Weakly Generalized M-Closed and Strongly M-Generalized Closed Sets in Fuzzy Neutrosophic Topological Spaces

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Abstract

The current study presents a new concept of sets and called fuzzy neutrosophic weakly generalized M-closed and fuzzy neutrosophic strongly M-generalized closed sets in fuzzy neutrosophic topological space. Actually, this study is an extended form of a research conducted by Z.A. Khalaf, F. M. Mohammed [1]. Moreover, it presents certain related relations of these notions as well as some theorems, propositions with some necessary examples.

Keywords: Fuzzy Neutrosophic Sets; Fuzzy Neutrosophic Weakly Generalized M-Closed Set; Fuzzy Neutrosophic Strongly M-Generalized Closed Set; Fuzzy Neutrosophic Topology.

1.Introduction

In 1965, Zadeh [2] created the fuzzy set (FS) concept in order to advance human understanding in various scientific fields. In order to expand the research of fuzzy sets, then in1968 Chang [3] established the idea of fuzzy topological space (FTS). Following that, the notion of a fuzzy set evolved into the idea of an intuitionistic fuzzy set (IFS), which was investigated in 1983 by Atanassov [4,5] wherever the fuzzy set offers the degree of membership of an element in the set, the intuitionistic fuzzy set provides a degree of membership function and a degree of non-membership function of an element in the set, and $0 \leq \mu_{\lambda 1}(\omega) + \nu\lambda_1(\omega) \leq 1$, for each $\omega \in \omega_N$ afterward, numerous studies on the generalizations of the concept of intuitionistic fuzzy set were carried out. In 1997, Intutionistic fuzzy topological space (IFTS) was first established by Coker [6].

Concerning the concepts of neutrosophy, neutrosophic set and neutrosophic component were all examined in 1995 by by Smarandche [7,8]. The concepts of neutrosophic set (NS) and neutrosophic topological space (NTS) were subsequently introduced in 2012 by Salama and Alblowi [9], followed by the definition of its continuous function in 2014. The phrase of neutrosophic set was explained with membership, non-membership and indeterminacy degrees. Fuzzy neutrosophic set (FNS) was first described in 2013 by Arockiarani, R. Sumathi, and Martina Jency [10]. Thereafter, Arockiarani and Martina Jency developed fuzzy neutrosophic topological space (FNTS) in 2014

DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

Since it is an extended form of a research conducted by F. Mohammed (see [1] and [11-14]); the current study introduces some new classes of sets. It discusses the relationships between fuzzy neutrosophic weakly generalized M-closed and fuzzy neutrosophic strongly M-generalized closed sets and compares them with the other sets.

2. Preliminaries

Definition 2.1[8] "Let ω_N be any non-empty fixed set. The fuzzy neutrosophic set (briefly, FNS), λ_F is a topic has the form $\lambda_F = \{< \omega, \mu_{\lambda F}(\omega), \sigma_{\lambda F}(\omega), \nu_{\lambda F}(\omega) >: \omega \in \omega_N\}$ where the functions $\mu_{\lambda F}, \sigma_{\lambda F}, \nu_{\lambda F} : \omega_N \rightarrow [0, 1]$ denote the degree of membership function (namely $\mu_{\lambda F}(\omega)$), also the degree of indeterminacy function (namely $\sigma_{\lambda F}(\omega)$) and the degree of non-membership (namely $\nu_{\lambda F}(\omega)$) respectively of each element $\omega \in \omega_N$ to the set λ_F and $0 \leq \mu_{\lambda F}(\omega) + \sigma_{\lambda F}(\omega) + \nu_{\lambda F}(\omega) \leq 3$, for each $\omega \in \omega_N$."

Remark 2.2 [8] FNS $\lambda_F = \{< \omega, \mu_{\lambda F}(\omega), \sigma_{\lambda F}(\omega), \nu_{\lambda F}(\omega)>: \omega \in \omega_N\}$ can be identified to an ordered triple $< \omega, \mu_{\lambda F}, \sigma_{\lambda F}, \nu_{\lambda F} >$ in [0, 1] on ω_N.

Definition 2.3 [8] "Let ω_N be a non-empty set and the FNSs λ_F and β_F be in the form:

$$\lambda_F = \{< \omega, \mu_{\lambda F}(\omega), \sigma_{\lambda F}(\omega), \nu_{\lambda F}(\omega)>: \omega \in \omega_N\} \text{ and}$$

$$\beta_F = \{< \omega, \mu_{\beta F}(\omega), \sigma_{\beta F}(\omega), \nu_{\beta F}(\omega)>: \omega \in \omega_N\} \text{ on } \omega_N.$$

Then,

- **i.** $\lambda_F \subseteq \beta_F$ iff $\mu_{\lambda F}(\omega) \leq \mu_{\beta F}(\omega)$, $\sigma_{\lambda F}(\omega) \leq \sigma_{\beta F}(\omega)$ and $\nu_{\lambda F}(\omega) \geq \nu_{\beta F}(\omega)$ for all $\omega \in \omega_N$,
- **ii.** $\lambda_F = \beta_F$ iff $\lambda_F \subseteq \beta_F$ and $\beta_F \subseteq \lambda_F$,
- **iii.** $(\lambda_F)^c = 1_F - \lambda_F = \{< \omega, \nu_{\lambda F}(\omega), 1- \sigma_{\lambda F}(\omega), \mu_{\lambda F}(\omega)>: \omega \in \omega_N\}$,
- **iv.** $\lambda_F \cup \beta_F = \{< \omega, \text{Max}(\mu_{\lambda F}(\omega), \mu_{\beta F}(\omega)), \text{Max}(\sigma_{\lambda F}(\omega), \sigma_{\beta F}(\omega)), \text{Min}(\nu_{\lambda F}(\omega), \nu_{\beta F}(\omega)) >: \omega \in \omega_N\}$,
- **v.** $\lambda_F \cap \beta_F = \{< \omega, \text{Min}(\mu_{\lambda F}(\omega), \mu_{\beta F}(\omega)), \text{Min}(\sigma_{\lambda F}(\omega), \sigma_{\beta F}(\omega)), \text{Max}(\nu_{\lambda F}(\omega), \nu_{\beta F}(\omega)) >: \omega \in \omega_N\}$,
- **vi.** $0_F = < \omega, 0, 0, 1>$ and $1_F = < \omega, 1, 1, 0>$. "

Definition 2.4 [8]"Fuzzy neutrosophic topology (briefly, FNT) on a non-empty set ω_N is a family Γ_N of fuzzy neutrosophic subsets in ω_N satisfying the following axioms:

- **i.** $0_F, 1_F \in \Gamma_N$,
- **ii.** $\lambda_{F1} \cap \lambda_{F2} \in \mathcal{T}_{FN}$ for any $\lambda_{F1}, \lambda_{F2} \in \Gamma_N$,
- **iii.** $\cup \lambda_{Fj} \in \Gamma_N$, $\forall \{ \lambda_{Fj}: j \in J\} \subseteq \Gamma_N$.

The pair (ω_N, Γ_N) is called fuzzy neutrosophic topological space (briefly, FNTS). The elements of Γ_N are called fuzzy neutrosophic-open sets (briefly, FN-Os).The complement of FN-open set in the FNTS (ω_N, Γ_N) is called FN-closed set (briefly, FN-Cs)."

Definition 2.5 [8]" Let (ω_N, Γ_N) is FNTS and $\lambda_F = <\omega, \mu_{\lambda F}, \sigma_{\lambda F}, \nu_{\lambda F}>$ is FNS in ω_N.Then, the fuzzy neutrosophic-closure (briefly, FN-cl) and the fuzzy neutrosophic-interior (briefly, FN-int) of λ_F are defined by:

$$\text{FN-cl}(\lambda_F) = \cap \{\beta_F: \beta_F \text{ is FN-Cs in } \omega_N \text{ and } \lambda_F \subseteq \beta_F\},$$

$$\text{FN-int}(\lambda_F) = \cup\{ \beta_F: \beta_F \text{ is FN-Os in } \omega_N \text{ and } \beta_F \subseteq \lambda_F \}.$$

Now, the FN-cl (λ_F) is FN-Cs and FN-int (λ_{FN}) is FN-Os in ω_N.

DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

Further,

i. λ_F is FN-Cs in ω_N iff FN-cl (λ_F) = λ_F
ii. λ_F is FN-Os in ω_N iff FN-int(λ_F) = λ_F ."

Proposition 2.6 [9,10] "For any FNS λ_F in FNTS (ω_N, Γ_N) we have,

i. FN-cl (1_F - λ_F) = 1_F - (FN-int (λ_F)),

ii. FN-int (1_F - λ_F) =1_F - (FN-cl (λ_F))".

Proposition 2.7 [1] "Let (ω_N, Γ_N) is FNTS and λ_F, β_F are FNSs in ω_N. Then the following properties hold:

i. FN-int (λ_F) $\subseteq$ λ_F and λ_F $\subseteq$ FN-cl (λ_F),
ii. $\lambda_F \subseteq \beta_F \Longrightarrow$ FN-int (λ_F) $\subseteq$ FN-int (β_F) and $\lambda_F \subseteq \beta_F \Longrightarrow$ FN-cl(λ_F) $\subseteq$ FN-cl(β_F),
iii. FN-int(FN-int(λ_F)) = FN-int(λ_F) and FN-cl(FN-cl(λ_F)) = FN-cl(λ_F),
iv. FNint($\lambda_F \cap \beta_F$) = FN-int(λ_F) $\cap$ FN-int(β_F) and FN-cl ($\lambda_F \cup \beta_F$) = FN-cl (λ_F) $\cup$ FN-cl (β_F),
v. FNint (1_F) = 1_F and FNcl (0_F) =0_F".

Definition 2.8: [1,11] " FNS λ_F in FNTS (ω_N, Γ_N)is called:

i. Fuzzy neutrosophic regular-open set (briefly, FN-RO set,) if λ_F= FN-int (FN-cl (λ_F)).
ii. Fuzzy neutrosophic regular-closed set (briefly, FN-RC set) if λ_F= FN-cl (FN-int (λ_F)).
iii. Fuzzy neutrosophic pre-open set (briefly, FN-PO set) if $\lambda_F\subseteq$ FN-int (FN-cl (λ_F)).
iv. Fuzzy neutrosophic pre-closed set (briefly, FN-PC set) if FN-cl (FN-int (λ_F)) $\subseteq \lambda_F$.
v. Fuzzy neutrosophic semi-open set (briefly, FN-SO set) if $\lambda_F\subseteq$ FN-cl (FN-int(λ_F)).
vi. Fuzzy neutrosophic semi-closed set (briefly, FN-SC set) if FN-int (FN-cl (λ_F))$\subseteq\lambda_F$.
vii. Fuzzy neutrosophicα-open set (briefly,FN-αO set) if $\lambda_F \subseteq$ FN-int(FN-cl(FN-int(λ_F))) .
viii. Fuzzy neutrosophic α-closed set (briefly,FN-αC set) if $\lambda_F \supseteq$ FN-cl(FN-int(FN-cl(λ_F))).
ix. FNM-open set if λ_F $\subseteq$ (FN-cl(FN-int$_\theta$(λ_F)) $\cup$ FN-int(FN-cl$_\delta$(λ_F))).
x. FNM-closed set if λ_F $\supseteq$ (FN-int(FN-cl$_\theta$(λ_F)) $\cap$ FN-cl(FN-int$_\delta$(λ_F)))."

Fuzzy Neutrosophic Weakly Generalized M-Closed Set

In this part we will study a new class of sets and called it fuzzy neutrosophic weakly generalized M-closed in fuzzy neutosophic topology and some relations with other sets in our studied space.

Definition 3.1: Let (ω_N,Γ_N) be FNTS. A fuzzy neutrosophic set λ_F is called:

i. Fuzzy neutrosophic weakly-closed set (briefly, FN-WCs) if FN-cl(λ_F) $\subseteq \upsilon_F$ wherever, $\lambda_F \subseteq \upsilon_F$ and υ_F is FNS-open set in ω_N. λ_F is said to be fuzzy neutrosophic weakly open set (briefly, FN-WOs) in if the complement 1_F- λ_F is FN-WCs in (ω_N,Γ_N).

ii. Fuzzy neutrosophic weakly M-closed set (FN-WMCs) if FN-cl(λ_F) $\subseteq \upsilon_F$ wherever, $\lambda_F \subseteq \upsilon_F$ and υ_F is FN-WOs in ω_N. λ_F is said to be fuzzy neutrosophic weakly M-open set (briefly, FN-WMOs) in (ω_N,Γ_N) if the complement 1_F-λ_F is FN-WMCs in (ω_N, Γ_N).

Theorem 3.2: Let (ω_N,Γ_N) be FNTS, then every FN-MCs is FN-WMCs but not conversely.

Proof. Let λ_F be any FN-MCs in FNTS (ω_N,Γ_N) and let β_F be any FN-WOs such that $\lambda_F \subseteq$ υ_F.

DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

Since λ_F is FN-MCs so we have, FN-Mcl(λ_F) = $\lambda_F \subseteq \upsilon_F$ and FN-Mcl(λ_F) $\subseteq \upsilon_F$. Hence, λ_F is FN-WMCs.

Example 3.3: Let ω_N = {a, b} define FNS λ_F in ω_N as follows:

$$\lambda_F = <\omega, a(0.5, 0.5, 0.4), b(0.5, 0.5, 0.5)>.$$

And the family, $\Gamma_N = \{0_F, 1_F, \lambda_F\}$ be FNT.

Now if, $\beta_F = <\omega$, a(0.3, 0.5, 0.6), b(0.4, 0.5, 0.5)> .

So, $\upsilon_F = \lambda_F$ is FN-open set such that, $\beta_F \subseteq \upsilon_F$,

But, $\lambda_F \subseteq$ Now, FN-cl(FN-int(λ_F))) such that, $\lambda_F \subseteq 1_F$ then, λ_F is FNS-open. Then, FN-cl(β_F) = $\lambda^c{}_F$.

Therefore, FN-cl(β_F) $\subseteq \upsilon_F$ then, β_F is FN-WMCs.

Hence, β_F is FN-WMCs and not FN-MCs.

Definition 3.4: Let (ω_N,Γ_N) be FNTS. A FNS λ_F is called fuzzy neutrosophic generalized M-closed set (briefly, FN-GMCs) if FN-int(FN-cl(λ_F)) $\subseteq \upsilon_F$, wherever $\lambda_F \subseteq \upsilon_F$ and υ_F is FNα-open set. λ_F is said to be fuzzy neutrosophic generalized M-open set (briefly, FN-GMOs) in (ω_N,Γ_N) if the complement 1_F-λ_F is FN-GMCs.

Example 3.5: Let $\omega_N = \{\omega\}$ define FNSs λ_F, β_F, η_F and ζ_F in ω_N as follows:

$$\lambda_F = <\omega, 1, 0.5, 0.7>, \ \beta_F = <\omega, 0, 0.9, 0.2>,$$

$$\eta_F = <\omega, 1, 0.9, 0.2>, \zeta_F = <\omega, 0, 0.5, 0.7>.$$

And the family, $\Gamma_N = \{0_F, 1_F, \lambda_F, \beta_F, \eta_F, \zeta_F\}$ be FNT.

Now if, $\xi_F = <\omega, 0, 0.4, 0.8>$ and $\upsilon_F = <\omega, 0, 0.5, 0.7>$ where, υ_F is FN-αopen set such that, $\xi_F \subseteq \upsilon_F$.

Then, FN-cl(ξ_F) = $<\omega, 0.7, 0.5, 0>$, FN-int(FN-cl(ξ_F)) = $<\omega, 0, 0.5, 0.7>$.

Therefore, FN-int(FN-cl(ξ_F)) $\subseteq \upsilon_F$. Hence, W_N is FN-GMOs .

Proposition 3.6: The concept of FN-MCs and FN-GMCs are independent. As shown by the following examples :

Example 3.7: i. Take Example 3.3. So, δ_F is FN-MCs, such that $\delta_F = \{1_F, 0_F, \lambda_F, \beta_F\}$.

Then, FN-int(FN-cl(λ_F)) $\subseteq \upsilon_F$, wherever $\lambda_F \subseteq \upsilon_F$ and υ_F is not FNα-open set.Therefore, it is not FN-GMCs.

ii. Take Example 3.3. We have, ξ_F is FN-GMOs. But, ξ_F is not FN-clopen set, then it's not FN-MOs.

Definition 3.8: Let (ω_N,Γ_N) be FNTS, a FNSλ_F is called fuzzy neutrosophic weakly generalized M-closed set (briefly, FN-WGMCs, for short) if FN-cl(FN-int(λ_F))) $\subseteq \upsilon_F$ wherever, $\lambda_F \subseteq \upsilon_F$ and υ_F is FNM-open set in ω_N. λ_F is said to be fuzzy neutrosophic weakly generalized M-open set (briefly, FN-WGMOs) in (ω_N,Γ_N) if the complement 1_F -λ_F is FN-WGCs in (ω_N,Γ_N).

Example 3.9: Let (ω_N,Γ_N) be FNTS and ω_N = {a, b, c} with $\Gamma_N = \{0_F, 1_F, \lambda_F, \upsilon_F\}$ where,

λ_F = <ω, a(0.3, 0.2, 0.5), b(0.4, 0.3, 0.5), c(0.5, 0.2, 0.5) >, and

υ_F = <ω, a(0.5, 0.5, 0.5), b(0.5, 0.5, 0.5), c(0.5, 0.5, 0.5) >.

DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

So, $\delta_F = \{1_F, 0_F, \upsilon_F\}$ is FN-MOs.

And $\lambda_F \subseteq \upsilon_F$, then we got FN-cl(FN-int(λ_F))) $\subseteq \upsilon_F \Longrightarrow \upsilon^c_F \subseteq \upsilon_F$. That means, λ_F will be FN-WGMCs .

Theorem 3.10: Every FN-WGMCs is FN-WMCs but, the converse is not true in general.

Proof: Let λ_F is FN-WGMC and υ_F is FNM-open set in $(\boldsymbol{\omega}_N, \Gamma_N)$ then,

FN-cl(FN-int(λ_F))) $\subseteq \upsilon_F$ wherever, $\lambda_F \subseteq \upsilon_F$

Since FN-cl(FN-int(λ_F))) $\subseteq$ FN-Mcl(λ_F). That means FN-Mcl(λ_F) = λ_F.

Also, since $\lambda_F \subseteq \upsilon_F$ then, FN-Mcl(λ_F) $\subseteq \upsilon_F$. Hence, λ_F is FN-WMCs.

Example 3.11: Take Example 3.5, then we have, λ_F is FN-WGMCs, and by Definition 3.1.

Also we have, FN-cl(λ_F) $\subseteq \upsilon_F$ wherever, $\lambda_F \subseteq \upsilon_F$ and υ_F is FNS-open, so λ_F is FN-WMCs.

Theorem 3.12: For each FNS, the following statements is satisfing :

i. Every FN$-\alpha$Cs is FN-WGMCs.
ii. Every FN-MCs is FN-WGMCs.
iii. Every FN-RCs is FN-WGMCs.
iv. Every FN-PCs is FN-WGMCs.

Proof: i. Let $\lambda_F = \{<\omega, , \mu_{\lambda F}(\omega), \sigma_{\lambda F}(\omega), \nu_{\lambda F}(\omega)>: \omega \in \omega_N\}$ be FNα-Cs in FNTS (ω_N, Γ_N).

Then, FN-cl(FN-int(FN-cl(λ_F) $\subseteq \lambda_F$.

Now, let β_F be FN-MOs such that $\lambda_F \subseteq \beta_F$.

Then, FN-cl(FN-int(λ_F) $\subseteq$ FN-cl(FN-int(FN-cl(λ_F) $\subseteq \lambda_F \subseteq \beta_F$.

Therefore, FN-cl(FN-int(λ_F)) $\subseteq \beta_F$. Hence, λ_F be FN-WGCs in (ω_N, Γ_N).

Let $\lambda_F = \{<\omega, , \mu_{\lambda F}(\omega), \sigma_{\lambda F}(\omega), \nu_{\lambda F}(\omega)>: \omega \in \omega_N\}$ be FN-MCs in FNTS (ω_N, Γ_N) then we get, λ_F = FN-Mcl(λ_F) ….(1)

Also we have, FN-int(λ_F) $\subseteq \lambda_F$ …..(2)

But, FN-cl(FN-int(λ_F)) $\subseteq$ FN-cl(λ_F) so by (1) we get, FN-cl(FN-int(λ_F)) $\subseteq \lambda_F$.

Now, let β_F FN-MOs such that, $\lambda_F \subseteq \beta_F$.

Then, FN-cl(FN-int(λ_F)) $\subseteq \lambda_F \subseteq \beta_F$.

Therefore, FN-cl(FN-int(λ_F)) $\subseteq \beta_F$. Hence, λ_F be FN-WGCs in (ω_N, Γ_N).

iii. Let $\lambda_F = \{<\omega, , \mu_{\lambda F}(\omega), \sigma_{\lambda F}(\omega), \nu_{\lambda F}(\omega)> : \omega \in \omega_N\}$ be FN-RCs in FNTS (ω_N, Γ_N).

Then, FN-cl(FN-int(λ_F)) = λ_F.

DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

Now, let β_F be FN-MOs such that, $\lambda_F \subseteq \beta_F$.

Then, FN-cl(FN-int(λ_F))= $\lambda_F \subseteq \beta_F$ therefore, FN-cl(FN-int(λ_F)) $\subseteq \beta_F$. Hence, λ_F be FN-WGCs in (ω_N, Γ_N).

iv. Let $\lambda_F = \{<\omega, , \mu_{\lambda F}(\omega), \sigma_{\lambda F}(\omega), \nu_{\lambda F}(\omega)> : \omega \in \omega_N\}$ be FN-PCs in FNTS (ω_N, Γ_N).

Then, FN-cl(FN-int(λ_F)) $\subseteq \lambda_F$.

Now, let β_F be FN-MOs such that, $\lambda_F \subseteq \beta_F$. Then, FN-cl(FN-int($\lambda_F$)) $\subseteq \lambda_F \subseteq \beta_F$.

Therefore, FN-cl(FN-int(λ_F)) $\subseteq \beta_F$. Hence, λ_F be FN-WGCs in (ω_N, Γ_N).

Remark 3.13: The converse of the last theorem is not true in general, it is significant to show it by the following examples :

Examples 3.14: i. Take the Example 3.5 then, we can see λ_F be FN-WGCs.

But, $\lambda_F \not\supseteq$ FN-cl(FN-int(FN-cl(λ_F))), $\lambda_F \not\supseteq \upsilon^c_F$, then λ_F is not FNα-closed set.

i. Take the Example 3.5, then, we can see λ_F be FN-WGCs but, λ_F not FN-MCs .

ii. Take again, the Example 3.5. Then, λ_F be FN-WGCs but, not FN-RCs because,

FN-cl(FN-int(λ_F))) = υ^c_F , $\lambda_F \neq \upsilon^c_F$.

iii. Take, the Example 3.5,then, λ_F is FN-WGCs.

But, FN-cl(FN-int(λ_F)) = υ^c_F, $\upsilon^c_F \nsubseteq \lambda_F$. Hence, λ_F is not FNP-closed set.

Remark 3.15: The relation between FNS-closed sets and FN-WGCs is independent, it is important to show it by the following examples.

Example 3.16: i. Let $\omega_N = \{a, b\}$ define FNS in ω_N as follows:

$\lambda_F = <\omega$, a(0.3, 0.5, 0.6), b(0.4, 0.5, 0.7)> and the family $\Gamma_N = \{0_F, 1_F, \lambda_F\}$ be FNTS.

Now if, $\zeta_F = <\omega$, a(0.3, 0.5, 0.6), b(0.4, 0.5, 0.7)>,

And , $\upsilon_F = \lambda_F$ where υ_F be FN-open set such that, $\zeta_F \subseteq \upsilon_F$

Then, FN-cl(ζ_F)= $1_F - \lambda_F$ and FN-int(FN-cl(ζ_F)))= λ_F .

Therefore, FN-int(FN-cl(ζ_F))) $\subseteq \zeta_F$. Hence, ζ_F is FNS-closed set.

But, FN-int(ζ_F) = λ_F and FN-cl(FN-int(ζ_F)))= $1_F - \lambda_F$.

Therefore, FN-cl(FN-int(ζ_F))) $\nsubseteq \upsilon_F$. Hence, ζ_F is not FN-WGCs .

ii. Take, Example 3.5 then, λ_F is FN-WGMCs.

But, $\lambda_F \not\supseteq$ FN-int(FN-cl(λ_F))). Hence, $\lambda_F \not\supseteq \upsilon_F$ is not FNS-closed set.

DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

Proposition 3.17: Let λ_F be FN-MCs in (ω_N, Γ_N) such that, FN-int$(\lambda_F) \subseteq \beta_F \subseteq \lambda_F$ then, β_F is FN-WGMCs on FNTS (ω_N, Γ_N).

Proof: Let $\lambda_F = \{<\omega, , \mu_{\lambda F}(\omega), \sigma_{\lambda F}(\omega), \nu_{\lambda F}(\omega)>: \omega \in \omega_F\}$ be FNs in FNTS (ω_N, Γ_N) such that,

FN-int$(\lambda_F) \subseteq \beta_F \subseteq \lambda_F$. So, there exists FN-MCs Z_F such that, Z_F (FN-int$(\lambda_F))) \subseteq \beta_F \subseteq \lambda_F \subseteq Z_F$.

Then, $\beta_F \subseteq Z_F$ and also FN-int$(\beta_F) \subseteq \beta_F \subseteq Z_F$. Thus, FN-cl(FN-int$(\beta_F) \subseteq \beta_F$.

Now, let H_F be FN-MOs such that, $\beta_F \subseteq H_F$. Then,FN-cl(FN-int$(\beta_F))) \subseteq \beta_F \subseteq H_F$.

Therefore, FN-cl(FN-int$(\beta_F))) \subseteq H_F$. Hence, β_F is FN-WGMCs in (ω_N, Γ_N) .

Theorem 3.18: Let (ω_N, Γ_N) be FNTS, then the union of any two FN-WGMC sets is also FN-WGMCs.

Proof: Let $\{\lambda_i , i \in I\}$ be a collection of FN-MCs .

Then, $\lambda_i \supseteq$ FN-cl(FN-int$_\theta$ $(\lambda_F))) \cap$ FN-int(FN-cl$_\delta$ $(\lambda_F)))$, hence

$\cup_i \lambda_i \supseteq \cup_i$ FN-cl(FN-int$_\theta$ $(\lambda_F))) \cap$ FN-int(FN-cl$_\delta$ $(\lambda_F)))$

$\supset$ FN-cl(FN-int$_\theta$ $(\cup_i \lambda_i))) \cap$ FN-int(FN-cl$_\delta$ $(\cup_i \lambda_i)))$ for all $i \in I$.

Thus $\cup_i \lambda_i$ is FN-MCs then, by Theorem 3.2 we got, every FN-MCs is FN-WGMCs then, $\cup_i \lambda_i$ is FN-WGMCs .

Remark 3.19: The intersection of any two FN-WGMC sets is not necessary to be FN-WGMCs.

Example 3.20: Let $\delta_{F1} = \{0_F , 1_F , \lambda_{F1}\}$ where,

$\lambda_{F1} = <\omega$, a(0.5, 0.5, 0.5), b(0.5, 0.5, 0.5),c(0.5, 0.5, 0.5) >.

And, $\delta_{F2} = \{0_F , 1_F , \lambda_{F2} , \beta_{F2}\}$

Where, $\lambda_{F2} = <\omega$, a(0.3, 0.5, 0.3), b(0.4, 0.2, 0.2), c(0.5, 0.3, 0.2) >,

And, $\beta_{F2} = <\omega$, a(0.3, 0.5, 0.3), b(0.2, 0.8, 0.4), c(0.2, 0.7, 0.5) >.

So, λ_{F1} and λ_{F2} are FN-WGMC sets

Now, $\lambda_{F1} \cap \lambda_{F2} = <\omega, (\lambda_{F1} \cap \lambda_{F2}), (\lambda_{F1} \cap \lambda_{F2}), (\lambda_{F1} \cup \lambda_{F2})>$

$= <\omega, \min(\lambda_{F1}, \lambda_{F2}), \min(\lambda_{F1}, \lambda_{F2}), \max(\lambda_{F1}, \lambda_{F2})>$,

$= <\omega$, a(0.3, 0.5, 0.5), b(0.4, 0.2, 0.5), c(0.5, 0.3, 0.4).

$= <\omega$, a(0.3, 0.5, 0.5), b(0.2, 0.5, 0.5), c(0.2, 0.5, 0.5) >.

That is $\lambda_{F1} \cap \lambda_{F2}$ is not FN-WGMCs

The Next figure shows some relations between different sets.

DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

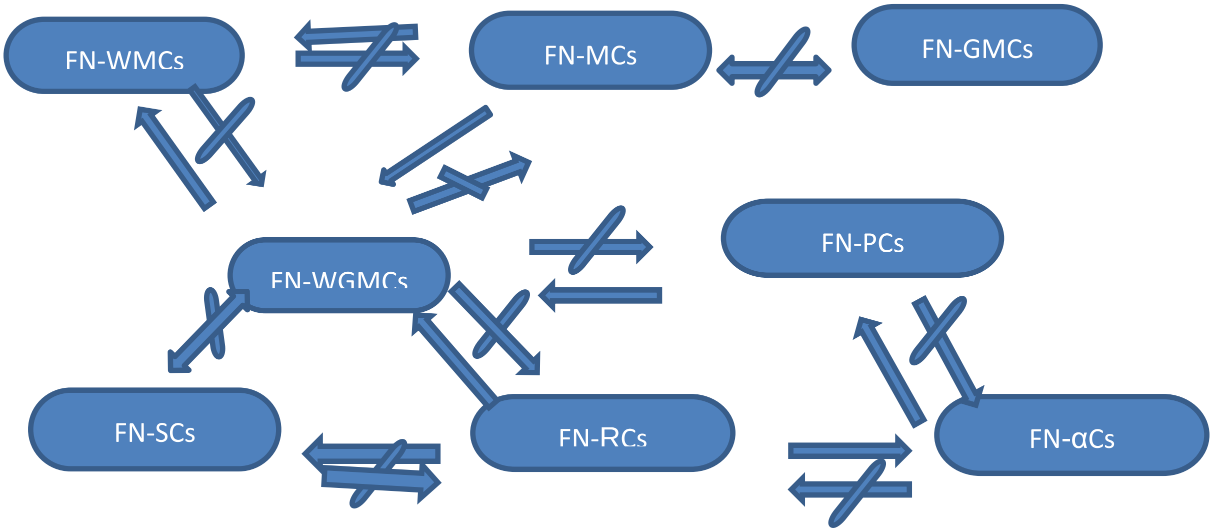

Figure 1

4.Fuzzy Neutrosophic Strongly M-Generalized Closed Set

In this part we will study another class of sets and called it strongly M-generalized closed sets in fuzzy neutosophic topology and some relations with other sets in studied space.

Definition 4.1: Fuzzy neutrosophic subset λ_F of FNTS ($\boldsymbol{\omega}_N$,Γ_N) is called fuzzy neutrosophic strongly M generalized closed set (briefly, FN-SMGCs) if FN-Mcl(λ_F) $\subseteq \upsilon_F$ wherever, $\lambda_F \subseteq \upsilon_F$ and υ_F is FN-GOs in $\boldsymbol{\omega}_N$.

Example 4.2: Let ($\boldsymbol{\omega}_N$, Γ_N) be FNTS and $\boldsymbol{\omega}_N$ = {a, b, c} where,

Γ_N = {0_F,1_F, λ_F, β_F} wherever,

λ_F = < $\boldsymbol{\omega}$, a(0.3, 0.2, 0.5), b(0.4, 0.3, 0.5), c(0.5, 0.2, 0.5) >,

β_F = < $\boldsymbol{\omega}$, a(0.5, 0.5, 0.5), b(0.5, 0.5, 0.5), c(0.5, 0.5, 0.5) >.

Now, let $\beta_F = \upsilon_F$, Then, $\lambda_F \subseteq \upsilon_F$, FN-Mcl($\lambda_F$) = υ_F and FN-Mcl(λ_F) = $\beta^c{}_F$, FN-Mcl(λ_F) $\subseteq \upsilon_F$.

So, λ_F is FN-SMGCs .

Example 4.3: Let ($\omega_{N,}$ Γ_N) be FNTS and $\omega_{N\,=}$ {a, b, c} and Γ_N= {0_F , 1_F , λ_F , β_F , υ_F }wherevers,

λ_F = < ω,a(0.3, 0.5 , 0.3), b(0.4, 0.2, 0.2), c(0.5, 0.3, 0.2) >,

$_{F\,=}$ < ω, a(0.3, 0.5, 0.3) , b(0.2, 0.8, 0.4), c(0.2, 0.7, 0.5) >, β

υ_F = < ω, a(0.3, 0.5, 0.3), b(0.4, 0.8, 0.2), c(0.5, 0.7, 0.2) >.

FN-Mcl(λ_F) $\subseteq \upsilon_F$, λ_F $\subseteq \upsilon_F$. Then λ_F is FN-SMGCs and FN-Mcl(β_F) $\subseteq$ U_N, β_F $\subseteq \upsilon_F$. So, β_F is FN-SMGCs.

Theorem 4.4: For any FNSs, the following statements are true in general:

DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

i. Every FN-open set is FN-GOs .

ii. Every FN-MCs is FN-closed set .

iii. Every FN-RCs is FN-SMGCs.

iv. Every FN-MCs is FN-SMGCs.

v. Every FN-PCs is FN-SMGCs .

vi. Every FN-RCs is FN-closed set .

Proof**:**

i. Let $\lambda_F = < \boldsymbol{\omega}, \mu_{\lambda F}(\boldsymbol{\omega}), \sigma_{\lambda F}(\boldsymbol{\omega}), \nu_{\lambda F}(\boldsymbol{\omega})>$ be FN-open set in FNTS $(\omega_{N,} \mathit{\Gamma}_N)$ then, we get FN-int(λ_F) = λ_F.

Now, let υ_F is FN-closed set such that, $\upsilon_F \subseteq \lambda_F$.

Therefore, FN-int(λ_F) = $\lambda_F \supseteq \upsilon_F$. Hence, λ_F is FN-GOs in $(\omega_{N\,,}\ \mathit{\Gamma}_N)$.

i. Let $\lambda_F = < \boldsymbol{\omega}, , \mu_{\lambda F}(\boldsymbol{\omega}), \sigma_{\lambda F}(\boldsymbol{\omega}), \nu_{\lambda F}(\boldsymbol{\omega})>$ be FN-MCs in FNTS $(\omega_{N\,,}\ \mathit{\Gamma}_N)$.

Then, FN-int(FN-cl$_\theta$(λ_F) ∩ FN-cl(FN-int$_\delta$(λ_F) ⊆ λ_F ... (1)

And, let υ_F be FN-GOs such that, $\lambda_F \subseteq \upsilon_F$. So, FN-int(FN-cl$_\theta$(λ_F) ∩ FN-cl(FN-int$_\delta$(λ_F) ⊆ FN-cl(λ_F)….(2)

Then, from (1) and (2) , we got FN-cl(λ_F) = λ_F .

Since FN-cl(λ_F) = λ_F is FN-Cs .

ii. Let $\lambda_F = < \boldsymbol{\omega}, , \mu_{\lambda F}(\boldsymbol{\omega}), \sigma_{\lambda F}(\boldsymbol{\omega}), \nu_{\lambda F}(\boldsymbol{\omega})>$ be FN-RCs in FNTS $(\omega_{N\,,}\ \mathit{\Gamma}_N)$. Then,

FN-cl(FN-int(λ_F) = λ_F ………(1)

This implies, FN-cl(FN-int(λ_F) = FN-cl(λ_F) ………(2)

Now, let υ_F be FN-GOs such that, $\lambda_F \subseteq \upsilon_F$.

From(1) and (2) we get, FN-cl(λ_F) = λ_F .

That means, λ_F is FN-Cs in ω_N .Then, by iii we get, FN-Mcl(λ_F) ⊆ FN-cl(λ_F) = $\lambda_F \subseteq \upsilon_F$.

Hence, λ_F is FN-SMGCs in $(\omega_{N\,,}\ \mathit{\Gamma}_N)$.

iii. Let $\lambda_F = < \boldsymbol{\omega}, , \mu_{\lambda F}(\boldsymbol{\omega}), \sigma_{\lambda F}(\boldsymbol{\omega}), \nu_{\lambda F}(\boldsymbol{\omega})>$ be FN-MCs in the FNTS $(\omega_{N\,,}\ \mathit{\Gamma}_N)$ then, we get,

FN-Mcl(λ_F) = λ_F. Now, let υ_F be FN-GOs such that, $\lambda_F \subseteq \upsilon_F$.

So, FN-Mcl(λ_F) = $\lambda_F \subseteq \upsilon_F$. Hence, λ_F is FN-SMGCs in $(\omega_{N,}\ \mathit{\Gamma}_N)$

iv. Let $\lambda_F = < \boldsymbol{\omega}, , \mu_{\lambda F}(\boldsymbol{\omega}), \sigma_{\lambda F}(\boldsymbol{\omega}), \nu_{\lambda F}(\boldsymbol{\omega})>$ be FN-PCs in FNTS $(\omega_{N\,,}\ \mathit{\Gamma}_N)$. Then,

FN-cl(FN-int(λ_F) ⊆ λ_F …… (1)

This implies, FN-cl(FN-int(λ_F) ⊆ FN-cl(λ_F) …..(2)

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Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

Now, let υ_F be FN-GOs such that, $\lambda_F \subseteq \upsilon_F$.

From(1) and (2) we get, FN-cl(λ_F) = λ_F .

That means λ_F is FN-Cs in ω_N.Then, by iii we get, FN-Mcl(λ_F) $\subseteq$ FN-cl(λ_F) = λ_F $\subseteq$

υ_F.

Hence, λ_F is FN-SMGCs in (ω_N, Γ_N).

Remark 4.5: The converse of Theorem 4.4 is not true and this can be clarified in the following examples:

Example 4.6: i. Let ω_N = {a, b} define the FNS λ_F in ω_N as follows:

λ_F = < ω, a(0.5, 0.5, 0.5), b(0.7, 0.5, 0.2)> and let the family Γ_N = { 0_F, 1_F, λ_F, } be FNT.

If we take, Ψ_F = < ω, a(0.1, 0.5, 0.9), b(0.6, 0.5, 0.3)> .

And put, υ_F = 0_F, where υ_F be FN-Cs such that, $\upsilon_F \subseteq \Psi_F$.

Then, FN-int(Ψ_F)) = < ω, a(0, 0, 1), b(0, 0, 1)>

$\subseteq$ < ω, a(0.1, 0.5, 0.9), b(0.6, 0.5, 0.3) > = 0_F .

So, FN-int(Ψ_F) $\supseteq$ υ_F, hence, Ψ_F is FN-GOs but, not FN-Os since $\Psi_F \notin \Gamma_N$.

ii. Take Example 3.3 we notice that υ^c_F is FN-Cs because

$\upsilon^c_F \in \Gamma^c_N$ but, not FN-MCs because it's not a FN-clopen set.

iii. Take Example 3.2 we got, λ_F is FN-SMGCs but, not FN-RCs because, FN-cl(FN-int(λ_F) $\neq$ λ_F.

iv. Take Example 3.2 we got, λ_F is FN-SMGCs but, not FN-MCs because it's not a FN-clopen set.

v. Take Example 3.2 we got, λ_F is FN-SMGCs but, not FN-SMGCs because, $\lambda_F \not\supseteq$ FN-cl(FN-int(λ_F)).

vi. Take Example 3.2 we got, υ^c_F is FN-closed set, such that $\upsilon^c_F \in \Gamma^c_N$, but υ^c_F is not FN-RCs because, FN-cl(FN-int(υ^c_F)) $\neq$ υ^c_F .

Theorem 4.7: λ_F is FN-MCs if and only if λ_F is FN-MGCs.

Proof: Let λ_F = < $\boldsymbol{\omega}$, $\mu_{\lambda F}(\boldsymbol{\omega})$, $\sigma_{\lambda F}(\boldsymbol{\omega})$, $\nu_{\lambda F}(\boldsymbol{\omega})$> be FN-MCs in the FNTS (ω_N, Γ_N). Then, FN-Mcl(λ_F) = λ_F .

Now, let υ_F be FN-Os such that, $\lambda_F \subseteq \upsilon_F$. So, FN-Mcl($\lambda_F$) = $\lambda_F \subseteq \upsilon_F$, hence, λ_F is FN-MGCs.

Conversely, Let λ_F = < $\boldsymbol{\omega}$,$\mu_{\lambda F}(\boldsymbol{\omega})$, $\sigma_{\lambda F}(\boldsymbol{\omega})$, $\nu_{\lambda F}(\boldsymbol{\omega})$> be FN-MGCs in the FNTS (ω_N, Γ_N).

Then, FN-Mcl(λ_F) $\subseteq \upsilon_F$, where $\lambda_F \subseteq \upsilon_F$, FN-Mcl($\lambda_F$) = λ_F .

Now, υ_F be FN-Os so, FN-Mcl(λ_F) = $\lambda_F \subseteq \upsilon_F$. Therefore, FN-Mcl($\lambda_F$) = λ_F then, λ_F is FN-MCs in (ω_N , Γ_N).

Remark 4.8: In general topology The intersection of two strongly generalized closed sets (SGCs) is not SGCs but in fuzzy neutrosophic satisfying the converse in the next theorem.

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Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

Theorem 4.9: The intersection of any two FN-SMGCs is FN-SMGCs in General.

Proof: Let λ_F and β_F are FN-SMGCs then, FN-cl(λ_F) $\subseteq \upsilon_F$ and FN-cl(β_F) $\subseteq \beta_F$ for all λ_F and $\beta_F \subseteq \upsilon_F$.

Consider, FN-cl(λ_F) $\cap$ FN-cl(β_F) $\subseteq \upsilon_F$.

That means, FN-cl($\lambda_F \cap \beta_F$) $\subseteq \upsilon_F$.

Since λ_F and β_F are FNC-sets, then FN-cl($\lambda_F \cap \beta_F$) = $\lambda_F \cap \beta_F$.Then, $\lambda_F \cap \beta_F \subseteq \upsilon_F$ is FN-SMGCs .

Remark 4.10: The union of any two FN-SMGCs is not FN-SMGCs in general and we explained it in the next example:

Example 4.11: Take Example 3.3, we got λ_F and β_F are both FN-SMGCs then,

$\lambda_F \cup \beta_F = <\omega$, a(0.3, 0.5, 0.3), b(0.4, 0.8, 0.2), c(0.5, 0.7, 0.2) $>$.

So, FN-cl($\lambda_F \cup \beta_F$) =1_F and $1_F \nsubseteq \upsilon_F$ then, the union of any two FN-SMGCs is not FN-SMGCs.

Remark 4.12: The next Figure shows the relations between different FNSs.

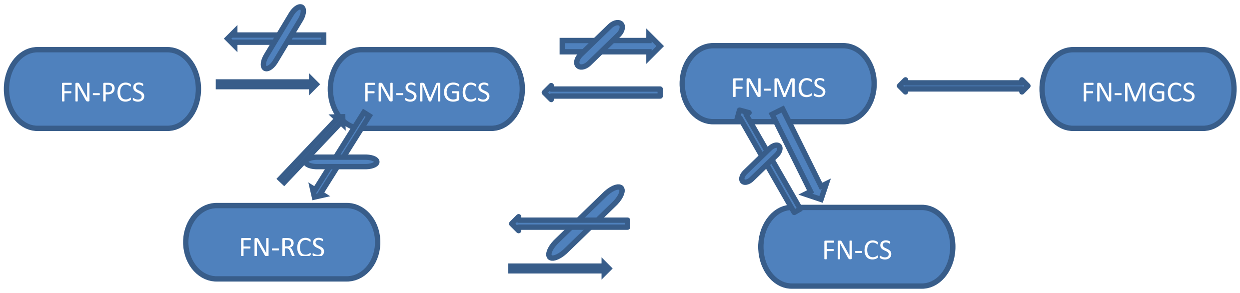

Figure 2

Theorem 4.13: Every FN-WMGCs is FN-SMGCs.

Proof: Let λ_F is FN-WMGC set then, FN-cl(FN-int(λ_F))) $\subseteq \upsilon_F$ wherever, $\lambda_F \subseteq \upsilon_F$ and υ_F is FNM-open set.

Since, by Theorem 3.10, then FN-cl(FN-int(λ_F))) $\subseteq$ FN-Mcl(λ_F) $\subseteq \upsilon_F$ wherever, $\lambda_F \subseteq \upsilon_F$.

So, FN-Mcl(λ_F) = λ_F, $\lambda_F \subseteq \upsilon_F$ and, FN-Mcl(λ_F) $\subseteq \upsilon_F$.Then, λ_F is FN-SMGCs in ω_N.

Remark 4.14: The converse of the last theorem is not true in the next example:

Example 4.15 By Example 3.3, we notice that λ_F is FN-SMGCs, then $\lambda_F \subseteq \upsilon_F$, but υ_F is not FN-MOs .

4.Conclusions

To conclude, the paper has introduced two new concepts of new classess of closed sets which include fuzzy neutrosophic weakly generalized M-closed and fuzzy neutrosophic strongly M-generalized closed sets via fuzzy neutrosophic topology. It has,also, clarified the relations between the essential studied concepts and other sets as it is

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Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023

shown in the illustrations (see Figure3.1 and Figure 4.1) . This study can be extended to several functions such as M-continuous and M-irresolute functions.

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DOI: https://doi.org/10.54216/IJNS.210201
Received: January 20, 2023 Revised: April 11, 2023 Accepted: May 09, 2023