On Some Analytical Relations Between Double Summability Methods of Abel-Natarajan

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Abstract

This paper is dedicated to study the analytical relations between Abel's double summability method and Natarajan's double summability method, where many theorems that draw a bridge between the mentioned methods will be obtained. The main result of our work is to prove that that summability by Natarajan's method implies summability by Abel's method in one or two variables. On the other hand, we illustrate some related examples to clarify the validity of our approach.

Keywords: Natarajan's method; Abel's method; summability.

1. Introduction and basic concepts

Natarajan's method is considered one of the most useful methods that helps in finding the sum of convergent and divergent series, with many applications in solving differential equations and approximation theory.

Abel's method is defined by using power series as a special case of matrix method.

This work concentrate on proving that summability by Natarajan's method implies summability by Abel's method in one or two variables.

Definition: [1,2]

The series $\sum_{n=0}^{\infty} u_n$ is a summable by Natarajan's method (μ, λ_n) to S if $\lim\limits_{n\to\infty} t_n{}^{(\mu,\lambda_n)} = S$, where:

$$t_n{}^{(\mu,\lambda_n)} = \sum_{k=0}^{\infty} a_{n,k}\, S_k,\; a_{n,k} = \begin{cases} \lambda_{n-k} & ; 0 \le k \le n \\ 0 & ; k > n \end{cases},\; S_n = \sum_{k=0}^{\infty} u_k$$

and $\{\lambda_n\}$ has the property $\sum_{k=0}^{\infty} |\lambda_n| < \infty$.

Theorem. [5]

The method (μ, λ_n) is regular if and only if $\sum_{n=0}^{\infty} \lambda_n = 1$

Definition. [7]

The two methods $(\mu, \lambda_n), (\mu, \mu_n)$ are called consistent if:

$\sum_{k=0}^{\infty} \lambda_k = S(\mu, \lambda_n),\; \sum_{k=0}^{\infty} u_k = \acute{S}(\mu, \mu_n) \Longrightarrow S = \acute{S}$.

Definition. [7]

$(\mu, \lambda_n), (\mu, \mu_n)$ are equivalent if:

$(\mu, \lambda_n) \subseteq (\mu, \mu_n), (\mu, \mu_n) \subseteq (\mu, \lambda_n)$

Definition. [4-6]

The series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m,n}$ is called summable by double Natarajan's method $(\mu, \lambda_{m,n})$ to S if:

$\lim\limits_{n\to\infty} t_{m,n}{}^{(\mu,\lambda_{m,n})} = S$, where $t_{m,n}{}^{(\mu,\lambda_{m,n})} = \sum_{j=0}^{m} \sum_{k=0}^{n} a_{m,n,j,k}\, S_{j,k}$.

$$a_{m,n,j,k} = \begin{cases} \lambda_{m-j,n-k} = \lambda_{m-j}\mu_{n-k} & ; 0 \le k \le n \\ 0 & ; k > n \end{cases}$$

$S_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} u_{j,k},\; \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\lambda_{m,n}| = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\lambda_m, \mu_n| < \infty$

Theorem. [3,4,5]

The method $(\mu, \lambda_{m,n})$ is regular if and only if $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n} = 1$.

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Definition. [8]

The series $\sum_{n=0}^{\infty} u_n$ is called summable by Abel's method to S if $\lim_{x\to 1^-} t_x^{A_1} = S$, where:

$t_x^{A_1} = \frac{\sum_{n=0}^{\infty} S_k x^n}{\sum_{n=0}^{\infty} x^n} = (1-x)\sum_{n=0}^{\infty} S_k\, x^n = \sum_{n=0}^{\infty} u_n\, x^n,\ S_n = \sum_{k=0}^{\infty} u_k$

Main discussion.

Definition.

The series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{m,n}$ is called summable by double Able's method (A_2) to S if:

$\lim_{x,y\to 1^-} t_{x,y}^{A_2} = S$, where:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{m,n} = S, t_{x,y}^{A_2} = \frac{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S_{m,n}\, x^m y^n}{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^m y^n} = (1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S_{m,n}\, x^m y^n\,, S_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} u_{i,j}$$

Theorem.

The Able's double method (A_2) transformation applied on the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{m,n}$ ha the following formula:

$$t_{x,y}^{A_2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S_{m,n}\, x^m y^n$$

Proof.

First, the method (A_2) is applicable if and only if the following series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S_{m,n}\, x^m y^n$is convergent.

We have:

$$t_{x,y}^{A_2} = (1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S_{m,n}\, x^m y^n = (1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{n} u_{i,j}\, x^m y^n$$

$$= (1-x)(1-y)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i,j}\sum_{m=i}^{\infty}\sum_{n=j}^{\infty} x^m y^n = (1-x)(1-y)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i,j}\sum_{m=i}^{\infty} x^m \sum_{n=j}^{\infty} y^n$$

$$= (1-x)(1-y)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i,j}\,(x^i + x^{i+1} + \cdots)(y^j + y^{j+1} + \cdots)$$

$$= (1-x)(1-y)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i,j}\, x^i y^j (1 + x + \cdots)(1 + y + \cdots)$$

$$= (1-x)(1-y)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i,j}\, x^i y^j \frac{1}{(1-x)(1-y)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i,j}\, x^i y^j$$

Example.

Consider the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{m+n}$, then:

$$\lim_{x,y\to 1^-} t_{x,y}^{A_2} = \lim_{x,y\to 1^-}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{m+n}\, x^m y^n = \frac{1}{4}$$

Example.

Consider the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} mn(-1)^{m+n+2}$, then:

$$\lim_{x,y\to 1^-} t_{x,y}^{A_2} = \lim_{x,y\to 1^-}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} mn(-1)^{m+n+2}\, x^m y^n = \frac{1}{16}$$

Theorem.

Let $\{a_{m,n}\}$ be a sequence, suppose that it is summable by double Nataraajan's method $(\mu, \lambda_{m,n})$ to a with regular $(\mu, \lambda_{m,n})$, then it is summable by (A_2) to a.

Proof.

Assume that $\{u_{m,n}\}$ is transformation of $(\mu, \lambda_{m,n})$ applied on $\{a_{m,n}\}$, then:

$$u_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} \lambda_{m-i,n-j} a_{i,j} = \sum_{i=0}^{m}\sum_{j=0}^{n} \lambda_{m-i} u_{n-j} a_{i,j} = \sum_{j=0}^{n} \lambda_{m-i}\sum_{j=0}^{n} u_{n-j} a_{i,j}$$

$$= \sum_{j=0}^{n} \lambda_{m-i}\left(\mu_0 a_{i,n} + \mu_1 a_{i,n-1} + \cdots + \mu_n a_{i,0}\right)$$

$$= \mu_0\left(\lambda_0 a_{m,n} + \lambda_1 a_{m-1,n} + \cdots + \lambda_m a_{0,n}\right) + \mu_1\left(\lambda_0 a_{m,n-1} + \lambda_1 a_{m-1,n-1} + \cdots + \lambda_m a_{0,n-1}\right)$$
$$+ \cdots + \mu_n\left(\lambda_0 a_{m,0} + \lambda_1 a_{m-1,0} + \cdots + \lambda_m a_{0,0}\right)$$

Thus $\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{m,n} x^m y^n\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} x^m y^n\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{m,n} x^m y^n$

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Hence, $\left[(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}x^my^n\right]\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}x^my^n\right)=(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{m,n}x^my^n$

Also,

$$\left[(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}x^my^n\right](1-x)(1-y)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}x^my^n\right).\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}x^my^n\right)$$
$$=(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{m,n}x^my^n$$

$\left[(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}x^my^n\right](1-x)(1-y)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Delta_{m,n}x^my^n\right)=(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{m,n}x^my^n$, where $\Delta_{m,n}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1)_{m-i,n-j}$, thus

$$\lim_{x,y\to1^-}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}x^my^n=\lim_{x,y\to1^-}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{m,n}x^my^n$$

Also, $\lim_{x,y\to1^-}(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Delta_{m,n}x^my^n=1$, $\lim_{m,n\to\infty}u_{m,n}=a$, hence $\lim_{x,y\to1^-}(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{m,n}x^my^n=a$, this implies:
$\lim_{x,y\to1^-}(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}x^my^n=a$, o that $\{a_{m,n}\}$ is summable by (A_2) to a.

Theorem.

If $\{a_{m,n}\}$ is summable by (A_2) to a, then $(\mu,\lambda_{m,n})(\{a_{m,n}\})$ is summable by (A_2) to a.

Proof.

We have:

$$\lim_{x,y\to1^-}(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}x^my^n=\lim_{x,y\to1^-}(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{m,n}x^my^n=1$$

But, $\lim_{x,y\to1^-}(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}x^my^n=a$, hence

$$\lim_{x,y\to1^-}(1-x)(1-y)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{m,n}x^my^n=a$$

And the proof is complete.

Remark.

$A_2(\mu,\lambda_{m,n})\neq(\mu,\lambda_{m,n})A_2$ in general.

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