphism of Hilbert algebras.

Keywords: Hilbert algebra; neutrosophic subalgebra; neutrosophic ideal; neutrosophic deductive system.

1 Introduction and Preliminaries

Smarandache[22,23] introduced the concept of neutrosophic sets (NSs) which is a more general platform to extend the notions of the classical sets and (intuitionistic, interval-valued) fuzzy sets. Diego[7] proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated Busneag[4,5] and Jun[12] and some of their filters forming deductive systems were recognized. Dudek[8] considered the fuzzification of subalgebras and deductive systems in Hilbert algebras. In 2021, Oner et al.[20] defined a neutrosophic $\mathcal{N}$-subalgebra and a level set of a neutrosophic $\mathcal{N}$-structure on Sheffer stroke Hilbert algebras. Iampan et al.[11] introduced the concept of interval-valued neutrosophic ideals of Hilbert algebras. There are many applications of NS theory, especially in neutrosophic graphs and fuzzy graphs, which can be seen in the following articles.[13–17]

In this paper, the notions of neutrosophic subalgebras (NSAs), neutrosophic ideals (NIDs), and neutrosophic deductive systems (NDSs) of Hilbert algebras are introduced, and some related properties are investigated. Relations between the notions are given. Finally, we study the properties of homomorphism of Hilbert algebras.

Before we begin the study, let's review the definition of Hilbert algebras, which was defined by Diego[7] in 1966.

Definition 1.1. A *Hilbert algebra* is a triplet $H = (H, \cdot, 1)$, where H is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of H such that the following axioms hold:

1. $(\forall x, y \in H)(x \cdot (y \cdot x) = 1)$,
2. $(\forall x, y, z \in H)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
3. $(\forall x, y \in H)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

https://doi.org/10.54216/IJNS.210409

Received: January 28, 2023 Revised: May 20, 2023 Accepted: July 15, 2023

The following result was proved in.[8]

Lemma 1.2. *Let $H = (H, \cdot, 1)$ be a Hilbert algebra. Then*

1. $(\forall x \in H)(x \cdot x = 1)$,
2. $(\forall x \in H)(1 \cdot x = x)$,
3. $(\forall x \in H)(x \cdot 1 = 1)$,
4. $(\forall x, y, z \in H)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $H = (H, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in H)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on H with 1 as the largest element.

Definition 1.3. [6] A nonempty subset S of a Hilbert algebra $H = (H, \cdot, 1)$ is called an *subalgebra* (SA) of H if $x \cdot y \in S, \forall x, y \in S$.

Definition 1.4. [6] A nonempty subset I of a Hilbert algebra $H = (H, \cdot, 1)$ is called an *ideal* (ID) of H if

1. $1 \in I$,
2. $(\forall x \in H, \forall y \in I)(x \cdot y \in I)$,
3. $(\forall x \in H, \forall y_1, y_2 \in I)((y_1 \cdot (y_2 \cdot x)) \cdot x \in I)$.

Definition 1.5. [9] A nonempty subset D of a Hilbert algebra $H = (H, \cdot, 1)$ is called a *deductive system* (DS) of H if

1. $1 \in D$,
2. $(\forall x \in D, \forall y \in H)(x \cdot y \in D \Rightarrow y \in D)$.

A *fuzzy set*[26] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 1.6. [8] A fuzzy set μ in a Hilbert algebra $H = (H, \cdot, 1)$ is said to be a *fuzzy subalgebra* (FSA) of H if the following condition holds:

$$(\forall x, y \in H)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 1.7. [10] A fuzzy set μ in a Hilbert algebra $H = (H, \cdot, 1)$ is said to be a *fuzzy ideal* (FID) of H if the following conditions hold:

1. $(\forall x \in H)(\mu(1) \geq \mu(x))$,
2. $(\forall x, y \in H)(\mu(x \cdot y) \geq \mu(y))$,
3. $(\forall x, y_1, y_2 \in H)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

Definition 1.8. A fuzzy set μ in a Hilbert algebra $H = (H, \cdot, 1)$ is said to be a *fuzzy deductive system* (FDS) of H if the following conditions hold:

1. $(\forall x \in H)(\mu(1) \geq \mu(x))$,
2. $(\forall x, y \in H)(\mu(y) \geq \min\{\mu(x \cdot y), \mu(x)\})$.

Definition 1.9. [3] A neutrosophic set (NS) in a nonempty set H is defined to be a structure

$$A := \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in H\}, \tag{1}$$

where $T_A : H \to [0, 1]$ is a truth membership function, $I_A : H \to [0, 1]$ is an indeterminate membership function, and $F_A : H \to [0, 1]$ is a false membership function. The NS in (1) is simply denoted by $A = (T_A, I_A, F_A)$.

Received: January 28, 2023 Revised: May 20, 2023 Accepted: July 15, 2023

2 Neutrosophic Sets in Hilbert Algebras

In this section, we introduce the notions of NSAs, NIDs, and NDSs of Hilbert algebras and investigate some related properties.

Definition 2.1. An NS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $H = (H, \cdot, 1)$ is called a *neutrosophic subalgebra* (NSA) of H if

$$(\forall x, y \in H) \left(\begin{array}{l} T_A(x \cdot y) \geq \min\{T_A(x), T_A(y)\} \\ I_A(x \cdot y) \leq \max\{I_A(x), I_A(y)\} \\ F_A(x \cdot y) \geq \min\{F_A(x), F_A(y)\} \end{array} \right). \tag{2}$$

Example 2.2. Let $H = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Table 1: Cayley table for Example 2.2

Then H is a Hilbert algebra. We define an NS $A = (T_A, I_A, F_A)$ as follows:

H	1	x	y	z	0
T_A	1	0.8	0.8	0.7	0.4
I_A	0.3	0.5	0.7	0.3	0.6
F_A	1	0.8	0.8	0.7	0.4

Table 2: NSA

Then A is an NSA of H.

Proposition 2.3. *Every NSA $A = (T_A, I_A, F_A)$ of a Hilbert algebra $H = (H, \cdot, 1)$ satisfies*

$$(\forall x \in H) \left(\begin{array}{l} T_A(1) \geq T_A(x) \\ I_A(1) \leq I_A(x) \\ F_A(1) \geq F_A(x) \end{array} \right). \tag{3}$$

Proof. Assume that $A = (T_A, I_A, F_A)$ is an NSA of a Hilbert algebra $H = (H, \cdot, 1)$. For any $x \in H$, we have

$$T_A(1) = T_A(x \cdot x) \geq \min\{T_A(x), T_A(x)\} = T_A(x),$$

$$I_A(1) = I_A(x \cdot x) \leq \max\{I_A(x), I_A(x)\} = I_A(x),$$

$$F_A(1) = F_A(x \cdot x) \geq \min\{F_A(x), F_A(x)\} = F_A(x).$$

□

Definition 2.4. An NS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $H = (H, \cdot, 1)$ is called a *neutrosophic ideal* (NID) of H if

$$(\forall x \in H) \left(\begin{array}{l} T_A(1) \geq T_A(x) \\ I_A(1) \leq I_A(x) \\ F_A(1) \geq F_A(x) \end{array} \right), \tag{4}$$

$$(\forall x, y \in H) \left(\begin{array}{l} T_A(x \cdot y) \geq T_A(y) \\ I_A(x \cdot y) \leq I_A(y) \\ F_A(x \cdot y) \geq F_A(y) \end{array} \right), \tag{5}$$

$$(\forall x, y_1, y_2 \in H) \left(\begin{array}{l} T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{T_A(y_1), T_A(y_2)\} \\ I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{I_A(y_1), I_A(y_2)\} \\ F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{F_A(y_1), F_A(y_2)\} \end{array} \right). \tag{6}$$

Received: January 28, 2023 Revised: May 20, 2023 Accepted: July 15, 2023

Example 2.5. Let $H = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Table 3: Cayley table for Example 2.5

Then H is a Hilbert algebra. We define an NS $A = (T_A, I_A, F_A)$ as follows:

H	1	x	y	z	0
T_A	1	0.8	0.8	0.7	0.4
I_A	0.3	0.5	0.7	0.3	0.6
F_A	1	0.8	0.8	0.7	0.4

Table 4: NID

Then A is an NID of H.

Proposition 2.6. *If $A = (T_A, I_A, F_A)$ is an NID of a Hilbert algebra $H = (H, \cdot, 1)$, then*

$$(\forall x, y \in H) \left(\begin{array}{l} T_A((y \cdot x) \cdot x) \geq T_A(y) \\ I_A((y \cdot x) \cdot x) \leq I_A(y) \\ F_A((y \cdot x) \cdot x) \geq F_A(y) \end{array} \right). \tag{7}$$

Proof. Assume that $A = (T_A, I_A, F_A)$ is an NID of a Hilbert algebra $H = (H, \cdot, 1)$. Putting $y_1 = y$ and $y_2 = 1$ in (6), we have

$$T_A((y \cdot x) \cdot x) \geq \min\{T_A(y), T_A(1)\} = T_A(y),$$
$$I_A((y \cdot x) \cdot x) \leq \max\{I_A(y), I_A(1)\} = I_A(y),$$
$$F_A((y \cdot x) \cdot x) \geq \min\{F_A(y), F_A(1)\} = F_A(y).$$

□

Lemma 2.7. *If $A = (T_A, I_A, F_A)$ is an NID of a Hilbert algebra $H = (H, \cdot, 1)$, then*

$$(\forall x, y \in H) \left(x \leq y \Rightarrow \left\{ \begin{array}{l} T_A(x) \leq T_A(y) \\ I_A(x) \geq I_A(y) \\ F_A(x) \leq F_A(y) \end{array} \right. \right). \tag{8}$$

Proof. Assume that $A = (T_A, I_A, F_A)$ is an NID of a Hilbert algebra $H = (H, \cdot, 1)$. Let $x, y \in H$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{array}{rcl} T_A(y) & = & T_A(1 \cdot y) \\ & = & T_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ & \geq & \min\{T_A(x \cdot y), T_A(x)\} \\ & \geq & \min\{T_A(1), T_A(x)\} \\ & = & T_A(x), \end{array}$$

$$\begin{array}{rcl} I_A(y) & = & I_A(1 \cdot y) \\ & = & I_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ & \leq & \max\{I_A(x \cdot y), I_A(x)\} \\ & \leq & \max\{I_A(1), I_A(x)\} \\ & = & I_A(x), \end{array}$$

$$\begin{array}{rcl} F_A(y) & = & F_A(1 \cdot y) \\ & = & F_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ & \geq & \min\{F_A(x \cdot y), F_A(x)\} \\ & \geq & \min\{F_A(1), F_A(x)\} \\ & = & F_A(x). \end{array}$$

□

Received: January 28, 2023 Revised: May 20, 2023 Accepted: July 15, 2023

Theorem 2.8. *Every NID of a Hilbert algebra $H = (H, \cdot, 1)$ is an NSA of H.*

Proof. Assume that $A = (T_A, I_A, F_A)$ is an NID of a Hilbert algebra $H = (H, \cdot, 1)$. Let $x, y \in H$. Then $y \leq x \cdot y$ as $y \cdot (x \cdot y) = 1$ and by Lemma 2.7, we have

$$T_A(y) \geq T_A(x \cdot y), I_A(y) \leq I_A(x \cdot y), \text{ and } F_A(y) \geq F_A(x \cdot y).$$

It follows from (5) that

$$\begin{aligned} T_A(x \cdot y) &\geq T_A(y) \\ &\geq \min\{T_A(x \cdot y), T_A(x)\} \\ &\geq \min\{T_A(x), T_A(y)\}, \\ I_A(x \cdot y) &\leq I_A(y) \\ &\leq \max\{I_A(x \cdot y), I_A(x)\} \\ &\leq \max\{I_A(x), I_A(y)\}, \\ F_A(x \cdot y) &\geq F_A(y) \\ &\geq \min\{F_A(x \cdot y), F_A(x)\} \\ &\geq \min\{F_A(x), F_A(y)\}. \end{aligned}$$

Hence, A is an NSA of H. □

Proposition 2.9. *If $\{A_i = (T_{A_i}, I_{A_i}, F_{A_i}) : i \in \Delta\}$ is a family of NIDs of a Hilbert algebra $H = (H, \cdot, 1)$, then $\bigwedge_{i \in \Delta} A_i$ is an NID of H.*

Proof. Assume that $\{A_i = (T_{A_i}, I_{A_i}, F_{A_i}) : i \in \Delta\}$ is a family of NIDs of a Hilbert algebra $H = (H, \cdot, 1)$. Let $x \in H$. Then

$$(\bigwedge_{i \in \Delta} T_{A_i})(1) = \inf_{i \in \Delta}\{T_{A_i}(1)\} \geq \inf_{i \in \Delta}\{T_{A_i}(x)\} = (\bigwedge_{i \in \Delta} T_{A_i})(x),$$

$$(\bigwedge_{i \in \Delta} I_{A_i})(1) = \sup_{i \in \Delta}\{I_{A_i}(1)\} \leq \sup_{i \in \Delta}\{I_{A_i}(x)\} = (\bigwedge_{i \in \Delta} I_{A_i})(x),$$

$$(\bigwedge_{i \in \Delta} F_{A_i})(1) = \inf_{i \in \Delta}\{F_{A_i}(1)\} \geq \inf_{i \in \Delta}\{F_{A_i}(x)\} = (\bigwedge_{i \in \Delta} F_{A_i})(x).$$

Let $x, y \in H$. Then

$$(\bigwedge_{i \in \Delta} T_{A_i})(x \cdot y) = \inf_{i \in \Delta}\{T_{A_i}(x \cdot y)\} \geq \inf_{i \in \Delta}\{T_{A_i}(y)\} = (\bigwedge_{i \in \Delta} T_{A_i})(y),$$

$$(\bigwedge_{i \in \Delta} I_{A_i})(x \cdot y) = \sup_{i \in \Delta}\{I_{A_i}(x \cdot y)\} \leq \sup_{i \in \Delta}\{I_{A_i}(y)\} = (\bigwedge_{i \in \Delta} I_{A_i})(y),$$

$$(\bigwedge_{i \in \Delta} F_{A_i})(x \cdot y) = \inf_{i \in \Delta}\{F_{A_i}(x \cdot y)\} \geq \inf_{i \in \Delta}\{F_{A_i}(y)\} = (\bigwedge_{i \in \Delta} F_{A_i})(y).$$

Let $x, y_1, y_2 \in H$. Then

$$\begin{aligned} (\bigwedge_{i \in \Delta} T_{A_i})((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \inf_{i \in \Delta}\{T_{A_i}((y_1 \cdot (y_2 \cdot x)) \cdot x)\} \\ &\geq \inf_{i \in \Delta}\{\min\{T_{A_i}(y_1), T_{A_i}(y_2)\}\} \\ &= \min\{\inf_{i \in \Delta} T_{A_i}(y_1), \inf_{i \in \Delta} T_{A_i}(y_2)\} \\ &= \min\{(\bigwedge_{i \in \Delta} T_{A_i})(y_1), (\bigwedge_{i \in \Delta} T_{A_i})(y_2)\}, \\ (\bigwedge_{i \in \Delta} I_{A_i})((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \sup_{i \in \Delta}\{I_{A_i}((y_1 \cdot (y_2 \cdot x)) \cdot x)\} \\ &\leq \sup_{i \in \Delta}\{\max\{I_{A_i}(y_1), I_{A_i}(y_2)\}\} \\ &= \max\{\sup_{i \in \Delta} I_{A_i}(y_1), \sup_{i \in \Delta} I_{A_i}(y_2)\} \\ &= \max\{(\bigwedge_{i \in \Delta} I_{A_i})(y_1), (\bigwedge_{i \in \Delta} I_{A_i})(y_2)\}, \end{aligned}$$

Received: January 28, 2023 Revised: May 20, 2023 Accepted: July 15, 2023

$$\begin{aligned}(\bigwedge_{i\in\Delta} F_{A_i})((y_1\cdot(y_2\cdot x))\cdot x) &= \inf_{i\in\Delta}\{F_{A_i}((y_1\cdot(y_2\cdot x))\cdot x)\}\\ &\geq \inf_{i\in\Delta}\{\min\{F_{A_i}(y_1),F_{A_i}(y_2)\}\}\\ &= \min\{\inf_{i\in\Delta} F_{A_i}(y_1),\inf_{i\in\Delta} F_{A_i}(y_2)\}\\ &= \min\{(\bigwedge_{i\in\Delta} F_{A_i})(y_1),(\bigwedge_{i\in\Delta} F_{A_i})(y_2)\}.\end{aligned}$$

Hence, $\bigwedge_{i\in\Delta} A_i$ is an NID of H. □

Proposition 2.10. *If* $\{A_i = (T_{A_i}, I_{A_i}, F_{A_i}) : i \in \Delta\}$ *is a family of NSAs of a Hilbert algebra* $H = (H,\cdot,1)$*, then* $\bigwedge_{i\in\Delta} A_i$ *is an NSA of* H*.*

Proof. Similar to the proof of Proposition 2.9. □

Definition 2.11. An NS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $H = (H,\cdot,1)$ is called a *neutrosophic deductive system* (NDS) of H if

$$(\forall x\in H)\begin{pmatrix} T_A(1)\geq T_A(x)\\ I_A(1)\leq I_A(x)\\ F_A(1)\geq F_A(x)\end{pmatrix}, \tag{9}$$

$$(\forall x,y\in H)\begin{pmatrix} T_A(y)\geq \min\{T_A(x\cdot y),T_A(x)\}\\ I_A(y)\leq \max\{I_A(x\cdot y),T_A(x)\}\\ F_A(y)\geq \min\{F_A(x\cdot y),F_A(x)\}\end{pmatrix}. \tag{10}$$

Proposition 2.12. *Every NID of a Hilbert algebra* $H = (H,\cdot,1)$ *is an NDS of* H*.*

Proof. Assume that $A = (T_A, I_A, F_A)$ is an NID of a Hilbert algebra $H = (H,\cdot,1)$. Let $x, y \in H$. If $y_1 = x\cdot y$, $y_2 = x$ and by (1), (2) of Lemma 1.2 and (6), we have

$$T_A(y) = T_A(1\cdot y) = T_A(((x\cdot y)\cdot(x\cdot y))\cdot y) \geq \min\{T_A(x\cdot y), T_A(x)\},$$

$$I_A(y) = I_A(1\cdot y) = I_A(((x\cdot y)\cdot(x\cdot y))\cdot y) \leq \max\{I_A(x\cdot y), I_A(x)\},$$

$$F_A(y) = F_A(1\cdot y) = F_A(((x\cdot y)\cdot(x\cdot y))\cdot y) \geq \min\{F_A(x\cdot y), F_A(x)\}.$$

Hence, A is an NDS of H. □

Proposition 2.13. *If* $\{A_i = (T_{A_i}, I_{A_i}, F_{A_i}) : i \in \Delta\}$ *is a family of NDSs of a Hilbert algebra* $H = (H,\cdot,1)$*, then* $\bigwedge_{i\in\Delta} A_i$ *is an NDS of* H*.*

Proof. Similar to the proof of Proposition 2.9. □

Definition 2.14. Let μ be a fuzzy set in a nonempty set H. For any $t \in [0,1]$, the sets

$$U(\mu,t) = \{x\in H : \mu(x)\geq t\},$$

$$L(\mu,t) = \{x\in H : \mu(x)\leq t\},$$

$$E(\mu,t) = \{x\in H : \mu(x) = t\},$$

are called an upper t-level subset, a lower t-level subset, and an equal t-level subset of μ, respectively.

Theorem 2.15. *An NS* $A = (T_A, I_A, F_A)$ *in a Hilbert algebra* $H = (H,\cdot,1)$ *is an NID of* H *if and only if for all* $t \in [0,1]$*, the sets* $U(T_A,t)$*,* $L(I_A,t)$*, and* $U(F_A,t)$ *are either empty or IDs of* H*.*

Proof. Assume that $A = (T_A, I_A, F_A)$ is an NID of a Hilbert algebra $H = (H, \cdot, 1)$ and let $t \in [0, 1]$ be such that $U(T_A, t)$, $L(I_A, t)$, and $U(F_A, t)$ are nonempty subsets of H. It is clear that $1 \in U(T_A, t) \cap L(I_A, t) \cap U(F_A, t)$ since $T_A(1) \geq T_A(x) \geq t, I_A(1) \leq I_A(x) \leq t$, and $F_A(1) \geq F_A(x) \geq t$ for all $x \in U(T_A, t), x \in L(I_A, t)$, and $x \in U(F_A, t)$. Let $x \in H$ and $y \in U(T_A, t)$. Then $T_A(y) \geq t$. It follows that $T_A(x \cdot y) \geq T_A(y) \geq t$ so that $x \cdot y \in U(T_A, t)$. Let $x \in H$ and $y_1, y_2 \in U(T_A, t)$. Then $T_A(y_1) \geq t$ and $T_A(y_2) \geq t$. Hence, $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{T_A(y_1), T_a(y_2)\} \geq t$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(T_A, t)$. Hence, $U(T_A, t)$ is an ID of H. Let $x \in H$ and $y \in L(I_A, t)$. Then $I_A(y) \leq t$. It follows that $I_A(x \cdot y) \leq I_A(y) \leq t$ so that $x \cdot y \in L(I_A, t)$. Let $x \in H$ and $y_1, y_2 \in L(I_A, t)$. Then $I_A(y_1) \leq t$ and $I_A(y_2) \leq t$. Hence, $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{I_A(y_1), I_A(y_2)\} \leq t$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(I_A, t)$. Hence, $L(I_A, s)$ is an ID of H. Let $x \in H$ and $y \in U(F_A, t)$. Then $F_A(y) \geq t$. It follows that $F_A(x \cdot y) \geq F_A(y) \geq t$ so that $x \cdot y \in U(F_A, t)$. Let $x \in H$ and $y_1, y_2 \in U(F_A, t)$. Then $F_A(y_1) \geq t$ and $F_A(y_2) \geq t$. Hence, $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{F_A(y_1), F_A(y_2)\} \geq t$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(F_A, t)$. Hence, $U(F_A, t)$ is an ID of H.

Conversely, assume now that every nonempty subsets $U(T_A, t), L(I_A, t)$, and $U(F_A, t)$ are IDs in H for all $t \in [0, 1]$. If $T_A(1) \geq T_A(x)$ is not true for all $x \in H$, then there exists $x_0 \in H$ such that $T_A(1) < T_A(x_0)$. Choose $t = \frac{1}{2}(T_A(1) + T_A(x_0))$. Then $x_0 \in U(T_A, t)$, that is, $U(T_A, t) \neq \emptyset$. Since by the assumption, $U(T_A, t)$ is an ID of H. Thus $T_A(1) \geq t$, which is impossible. Hence, $T_A(1) \geq T_A(x)$ for all $x \in H$. If $I_A(1) \leq I_A(x)$ is not true for all $x \in H$, then there exists $x_0 \in H$ such that $I_A(1) > I_A(x_0)$. Choose $t = \frac{1}{2}(I_A(1) + I_A(x_0))$. Then $x_0 \in L(I_A, t)$, that is, $L(I_A, t) \neq \emptyset$. Since by the assumption, $L(I_A, t)$ is an ID of H. Thus $I_A(1) \leq t$, which is impossible. Hence, $I_A(1) \leq I_A(x)$ for all $x \in H$. If $F_A(1) \geq F_A(x)$ is not true for all $x \in H$, then there exists $x_0 \in H$ such that $F_A(1) < F_A(x_0)$. Choose $t = \frac{1}{2}(F_A(1) + F_A(x_0))$. Then $x_0 \in U(F_A, t)$, that is, $U(F_A, t) \neq \emptyset$. Since by the assumption, $U(F_A, t)$ is an ID of H, then $F_A(1) \geq t$, which is impossible. Hence, $F_A(1) \geq F_A(x)$ for all $x \in H$. If $T_A(x \cdot y) \geq T_A(y)$ is not true for all $x, y \in H$, then there exist $x_0, y_0 \in H$ such that $T_A(x_0 \cdot y_0) < T_A(y_0)$. Let $t = \frac{1}{2}(T_A(x_0 \cdot y_0) + T_A(y_0))$. Then $t \in [0, 1]$ and $T_A(x_0 \cdot y_0) < t < T_A(y_0)$, which prove that $y_0 \in U(T_A, t)$. Since $U(T_A, t)$ is an ID of H, $x_0 \cdot y_0 \in U(T_A, t)$. Hence, $T_A(x_0 \cdot y_0) \geq t$, a contradiction. Thus $T_A(x \cdot y) \geq T_A(y)$ is true for all $x, y \in H$. If $I_A(x \cdot y) \leq I_A(y)$ is not true for all $x, y \in H$, then there exist $x_0, y_0 \in H$ such that $I_A(x_0 \cdot y_0) > I_A(y_0)$. Let $t = \frac{1}{2}(I_A(x_0 \cdot y_0) + I_A(y_0))$. Then $t \in [0, 1]$ and $I_A(x_0 \cdot y_0) > t > I_A(y_0)$, which prove that $y_0 \in L(I_A, t)$. Since $L(I_A, t)$ is an ID of H, $x_0 \cdot y_0 \in L(I_A, t)$. Hence, $I_A(x_0 \cdot y_0) \leq t$, a contradiction. Thus $I_A(x \cdot y) \leq I_A(y)$ is true for all $x, y \in H$. If $F_A(x \cdot y) \geq F_A(y)$ is not true for all $x, y \in H$, then there exist $x_0, y_0 \in H$ such that $F_A(x_0 \cdot y_0) < F_A(y_0)$. Let $t = \frac{1}{2}(F_A(x_0 \cdot y_0) + F_A(y_0))$. Then $t \in [0, 1]$ and $F_A(x_0 \cdot y_0) < t < F_A(y_0)$, which prove that $y_0 \in U(F_A, t)$. Since $U(F_A, t)$ is an ID of H, $x_0 \cdot y_0 \in U(F_A, t)$. Hence, $F_A(x_0 \cdot y_0) \geq t$, a contradiction. Thus $F_A(x \cdot y) \geq F_A(y)$ is true for all $x, y \in H$. Suppose that $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{T_A(y_1), T_A(y_2)\}$ is not true for all $x, y_1, y_2 \in H$. Then there exist $u_0, v_0, x_0 \in H$ such that $T_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) < \min\{T_A(u_0), T_A(v_0)\}$. Taking $t = \frac{1}{2}(T_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) + \min\{T_A(u_0), T_A(v_0)\})$. Then we have $T_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) < t < \min\{T_A(u_0), T_A(v_0)\}$, which prove that $u_0, v_0 \in U(T_A, t)$. Since $U(T_A, t)$ is an ID of H, $(u_0 \cdot (v_0 \cdot x_0))x_0 \in U(T_A, t)$, a contradiction. Thus $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{T_A(y_1), T_A(y_2)\}$ is true for all $x, y_1, y_2 \in H$. Suppose that $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{I_A(y_1), I_A(y_2)\}$ is not true for all $x, y_1, y_2 \in H$. Then there exist $u_0, v_0, x_0 \in H$ such that $I_A((u_0 \cdot (v_0 \cdot x_0) \cdot x_0) > \max\{I_A(u_0), I_A(v_0)\}$. Taking $t = \frac{1}{2}(I_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) + \max\{I_A(u_0), I_A(v_0)\})$. Then we have $I_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) > t > \max\{I_A(u_0), I_A(v_0)\}$, which prove that $u_0, v_0 \in L(I_A, t)$. Since $L(I_A, t)$ is an ID of H, $(u_0 \cdot (v_0 \cdot x_0)) \cdot x_0 \in L(I_A, t)$, a contradiction. Thus $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{I_A(y_1), I_A(y_2)\}$ is true for all $x, y_1, y_2 \in H$. Suppose that $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{F_A(y_1), F_A(y_2)\}$ is not true for all $x, y_1, y_2 \in H$. Then there exist $u_0, v_0, x_0 \in H$ such that $F_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) < \min\{F_A(u_0), F_A(v_0)\}$. Taking $t = \frac{1}{2}(F_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) + \min\{F_A(u_0), F_A(v_0)\})$. Then we have $F_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) < t < \min\{F_A(u_0), F_A(v_0)\}$, which prove that $u_0, v_0 \in U(F_A, t)$. Since $U(F_A, t)$ is an ID of H, $(u_0 \cdot (v_0 \cdot x_0))x_0 \in U(F_A, t)$, a contradiction. Thus $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{F_A(y_1), F_A(y_2)\}$ is true for all $x, y_1, y_2 \in H$. Hence, A is an NID of H. □

Theorem 2.16. *An NS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $H = (H, \cdot, 1)$ is an NSA of H if and only if for all $t \in [0, 1]$, the sets $U(T_A, t)$, $L(I_A, t)$, and $U(F_A, t)$ are either empty or SAs of H.*

Proof. Similar to the proof of Theorem 2.15. □

Theorem 2.17. *An NS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $H = (H, \cdot, 1)$ is an NDS of H if and only if for all $t \in [0, 1]$, the sets $U(T_A, t)$, $L(I_A, t)$, and $U(F_A, t)$ are either empty or DSs of H.*

Received: January 28, 2023 Revised: May 20, 2023 Accepted: July 15, 2023

Proof. Similar to the proof of Theorem 2.15. ☐

A mapping $f : X \to Y$ of Hilbert algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1) = 1$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras. For any NS $A = (T_A, I_A, F_A)$ in Y, we define a new NS $f^{-1}(A) = (T_{f^{-1}(A)}, I_{f^{-1}(A)}, F_{f^{-1}(A)})$ in X by

$$(\forall x \in X) \left(\begin{array}{l} T_{f^{-1}(A)}(x) = T_A(f(x)) \\ I_{f^{-1}(A)}(x) = I_A(f(x)) \\ F_{f^{-1}(A)}(x) = F_A(f(x)) \end{array} \right).$$

Theorem 2.18. *Let $f : X \to Y$ be a homomorphism of Hilbert algebras and $A = (T_A, I_A, F_A)$ be an NS in Y. If $A = (T_A, I_A, F_A)$ is an NID of Y, then $f^{-1}(A) = (T_{f^{-1}(A)}, I_{f^{-1}(A)}, F_{f^{-1}(A)})$ is an NID of X.*

Proof. Assume that $A = (T_A, I_A, F_A)$ is an NID of Y. Since f is a homomorphism of X into Y, then $f(1) = 1 \in Y$ and, by the assumption, $T_A(f(1)) = T_A(1) \geq T_A(y)$ for every $y \in Y$. In particular, $T_A(f(1)) \geq T_A(f(x))$ for all $x \in X$. Hence, $T_{f^{-1}(A)}(1) \geq T_{f^{-1}(A)}(x)$. Also $I_A(f(1)) = I_A(1) \leq I_A(y)$ for every $y \in Y$. In particular, $I_A(f(1)) \leq I_A(f(x))$ for all $x \in X$. Hence, $I_{f^{-1}(A)}(1) \leq I_{f^{-1}(A)}(x)$ for all $x \in X$. Furthermore, $F_A(f(1)) = F_A(1) \geq F_A(y)$ for all $y \in Y$. In particular, $F_A(f(1)) \geq F_A(f(x))$ for all $x \in X$. Hence, $F_{f^{-1}(A)}(1) \geq F_{f^{-1}(A)}(x)$ or all $x \in X$, which proves (4). Let $x, y \in X$. Then, by the assumption, we have

$$T_{f^{-1}(A)}(x \cdot y) = T_A(f(x \cdot y)) = T_A(f(x) \cdot f(y) \geq T_A(f(y)) = T_{f^{-1}(A)}(y),$$

$$I_{f^{-1}(A)}(x \cdot y) = I_A(f(x \cdot y)) = I_A(f(x) \cdot f(y) \leq I_A(f(y)) = I_{f^{-1}(A)}(y),$$

$$F_{f^{-1}(A)}(x \cdot y) = F_A(f(x \cdot y)) = F_A(f(x) \cdot f(y) \geq F_A(f(y)) = F_{f^{-1}(A)}(y),$$

which proves (5). Let $x, y_1, y_2 \in X$. Then, by the assumption, we have

$$\begin{aligned} T_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x)) &= T_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\ &= T_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\ &= T_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\ &= T_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\ &\geq \min\{T_A(f(y_1)), T_A(f(y_2))\} \\ &= \min\{T_{f^{-1}(A)}(y_1), T_{f^{-1}(A)}(y_2)\}, \end{aligned}$$

$$\begin{aligned} I_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x)) &= I_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\ &= I_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\ &= I_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\ &= I_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\ &\leq \max\{I_A(f(y_1)), I_A(f(y_2))\} \\ &= \max\{I_{f^{-1}(A)}(y_1), I_{f^{-1}(A)}(y_2)\}, \end{aligned}$$

$$\begin{aligned} F_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x)) &= F_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\ &= F_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\ &= F_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\ &= F_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\ &\geq \min\{F_A(f(y_1)), F_A(f(y_2))\} \\ &= \min\{F_{f^{-1}(A)}(y_1), F_{f^{-1}(A)}(y_2)\}, \end{aligned}$$

which proves (6). Hence, $f^{-1}(A)$ is an NID of X. ☐

Theorem 2.19. *Let $f : X \to Y$ be a homomorphism of Hilbert algebras and $A = (T_A, I_A, F_A)$ be an NS in Y. If $A = (T_A, I_A, F_A)$ is an NSA (resp., NDS) of Y, then $f^{-1}(A) = (T_{f^{-1}(A)}, I_{f^{-1}(A)}, F_{f^{-1}(A)})$ is an NSA (resp., NDS) of X.*

Proof. Similar to the proof of Theorem 2.18. ☐

3 Conclusions and Future Works

We have introduced and studied the concepts of NSAs, NIDs, and NDSs in Hilbert algebras and investigated some of their properties. We also studied inverse images of homomorphisms under an NSA, an NID, and an NDS.

The research topics of interest by our research team being studied in Hilbert algebras are as follows:

(1) to study int-soft ideals over the soft sets in Hilbert algebras based on the concept of Muhiuddin and Mahboob,[19]

(2) to study $\mathcal{N}$-ideals theory in Hilbert algebras based on $\mathcal{N}$-structures using the concept of Muhiuddin et al.,[1,18]

(3) to introduce the concept of bipolar (λ, δ)-fuzzy subalgebras and bipolar (λ, δ)-fuzzy ideals based on the concept of Ansari et al.,[2,25]

(4) to introduce the concept of neutrosophic $\mathcal{N}$-structures based on the concept of Rangsuk et al.,[21]

(5) to introduce the concept of implicative, comparative, and shift of NSs based on the concept of Songsaeng et al.[24]

Acknowledgment

This research project was supported by the Thailand Science Research and Innovation Fund and the University of Phayao (Grant No. FF66-UoE017).

Conflicts of Interest: The authors declare no conflict of interest.

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https://doi.org/10.54216/IJNS.210409

Received: January 28, 2023 Revised: May 20, 2023 Accepted: July 15, 2023