On Some Novel Results About Weak Fuzzy Complex Matrices

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Abstract

The objective of this paper is to study the algebraic properties of weak fuzzy complex matrices, where many elementary properties will be obtained such as the invertibility, the determinants, and the eigen values and vectors. In addition, a full solution of linear systems of weak fuzzy complex equations will be provided as an effective and easy algorithm. Also, many examples to clarify the validity of our approach.

Keywords: weak fuzzy complex number; weak fuzzy complex vector space; weak fuzzy complex matrix

1. Introduction and Preliminaries

The fuzzy logic as a generalization of classical logic deals with many scientific fields. Fuzzy ideas were generalized to neutrosophic ideas in many references [1-2, 9-12].

Weak fuzzy complex numbers were defined firstly in [3], as a new generalization of classical real numbers by applying fuzzy operators. This class of numbers was used to study many corresponding algebraic structures such as vector spaces [5-7].

A research gap is still open, where the matrices with weak fuzzy complex entries are still unstudied. From this point of view, we continuo the previous efforts to prove and find the properties of weak fuzzy complex algebraic structures, in a similar way of neutrosophic matrices [10, 13].

Many related points will be handled, such as the invertibility, the computing of determinants, and the finding of eigen values with applications in weak fuzzy linear systems of equations.

Definition.
The ring of weak fuzzy complex numbers is defined as follows:

$$C_w = \{a + bJ; a, b \in R, J^2 = t \in]0,1[\}$$

Example.
For $J^2 = t = \frac{1}{\sqrt{3}}$, we have:

$$C_w = \left\{a + bJ, J^2 = \frac{1}{\sqrt{3}}; a, b \in R\right\}$$

For $X = 1 + 3J, Y = -2 + J$, we have:

$$X + Y = -1 + 4J, X.Y = -2 + J - 6J + 3J^2 = \left(-2 + \frac{3}{\sqrt{3}}\right) - 5J^2 = \left(-2 + \sqrt{3}\right) - 5J$$

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Definition.

Let V be a vector space over R, the corresponding weak fuzzy vector space is defined as follows:

$$V_w = \{x + yJ; x, y \in V, J^2 = t \in]0,1[\}$$

Example.

Take $V = R^2$, then

$V_w = \left\{(x_1, y_1, z_1) + (x_2, y_2, z_2)J; x_i, y_i, z_i \in R, J^2 = \frac{1}{4}\right\}$ is a weak fuzzy complex vector space.

2. Main Discussion

Definition.

Let $A_{n\times m} = (a_{ij})$ be a matrix, A is called weak fuzzy complex matrix if $a_{ij} \in C_w$, for $J^2 = t \in]0,1[$.

Example:

Consider C_w, for $J^2 = \frac{1}{2}$, take the following:

$A = \begin{pmatrix} 1+J & 1-J \\ J & J \end{pmatrix}, B = \begin{pmatrix} 2-3J & -J \\ J & 4J \end{pmatrix}$, we have:

$A + B = \begin{pmatrix} 2-3J & 1-2J \\ 2J & 5J \end{pmatrix}, A.B = \begin{pmatrix} 2-3J+2J-3J^2+J-J^2 & -J-J^2+4J-4J^2 \\ 2J-3J^2+J^2 & -J^2+4J^2 \end{pmatrix}$

So that $A.B = \begin{pmatrix} 0 & \frac{-5}{2}+3J \\ -1+2J & \frac{3}{2} \end{pmatrix}$

We denote the set of all weak fuzzy complex matrices with $J^2 = t \in]0,1[$ by μ_{C_t}.

Remark.

$(\mu_{C_t}, +, .)$ Is an element of μ_{C_t}, then A can be written as follows $A = A_1 + A_2J$; $(A_1)_{n\times m}, (A_2)_{n\times m} \in \mu_{n\times m}$.

Example.

Consider $A = \begin{pmatrix} 1+J & 2-J & J \\ J & 4+10J & 1+8J \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} + J\begin{pmatrix} 1 & -1 & 1 \\ 1 & 10 & 8 \end{pmatrix} = A_1 + A_2J$.

Remark.

Let $A_{n\times m}, B_{n\times m} \in \mu_{C_t}$, then $A = A_1 + A_2J, B = B_1 + B_2J$, then:

$A + B = (A_1 + A_2) + (B_1 + B_2)J$.

Remark.

Let $A_{n\times m}, B_{m\times k} \in \mu_{C_t}$, then $A = A_1 + A_2J, B = B_1 + B_2J$, then:

$A \times B = [A_1 \times B_1 + (A_2 \times B_2)t] + [A_1 \times B_2 + A_2 \times B_1]J$.

For example: $A = \begin{pmatrix} 1+J & 1-J \\ J & J \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + J\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = A_1 + A_2J$

$B = \begin{pmatrix} 2-3J & -J \\ J & 4J \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + J\begin{pmatrix} -3 & -1 \\ 1 & 4 \end{pmatrix} = B_1 + B_2J$.

For $J^2 = t = \frac{1}{2}$, we have:

$$A_1 \times B_1 + (A_2 \times B_2)t = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -4 & -5 \\ -2 & 3 \end{pmatrix}\left(\frac{1}{2}\right) = \begin{pmatrix} 0 & \frac{-5}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$$

$A_1 \times B_2 + A_2 \times B_1 = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$, so that $A \times B = \begin{pmatrix} 0 & \frac{-5}{2}+3J \\ -2+2J & \frac{3}{2} \end{pmatrix}$.

Invertibility:

Let $A = A_1 + A_2J \in \mu_{C_t}$ be an n-square weak fuzzy complex matrix, then A is invertible if and only if there exists $B = B_1 + B_2J \in \mu_{C_t}$ such that $A \times B = U_{n\times n} = B \times A$.

Theorem.

Let $A = A_1 + A_2J \in \mu_{C_t}$ be an n-square weak fuzzy complex matrix, then A is invertible if and only if $A_1 - \sqrt{t}A_2, A_1 + \sqrt{t}A_2$ are invertible, and:

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$A^{-1} = \frac{1}{2}\left[\left(A_1 - \sqrt{t}A_2\right)^{-1} + \left(A_1 + \sqrt{t}A_2\right)^{-1}\right] + \frac{1}{2\sqrt{t}}J\left[\left(A_1 + \sqrt{t}A_2\right)^{-1} - \left(A_1 - \sqrt{t}A_2\right)^{-1}\right].$

Proof.

A is invertible if and only if there exists $A^{-1} = B = B_1 + B_2J$, such that $A \times B = B \times A = U_{n\times n}$.

These equivalents:

$$\begin{cases} A_1 \times B_1 + A_2 \times B_2 t = B_1 \times A_1 + B_2 \times A_2 t \ldots (1) \\ A_1 \times B_2 + A_2 \times B_1 = O_{n\times n} = A_2 \times B_1 + A_1 \times B_2 \ldots (2) \end{cases}$$

We multiply (2) by $\sqrt{t}$ and the subtract (2) from (1), then:

$A_1 \times B_1 + A_2 \times B_2 t - \sqrt{t}(A_1 \times B_2 + A_2 \times B_1) = U_{n\times n}$, thus:

$\left(A_1 - \sqrt{t}A_2\right)\left(B_1 - \sqrt{t}B_2\right) = U_{n\times n}$, so that $A_1 - \sqrt{t}A_2$ is invertible and $\left(A_1 - \sqrt{t}A_2\right)^{-1} = B_1 - \sqrt{t}B_2$.

We multiply (2) by $\sqrt{t}$ and add (1) to (2), then:

$A_1 \times B_1 + A_2 \times B_2 t - \sqrt{t}(A_1 \times B_2 + A_2 \times B_1) = U_{n\times n}$, thus:

$\left(A_1 + \sqrt{t}A_2\right)\left(B_1 + \sqrt{t}B_2\right) = U_{n\times n}$, so that $A_1 + \sqrt{t}A_2$ is invertible and $\left(A_1 + \sqrt{t}A_2\right)^{-1} = B_1 + \sqrt{t}B_2$.

This implies that $B_1 = \frac{1}{2}\left[\left(A_1 - \sqrt{t}A_2\right)^{-1} + \left(A_1 + \sqrt{t}A_2\right)^{-1}\right], B_2 = \frac{1}{2\sqrt{t}}J\left[\left(A_1 + \sqrt{t}A_2\right)^{-1} - \left(A_1 - \sqrt{t}A_2\right)^{-1}\right]$

Example.

Take $J^2 = t = \frac{1}{2}$, $A = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ 1 & 1 \end{pmatrix} + J\begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} + \frac{3}{\sqrt{2}}J & \frac{1}{2} - \frac{1}{\sqrt{2}}J \\ 1 + \sqrt{2}J & 1 \end{pmatrix}$.

$A_1 - \sqrt{\frac{1}{2}}A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\left(A_1 - \sqrt{\frac{1}{2}}A_2\right)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

$A_1 + \sqrt{\frac{1}{2}}A_2 = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$, $\left(A_1 + \sqrt{\frac{1}{2}}A_2\right)^{-1} = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{-1}{2} & 1 \end{pmatrix}$

Thus $A^{-1} = \frac{1}{2}\left[\begin{pmatrix} \frac{5}{4} & -1 \\ \frac{-1}{2} & 2 \end{pmatrix}\right] + \frac{1}{2\sqrt{\frac{1}{2}}}J\left[\begin{pmatrix} \frac{-3}{4} & 1 \\ \frac{-1}{2} & 0 \end{pmatrix}\right] = \begin{pmatrix} \frac{5}{8} & -\frac{1}{2} \\ -\frac{1}{4} & 1 \end{pmatrix} + J\begin{pmatrix} -\frac{3}{4\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 \end{pmatrix} =$

$\begin{pmatrix} \frac{5}{8} - \frac{3}{4\sqrt{2}}J & -\frac{1}{2} + \frac{1}{2\sqrt{2}}J \\ -\frac{1}{4} - \frac{1}{2\sqrt{2}}J & 1 \end{pmatrix}$

Definition.

Let $A = A_1 + A_2J \in \mu_{C_t}$, then we define:

$det(A) = \frac{1}{2}\left[det\left(A_1 - \sqrt{t}A_2\right) + det\left(A_1 + \sqrt{t}A_2\right)\right] + \frac{1}{2\sqrt{t}}J\left[det\left(A_1 + \sqrt{t}A_2\right) - det\left(A_1 - \sqrt{t}A_2\right)\right]$ where A is a square matrix.

Theorem.

Let $A = A_1 + A_2J, B = B_1 + B_2J \in \mu_{C_t}$ be two n-square matrices, then:

1. $det(A.B) = det(A).det(B)$.
2. $det(A^T) = det(A)$.
3. A is invertible if and only if $det(A)$ is invertible in C_w.

Proof.

1. $A \times B = A_1 \times B_1 + A_2 \times B_2 t + [A_1 \times B_2 + A_2 \times B_1]J$

$$\begin{aligned} det(A \times B) &= \frac{1}{2}\left[det\left(A_1B_1 + A_2B_2t - \sqrt{t}(A_1B_2 + A_2B_1)\right) + det\left(A_1B_1 + A_2B_2t + \sqrt{t}(A_1B_2 + A_2B_1)\right)\right] \\ &+ \frac{1}{2\sqrt{t}}J\left[det\left(A_1B_1 + A_2B_2t + \sqrt{t}(A_1B_2 + A_2B_1)\right)\right. \\ &\left.- det\left(A_1B_1 + A_2B_2t - \sqrt{t}(A_1B_2 + A_2B_1)\right)\right] \\ &= \frac{1}{2}\left[det\left[\left(A_1 - \sqrt{t}A_2\right)\left(B_1 - \sqrt{t}B_2\right)\right] + det\left[\left(A_1 + \sqrt{t}A_2\right)\left(B_1 + \sqrt{t}B_2\right)\right]\right] \\ &+ \frac{1}{2\sqrt{t}}J\left[det\left[\left(A_1 + \sqrt{t}A_2\right)\left(B_1 + \sqrt{t}B_2\right)\right] - det\left[\left(A_1 - \sqrt{t}A_2\right)\left(B_1 - \sqrt{t}B_2\right)\right]\right] \end{aligned}$$

On the other hand, we have:

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$det(A).det(B) = \left(\frac{1}{2}\left[det\left(A_1 - \sqrt{t}A_2\right) + det\left(A_1 + \sqrt{t}A_2\right)\right] + \frac{1}{2\sqrt{t}}J\left[det\left(A_1 + \sqrt{t}A_2\right) - det\left(A_1 - \sqrt{t}A_2\right)\right]\right) \times \left(\frac{1}{2}\left[det\left(B_1 - \sqrt{t}B_2\right) + det\left(B_1 + \sqrt{t}B_2\right)\right] + \frac{1}{2\sqrt{t}}J\left[det\left(B_1 + \sqrt{t}B_2\right) - det\left(B_1 - \sqrt{t}B_2\right)\right]\right) = \frac{1}{4}det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) + \frac{1}{4}det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right) + \frac{1}{4}det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) + \frac{1}{4}det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right) + \frac{1}{4}det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right) - \frac{1}{4}det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) - \frac{1}{4}det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right) + \frac{1}{4}det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) + \frac{1}{4\sqrt{t}}J\left[det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right) - det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) + det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) - det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) + det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) + det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right) - det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right) - det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right)\right] = \frac{1}{2}\left[det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right) + det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right)\right] + \frac{1}{2\sqrt{t}}J\left[det\left(A_1 + \sqrt{t}A_2\right)det\left(B_1 + \sqrt{t}B_2\right) - det\left(A_1 - \sqrt{t}A_2\right)det\left(A_1 - \sqrt{t}A_2\right)det\left(B_1 - \sqrt{t}B_2\right)\right] = det(A \times B).$

2. $det(A^T) = det({A_1}^T + {A_2}^T J) = \frac{1}{2}\left[det\left(A_1 - \sqrt{t}A_2\right)^T + det\left(A_1 + \sqrt{t}A_2\right)^T\right] + \frac{1}{2\sqrt{t}}J\left[det\left(A_1 + \sqrt{t}A_2\right)^T - det\left(A_1 - \sqrt{t}A_2\right)^T\right] = det(A)$

3. A is invertible if and only if $A_1 - \sqrt{t}A_2, A_1 + \sqrt{t}A_2$ are invertible, hence $det\left(A_1 - \sqrt{t}A_2\right) \neq 0$, $det\left(A_1 + \sqrt{t}A_2\right) \neq 0$, thus $det(A) = \frac{1}{2}\left[det\left(A_1 - \sqrt{t}A_2\right) + det\left(A_1 + \sqrt{t}A_2\right)\right] + \frac{1}{2\sqrt{t}}J\left[det\left(A_1 + \sqrt{t}A_2\right) - det\left(A_1 - \sqrt{t}A_2\right)\right]$ is invertible in C_w.

Example.

For $J^2 = t = \frac{1}{4}$, we consider:

$A = \begin{pmatrix} 2+4J & 4-6J \\ 4+10J & 5J \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 0 \end{pmatrix} + J\begin{pmatrix} 4 & -6 \\ 10 & 5 \end{pmatrix} = A_1 + A_2 J$

$A_1 + \sqrt{t}A_2 = A_1 + \frac{1}{2}A_2 = \begin{pmatrix} 4 & 1 \\ 9 & 4 \end{pmatrix}, det\left(A_1 + \sqrt{t}A_2\right) = 16 - 9 = 7$

$A_1 - \sqrt{t}A_2 = A_1 - \frac{1}{2}A_2 = \begin{pmatrix} 0 & 7 \\ -1 & -4 \end{pmatrix}, det\left(A_1 - \sqrt{t}A_2\right) = 7$

$det(A) = \frac{1}{2}[7+7] + \frac{1}{4}J[7-7] = 7.$

On the other hand, we compute $det(A)$ as follows:

$det(A) = (2+4J)(8J) - (4-6J)(4+10J) = 16J + 8 - (16 + 40J - 24J - 15) = 7.$

3. The Applications To Linear Systems of Equations.

We can solve it by $X = A^{-1}.B$ if A is invertible.

Example.

Consider the following system:

$(1+2J)X_1 + (3-J)X_2 = \frac{1}{9} + 4J$, where $J^2 = t = \frac{1}{9}$

$(4+J)X_1 + (1+J)X_2 = \frac{2}{9} + 5J$

It can be written as follows $A.X = B$, where:

$A = \begin{pmatrix} 1+2J & 3-J \\ 4+J & 1+J \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix} + J\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = A_1 + A_2 J$

$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} x_1 + \acute{x}_1 J \\ x_2 + \acute{x}_2 J \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + J\begin{pmatrix} \acute{x}_1 \\ \acute{x}_2 \end{pmatrix}$

$B = \begin{pmatrix} \frac{1}{9} + 4J \\ \frac{2}{9} + 5J \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ \frac{2}{9} \end{pmatrix} + J\begin{pmatrix} 4 \\ 5 \end{pmatrix} = B_1 + B_2 J$

$A_1 - \sqrt{t}A_2 = A_1 - \frac{1}{3}A_2 = \begin{pmatrix} \frac{1}{3} & \frac{10}{3} \\ \frac{11}{3} & \frac{2}{3} \end{pmatrix}, det\left(A_1 - \sqrt{t}A_2\right) = \frac{2}{9} - \frac{110}{9} = -\frac{108}{9} = -12$

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$$A_1+\sqrt{t}A_2=A_1+\frac{1}{3}A_2=\begin{pmatrix}\frac{5}{3} & \frac{8}{3}\\ \frac{13}{3} & \frac{4}{3}\end{pmatrix}, det(A_1+\sqrt{t}A_2)=\frac{20}{9}-\frac{104}{9}=-\frac{84}{9}=-\frac{28}{3}$$

Thus A is invertible:

$$(A_1-\sqrt{t}A_2)^{-1}=-\frac{1}{12}\begin{pmatrix}\frac{2}{3} & \frac{-10}{3}\\ \frac{-11}{3} & \frac{1}{3}\end{pmatrix}=\begin{pmatrix}\frac{-2}{36} & \frac{10}{36}\\ \frac{11}{36} & \frac{-1}{36}\end{pmatrix}$$

$$(A_1+\sqrt{t}A_2)^{-1}=-\frac{3}{28}\begin{pmatrix}\frac{4}{3} & \frac{-8}{3}\\ \frac{-13}{3} & \frac{5}{3}\end{pmatrix}=\begin{pmatrix}\frac{-12}{84} & \frac{24}{84}\\ \frac{39}{84} & \frac{-15}{84}\end{pmatrix}$$

$$A^{-1}=\frac{1}{2}\left[\begin{pmatrix}\frac{-50}{252} & \frac{142}{252}\\ \frac{194}{252} & \frac{-52}{252}\end{pmatrix}+\begin{pmatrix}\frac{-12}{84} & \frac{24}{84}\\ \frac{39}{84} & \frac{-15}{84}\end{pmatrix}\right]+\frac{1}{2\times\frac{1}{3}}J\left[\begin{pmatrix}\frac{-22}{252} & \frac{2}{252}\\ \frac{40}{252} & \frac{-35}{252}\end{pmatrix}-\begin{pmatrix}\frac{-2}{36} & \frac{10}{36}\\ \frac{11}{36} & \frac{-1}{36}\end{pmatrix}\right]$$

$$A^{-1}=\begin{pmatrix}\frac{-50}{504} & \frac{142}{504}\\ \frac{194}{504} & \frac{-52}{504}\end{pmatrix}+J\begin{pmatrix}\frac{-66}{504} & \frac{6}{504}\\ \frac{120}{504} & \frac{-114}{504}\end{pmatrix}$$

$$\begin{aligned}A^{-1}\times B&=\begin{pmatrix}\frac{-50}{504} & \frac{142}{504}\\ \frac{194}{504} & \frac{-52}{504}\end{pmatrix}\begin{pmatrix}\frac{1}{9}\\ \frac{2}{9}\end{pmatrix}+\frac{1}{9}\begin{pmatrix}\frac{-66}{504} & \frac{6}{504}\\ \frac{120}{504} & \frac{-114}{504}\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}\\ &+J\left[\begin{pmatrix}\frac{-50}{504} & \frac{142}{504}\\ \frac{194}{504} & \frac{-52}{504}\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}+\begin{pmatrix}\frac{-66}{504} & \frac{6}{504}\\ \frac{120}{504} & \frac{-114}{504}\end{pmatrix}\begin{pmatrix}\frac{1}{9}\\ \frac{2}{9}\end{pmatrix}\right]\\ &=\begin{pmatrix}\frac{-50}{4536} & \frac{284}{4536}\\ \frac{194}{4536} & -\frac{104}{4536}\end{pmatrix}+\begin{pmatrix}\frac{-264}{4536} & \frac{30}{4536}\\ \frac{480}{4536} & -\frac{570}{4536}\end{pmatrix}+J\left[\begin{pmatrix}\frac{-200}{504} & \frac{710}{504}\\ \frac{776}{504} & \frac{-260}{504}\end{pmatrix}+\begin{pmatrix}\frac{-66}{4536} & \frac{12}{4536}\\ \frac{120}{504} & \frac{-228}{504}\end{pmatrix}\right]\\ &=\begin{pmatrix}0\\0\end{pmatrix}+J\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}J\\J\end{pmatrix}\end{aligned}$$

Thus, the solution of the system is:

$$X_1=J, X_2=J$$

Example.

Consider $J^2=t=\frac{1}{4}$, take the following linear system:

$$\begin{cases}(4+4J)X_1+(8-8J)X_2=3+16J\\ JX_1+15JX_2=4+J\end{cases}$$

We have:

$$A=\begin{pmatrix}4+4J & 8-8J\\ J & 15J\end{pmatrix}=\begin{pmatrix}4 & 8\\ 0 & 0\end{pmatrix}+J\begin{pmatrix}4 & -8\\ 1 & 15\end{pmatrix}=A_1+A_2J$$

$$B=\begin{pmatrix}3+16J\\ 4+J\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}+J\begin{pmatrix}16\\1\end{pmatrix}=B_1+B_2J$$

$$A_1-\sqrt{t}A_2=A_1-\frac{1}{2}A_2=\begin{pmatrix}2 & 12\\ \frac{-1}{2} & \frac{-15}{2}\end{pmatrix}, det(A_1-\sqrt{t}A_2)=-15+6=-9$$

$$A_1+\sqrt{t}A_2=A_1+\frac{1}{2}A_2=\begin{pmatrix}6 & 4\\ \frac{1}{2} & \frac{15}{2}\end{pmatrix}, det(A_1+\sqrt{t}A_2)=45-2=43$$

$$(A_1-\sqrt{t}A_2)^{-1}=-\frac{1}{9}\begin{pmatrix}\frac{-15}{2} & -12\\ \frac{1}{2} & 2\end{pmatrix}=\begin{pmatrix}\frac{15}{18} & \frac{12}{9}\\ \frac{-1}{18} & -\frac{2}{9}\end{pmatrix}$$

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$$(A_1+\sqrt{t}A_2)^{-1}=\frac{1}{43}\begin{pmatrix}\frac{15}{2} & -4\\ \frac{-1}{2} & 6\end{pmatrix}=\begin{pmatrix}\frac{15}{86} & \frac{-4}{43}\\ \frac{-1}{86} & \frac{6}{43}\end{pmatrix}$$

$$A^{-1}=\frac{1}{2}\left[\begin{pmatrix}\frac{15}{18} & \frac{12}{9}\\ \frac{-1}{18} & -\frac{2}{9}\end{pmatrix}+\begin{pmatrix}\frac{15}{86} & \frac{-4}{43}\\ \frac{-1}{86} & \frac{6}{43}\end{pmatrix}\right]+\frac{1}{2\times\frac{1}{3}}J\left[\begin{pmatrix}\frac{15}{86} & \frac{-4}{43}\\ \frac{-1}{86} & \frac{6}{43}\end{pmatrix}-\begin{pmatrix}\frac{15}{18} & \frac{12}{9}\\ \frac{-1}{18} & -\frac{2}{9}\end{pmatrix}\right].$$

$X=A^{-1}\times B.$

4. Natural Powers.

Theorem.

Let $A=A_1+A_2J\in\mu_{C_t}$, then $A^n=\frac{1}{2}\left[(A_1-\sqrt{t}A_2)^n+(A_1+\sqrt{t}A_2)^n\right]+\frac{1}{2\sqrt{t}}J\left[(A_1+\sqrt{t}A_2)^n-(A_1-\sqrt{t}A_2)^n\right]$.

Proof.

For $n=1$, it is clear. We assume that it holds for $n=k$, we will prove it for $n=k+1$.

$$\begin{aligned}A^{k+1}=A.A^k&=(A_1+A_2J)\left(\frac{1}{2}\left[(A_1-\sqrt{t}A_2)^k+(A_1+\sqrt{t}A_2)^k\right]+\frac{1}{2\sqrt{t}}J\left[(A_1+\sqrt{t}A_2)^k-(A_1-\sqrt{t}A_2)^k\right]\right)\\
&=\frac{1}{2}A_1(A_1-\sqrt{t}A_2)^k+\frac{1}{2}A_1(A_1+\sqrt{t}A_2)^k+\frac{1}{2\sqrt{t}}tA_2(A_1+\sqrt{t}A_2)^k\\
&-\frac{1}{2\sqrt{t}}tA_2(A_1-\sqrt{t}A_2)^k\\
&+J\left[\frac{1}{2}A_2(A_1-\sqrt{t}A_2)^k+\frac{1}{2}A_2(A_1+\sqrt{t}A_2)^k+\frac{1}{2\sqrt{t}}A_1(A_1+\sqrt{t}A_2)^k\right.\\
&\left.-\frac{1}{2\sqrt{t}}A_1(A_1-\sqrt{t}A_2)^k\right]\\
&=\frac{1}{2}(A_1+\sqrt{t}A_2)(A_1+\sqrt{t}A_2)^k+\frac{1}{2}(A_1-\sqrt{t}A_2)(A_1-\sqrt{t}A_2)^k\\
&+J\left[\frac{1}{2}\left(A_1+\frac{1}{\sqrt{t}}A_2\right)(A_1+\sqrt{t}A_2)^k+\frac{1}{2}\left(A_1-\frac{1}{\sqrt{t}}A_2\right)(A_1-\sqrt{t}A_2)^k\right]\\
&=\frac{1}{2}\left[(A_1+\sqrt{t}A_2)^{k+1}+(A_1-\sqrt{t}A_2)^{k+1}\right]+\frac{1}{2\sqrt{t}}J\left[(A_1+\sqrt{t}A_2)^{k+1}-(A_1-\sqrt{t}A_2)^{k+1}\right]\end{aligned}$$

Thus, the proof is complete by induction.

Theorem.

Let $A=A_1+A_2J\in\mu_{C_t}$, then:

1. A is idempotent if and only if $A_1-\sqrt{t}A_2, A_1+\sqrt{t}A_2$ are idempotents.
2. A is nilpotent if and only if $A_1-\sqrt{t}A_2, A_1+\sqrt{t}A_2$ are nilpotents.

Proof.

A is idempotent matrix if and only if $A^2=A$, thus:

$\frac{1}{2}\left[(A_1+\sqrt{t}A_2)^2+(A_1-\sqrt{t}A_2)^2\right]+\frac{1}{2\sqrt{t}}J\left[(A_1+\sqrt{t}A_2)^2-(A_1-\sqrt{t}A_2)^2\right]=A_1+A_2J$, this means that:

$$\begin{cases}A_1=\frac{1}{2}\left[(A_1+\sqrt{t}A_2)^2+(A_1-\sqrt{t}A_2)^2\right]\ldots(1)\\ A_2=\frac{1}{2\sqrt{t}}J\left[(A_1+\sqrt{t}A_2)^2-(A_1-\sqrt{t}A_2)^2\right]\ldots(2)\end{cases}$$

We multiply (2) by $\sqrt{t}$, and then add (1) to (2).

$A_1+\sqrt{t}A_2=(A_1+\sqrt{t}A_2)^2$, thus $A_1+\sqrt{t}A_2$ is idempotent.

Also, we multiply (2) by $\sqrt{t}$, and then compute (1) - (2).

$A_1-\sqrt{t}A_2=(A_1-\sqrt{t}A_2)^2$, therefor $A_1+\sqrt{t}A_2$ is idempotent.

A is nilpotent if and only if there exists $n\in Z^+$ such that $A^n=O_{n\times n}$, which is equivalent to:

$$\begin{cases}\frac{1}{2}\left[(A_1+\sqrt{t}A_2)^n+(A_1-\sqrt{t}A_2)^n\right]=O_{n\times n}\\ \frac{1}{2\sqrt{t}}J\left[(A_1+\sqrt{t}A_2)^n-(A_1-\sqrt{t}A_2)^n\right]=O_{n\times n}\end{cases}$$

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Thus $\left(A_1-\sqrt{t}A_2\right)^n=\left(A_1+\sqrt{t}A_2\right)^n=O_{n\times n}$, therefor $A_1-\sqrt{t}A_2, A_1+\sqrt{t}A_2$ are nilpotent.

5. Conclusion

In this paper, we have studied for the first time the concept of weak fuzzy complex matrices, and we have obtained many results about the invertibility, the eigen values/vectors of these matrices. On the other hand, we have found a formula for computing the solutions of linear systems with fuzzy weak complex coefficients by applying this class of matrices.

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