



A Short Contribution to Split-Complex Linear Diophantine Equations in Two Variables

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Abstract

In this work, we study the split-complex integer solutions for the split-complex linear Diophantine equation in two variables $AX + BY = C$ where A, B, C, X, Y are split-complex integers. An algorithm for generating all solutions will be obtained by transforming the split-complex equation to a classical equivalent system of linear Diophantine equations in four variables.

Keywords: split-complex integer; split-complex Diophantine equation; general solution.

1. Introduction and preliminaries

Split-complex numbers are considered as a generalization of real numbers. The commutative ring of split-complex numbers is defined as follows:

$$S = \{a + bJ; a, b \in R, J^2 = 1\}.$$

This algebraic class of numbers was studied by many authors from many different sides such as geometry, physics, and linear spaces [1,3,4]. It is good to for the reader to check another similar structure (weal fuzzy complex numbers) [2,5].

The set of split-complex integers makes a commutative subring of S , where it is defined as:

$$S - I = \{a + bJ; a, b \in Z, J^2 = 1\}. [6]$$

The split-complex integers and number theory were studied in [6], where many concepts such as division, congruencies, and Euler's function were handled in the split-complex integer version.

These concepts have helped with generalizing RSA algorithm into a novel version depending on split-complex integers [6].

In this paper, we study the split-complex linear Diophantine equation in two variables, where a general algorithm for its solutions and solvability will be provided with an example.

This work lies under the section of number theory (non-classical number theory), and it can be considered as the first try with Diophantine equations defined over split-complex integer ring.

Main discussion.

Definition:

Let $A = a_1 + a_2J, B = b_1 + b_2J, C = c_1 + c_2J$ are three split-complex integers, the split-complex linear Diophantine equation in two variables is defined as follows:

$$AX + BY = C; X = x_1 + x_2J, Y = y_1 + y_2J \text{ are split-complex integers.}$$

Theorem.

Let $AX + BY = C$ be a split-complex linear Diophantine equation in two variables, then it is solvable if and only if:

$$\begin{cases} gcd(a_1 + a_2, b_1 + b_2) \mid c_1 + c_2 \\ gcd(a_1 - a_2, b_1 - b_2) \mid c_1 - c_2 \end{cases}$$

Proof.

The equation $AX + BY = C$ is equivalent to:

$$(a_1 + a_2J)(x_1 + x_2J) + (b_1 + b_2J)(y_1 + y_2J) = c_1 + c_2J$$

$$\Rightarrow \begin{cases} a_1x_1 + b_1y_1 + a_2x_2 + b_2y_2 = c_1 & (1) \\ a_2x_1 + a_1x_2 + b_1y_2 + b_2y_1 = c_2 & (2) \end{cases}$$

Assume that $AX + BY = C$ is solvable, then (1), (2) are solvable.

By adding (1) to (2) and subtracting (2) from (1), we get:

$$\begin{cases} (a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2 \\ (a_1 - a_2)(x_1 - x_2) + (b_1 - b_2)(y_1 - y_2) = c_1 - c_2 \end{cases}$$

$$\begin{cases} (a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2 \\ (a_1 - a_2)(x_1 - x_2) + (b_1 - b_2)(y_1 - y_2) = c_1 - c_2 \end{cases}$$

Thus, $\gcd(a_1 + a_2, b_1 + b_2) \mid c_1 + c_2$ and $\gcd(a_1 - a_2, b_1 - b_2) \mid c_1 - c_2$.

For the converse, suppose that $\gcd(a_1 + a_2, b_1 + b_2) \mid c_1 + c_2$, $\gcd(a_1 - a_2, b_1 - b_2) \mid c_1 - c_2$, then the following two Diophantine equations are solvable:

$$\begin{cases} (a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2 \\ (a_1 - a_2)(x_1 - x_2) + (b_1 - b_2)(y_1 - y_2) = c_1 - c_2 \end{cases}$$

$$\begin{cases} (a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2 \\ (a_1 - a_2)(x_1 - x_2) + (b_1 - b_2)(y_1 - y_2) = c_1 - c_2 \end{cases}$$

Thus, the Diophantine equation are solvable:

$$\begin{cases} a_1x_1 + a_1x_2 + a_2x_1 + a_2x_2 + b_1y_1 + b_1y_2 + b_2y_1 + b_2y_2 = c_1 + c_2 & (1) \\ a_1x_1 + a_2x_2 - a_1x_2 - a_2x_1 + b_1y_1 - b_1y_2 - b_2y_1 + b_2y_2 = c_1 - c_2 & (2) \end{cases}$$

$$\begin{cases} a_1x_1 + a_2x_2 - a_1x_2 - a_2x_1 + b_1y_1 - b_1y_2 - b_2y_1 + b_2y_2 = c_1 - c_2 & (2) \end{cases}$$

Adding (1) to (2), and subtracting (2) from (1) gives:

$$\begin{cases} a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2 = c_1 \\ a_2x_1 + a_1x_2 + b_1y_2 + b_2y_1 = c_2 \end{cases}$$

$$\begin{cases} a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2 = c_1 \\ a_2x_1 + a_1x_2 + b_1y_2 + b_2y_1 = c_2 \end{cases}$$

This implies:

$$(a_1 + a_2J)(x_1 + x_2J) + (b_1 + b_2J)(y_1 + y_2J) = c_1 + c_2J \text{ is solvable.}$$

Hence, our proof is complete.

Algorithm for the solution.

To solve the split-complex linear Diophantine equation in two variables $AX + BY = C$, follow these steps:

Step (1).

Write the equivalent system:

$$\begin{cases} (a_1 + a_2)Z_1 + (b_1 + b_2)T_1 = c_1 + c_2 & (1) \\ (a_1 - a_2)Z_2 + (b_1 - b_2)T_2 = c_1 - c_2 & (2) \end{cases}; Z_1 = x_1 + x_2, Z_2 = x_1 - x_2, T_1 = y_1 + y_2, T_2 = y_1 - y_2$$

$$\begin{cases} (a_1 + a_2)Z_1 + (b_1 + b_2)T_1 = c_1 + c_2 & (1) \\ (a_1 - a_2)Z_2 + (b_1 - b_2)T_2 = c_1 - c_2 & (2) \end{cases}; Z_1 = x_1 + x_2, Z_2 = x_1 - x_2, T_1 = y_1 + y_2, T_2 = y_1 - y_2$$

Step (2).

Let (Z_1^*, T_1^*) be a solution for (1), and (Z_2^*, T_2^*) be a solution for (2), then:

$$d_1 = \gcd(a_1 + a_2, b_1 + b_2), d_2 = \gcd(a_1 - a_2, b_1 - b_2)$$

$$\begin{cases} Z_1 = Z_1^* + k_1 \frac{b_1 + b_2}{d_1} \\ T_1 = T_1^* - k_1 \frac{a_1 + a_2}{d_1} \end{cases}; k_1 \in \mathbb{Z}$$

Also,

$$\begin{cases} Z_2 = Z_2^* + k_2 \frac{b_1 - b_2}{d_2} \\ T_2 = T_2^* - k_2 \frac{a_1 - a_2}{d_2} \end{cases}; k_2 \in \mathbb{Z}$$

Step (3).

The solution of $AX + BY = C$ is:

$$X = \frac{1}{2} \left[Z_1^* + Z_2^* + k_1 \frac{b_1 + b_2}{d_1} + k_2 \frac{b_1 - b_2}{d_2} \right] + \frac{1}{2} J \left[T_1^* + T_2^* - k_1 \frac{a_1 + a_2}{d_1} - k_2 \frac{a_1 - a_2}{d_2} \right]$$

$$Y = \frac{1}{2} \left[Z_1^* - Z_2^* + k_1 \frac{b_1 + b_2}{d_1} - k_2 \frac{b_1 - b_2}{d_2} \right] + \frac{1}{2} J \left[T_1^* - T_2^* - k_1 \frac{a_1 + a_2}{d_1} + k_2 \frac{a_1 - a_2}{d_2} \right]$$

With $k_1, k_2 \in \mathbb{Z}$.

Example.

Consider the following split-complex linear Diophantine equation:

$$(2 + J)X + (5 - J)Y = 14 + 23J, \text{ we have:}$$

$$\begin{cases} a_1 = 2, a_2 = 1 \\ b_1 = 5, b_2 = -1 \\ c_1 = 14, c_2 = 23 \end{cases}$$

Also,

$$\begin{cases} a_1 + a_2 = 3, a_1 - a_2 = 1 \\ b_1 + b_2 = 4, b_1 - b_2 = 6 \\ c_1 + c_2 = 37, c_1 - c_2 = -9 \end{cases}$$

$$\gcd(a_1 + a_2, b_1 + b_2) = \gcd(3, 4) = 1 \mid c_1 + c_2$$

$$\gcd(a_1 - a_2, b_1 - b_2) = \gcd(1, 6) = 1 \mid c_1 - c_2$$

Thus, it is solvable.

The equivalent system is:

$$\begin{cases} 3Z_1 + 4T_1 = 37 \\ Z_2 + 6T_2 = -9 \end{cases}$$

We have: $Z_1^* = 3, T_1^* = 7, Z_2^* = 3, T_2^* = -2$.

$$\begin{cases} Z_1 = 3 + k_1 \frac{4}{1} = 3 + 4k_1 \\ T_1 = 7 - k_1 \frac{3}{1} = 7 - 3k_1 \end{cases}; k_1 \in \mathbb{Z}$$

$$\begin{cases} Z_2 = 3 + k_2 \frac{6}{1} = 3 + 6k_2 \\ T_2 = -2 - k_2 \frac{1}{1} = -2 - k_2 \end{cases}; k_2 \in \mathbb{Z}$$

Thus:

$$X = \frac{1}{2}[6 + 4k_1 + 6k_2] + \frac{1}{2}J[4k_1 - 6k_2]$$

$$Y = \frac{1}{2}[5 - 3k_1 - k_2] + \frac{1}{2}J[9 - 3k_1 + k_2]; k_1, k_2 \in \mathbb{Z} \text{ and } 5 - 3k_1 - k_2 \equiv 0(\text{mod}2), 9 - 3k_1 + k_2 \equiv 0(\text{mod}2).$$

for example, we can generate a solution for:

$$k_1 = 2, k_2 = 1.$$

$$X = \frac{1}{2}[6 + 8 + 6] + \frac{1}{2}J[8 - 6] = 10 + J.$$

$$Y = \frac{1}{2}[-2] + \frac{1}{2}J[9 - 6 + 1] = -1 + 2J.$$

The all solution of the original split-complex Diophantine equation can be obtained by taking k_1, k_2 such that $k_1 + k_2 \equiv 1(\text{mod}2)$.

References

- [1] Deckelman, S.; Robson, B. Split-Complex Numbers and Dirac Bra-kets. Communications In Information and Systems **2014**, *14*, 135-159.
- [2] Hatip, A., "An Introduction To Weak Fuzzy Complex Numbers ", Galoitica Journal Of Mathematical Structures and Applications, Vol.3, 2023.
- [3] Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.
- [4] Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
- [5] Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.
- [6] Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.