On a Novel Generalization of p-Quasi-λ-Nuclear Operators

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Abstract

In this paper we generalize the concept of 2-quasi -λ- nuclear operators between Normed spaces ($\lambda \subseteq l_1$) to P-quasi-λ-nuclear operators between locally convex spaces ($P > 0, \lambda \subseteq l_\infty$) and we study the relationship between p-quasi-λ- nuclear, nuclear operators, λ-nuclear, quasi-nuclear and quasi-λ- nuclear. Also, we prove that the composition of two operators, one of them is a P-quasi-λ-nuclear, is again a p-quasi-λ-nuclear operator.

Keywords: Nuclear operators; quasi nuclear operators; normed spaces

1. Introductio

By Shatanawi [5], the operator T from a normed space E into a normed space F is said to be 2-quasi- λ -nuclear if there is a sequence $(\alpha_n) \in \lambda(\lambda \subseteq l_1)$ and a bounded sequence (α_n) in E' such that

$$\|Tx\| \leq \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, \alpha_n\rangle|^2\right)^{1/2} < \infty, \quad \forall x \in E.$$

In this paper, we generalize this definition to p-quasi-λ-nuclear operator between locally convex spaces where$\lambda \subseteq l_\infty$and $P > 0$.

By locally convex spaces E, we mean a locally convex spaces E with a topology induced by a continuous sequence of seminorms. Throughout this paper E, F,… will denote locally convex spaces over the same field **K**. By E' we mean the set of all continuous linear operators from E into **K**. We let $\langle x, a\rangle = a(x)$ for all x in E and a in E'.

By l_p [respectively, c_0, l_∞], we mean the usual Banach space of all scalarvalued, p-power summable [respectively, zero-convergent, bounded]
sequences. By λ we mean any sequence space subset of l_∞.

We say that the sequence $a = (a_n)$ *dominates* the sequence $b = (b_n)$, written $b_n = O(a_n)$, if there exists a real number $M > 0$ such that $b_n \leq M a_n$ for all $n \in \square$. A set A of sequences of non-negative real numbers is called a *Köthe set*, if it satisfies the following conditions:

(1) $\forall a, b \in A$ there is $c \in A$ with $a_n = O(c_n)$ and $b_n = O(c_n)$.
(2) $\forall r \in \square$ there exists $a \in A$ with $a_r > 0$.

The sequence space $\lambda(A)$ defined by

$$\lambda(A) = \{x = (x_n) : q_a(x) = \sum_{n=1}^{\infty} |x_n| a_n < \infty\}$$

is called a *Köthe space* generated by A (see [2]).

The space s of rapidly decreasing sequences is a Köthe space which is generated by the set $A = \{(n^k) : k \in \square\}$, so

$$s = \left\{(a_n) : \sum_{n=1}^{\infty} |a_n| n^k < \infty \quad \forall k \in \square\right\} \quad \text{(see[13])}$$

For each closed and absolutely convex bounded subset B of a locally convex space E

$$E_B = \{x \in E : x \in \rho B \quad \text{for some } \rho > 0\} = \bigcup_{\rho > 0} \rho B$$

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is a linear subspace of E which is the linear span of B and B is absorbent in E_B. Also we have $r_B: E_B \to \square$ defined by $r_B(x) = \inf\{\rho > 0: x \in \rho B\}$ is a norm, so we shall always consider E_B as a normed space with respect to the norm r_B. We call the disk B_a *Banach disk* if E_B happens to be a Banach space (see [1]).

2. Main Results

Definition 2.1 [1] A continuous operator $T: E \to F$ is said to be *nuclear* iff there is a sequence $(\alpha_n) \in l_1$, an equicontinuous sequence (α_n) in E', a Banach disk B in F and a bounded sequence (y_n) in F_B such that

$$Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, a_n \rangle y_n, \qquad \forall x \in E$$

Definition 2.2 A continuous operator $T: E \to F$ is said to be *p-quasi-nuclear* iff for each continuous seminorm q on F there is an equicontinuous sequence (a_n) in E' such that

$$q(Tx) \leq \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^P \right)^{1/P} < \infty, \quad \forall x \in E$$

Definition 2.3 A continuous operator $T: E \to F$ is said to be *p-quasi-λ- nuclear* iff for each continuous seminorm q on F there is a sequence $(\alpha_n) \in \lambda$ and an equicontinuous sequence (a_n) in E' such that

$$q(Tx) \leq \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n \rangle|^P \right)^{1/P} < \infty, \quad \forall x \in E$$

The operator $T: E \to F$ is said to be *quasi-nuclear* (respectively, *quasi-λ-nuclear*) if it is 1-quasi-nuclear (respectively, 1-quasi-λ-nuclear**)**

Note: If $(F, \|.\|)$. is a normed space, then $T: E \to F$ is called a *p*quasi-λ nuclear operator if there is a sequence (α_n)in λ and a bounded sequence (a_n)in E' such that

$$\|Tx\| \leq \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n \rangle|^P \right)^{1/P}, \quad \forall x \in E$$

Proposition 2.4 *If $\lambda = l_1$and p >1 , then each quasi-λ-nuclear operator is p-quasi-λ-nuclear.*

Proof: Let $T: E \to F$ be a quasi-λ-nuclear operator. Then for each continuous seminorm q on F, there exist an $(\alpha_n) \in \lambda$ and an equicontinuous sequence(a_n) in E' such that

$$q(Tx) \leq \sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n \rangle|, \quad \forall x \in E$$

Let $\beta = \sum_n |\alpha_n|$. Since p >1 , there exists r >1 such that $\frac{1}{p} + \frac{1}{r} = 1$, and so by Hölder's inequality we have

$$q(Tx) \leq \sum_{n=1}^{\infty} |\alpha_n|^{\frac{1}{p}+\frac{1}{r}} |\langle x, a_n \rangle| = \left(\sum_{n=1}^{\infty} (|\alpha_n|)^{\frac{1}{r}} (|\alpha_n|)^{\frac{1}{p}} |\langle x, a_n \rangle| \right) \leq$$

$$\left(\sum_{n=1}^{\infty} |\alpha_n| \right)^{\frac{1}{r}} \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n \rangle|^P \right)^{1/P}$$

$$= \beta^{1/r} \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n \rangle|^P \right)^{1/P} = \left(\sum_{n=1}^{\infty} |\alpha_n| \left| \langle x, \beta^{\frac{1}{r}} a_n \rangle \right|^P \right)^{1/P}$$

Since $(\alpha_n) \in \lambda$and $(\beta^{\frac{1}{r}} a_n)$is an equicontinuous sequence in E ', T is a p-quasi-λ-nuclear operator. ⊔

Proposition 2.5 *If p >1 , then every p-quasi-nuclear operator is a p-quasi-l_p-nuclear operator.*

Proof: Assume $T: E \to F$ is p-quasi-nuclear operator. Then for each continuous seminorm q on F there exists an equicontinuous sequence (a_n)in E' such that $\forall x \in E$we have

$$q(Tx) \leq \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^P \right)^{1/P} = \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle| \left| \langle x, \frac{a_n}{|\langle x, a_n \rangle|^{1/P}} \rangle \right|^P \right)^{1/P}$$

without lose of generality we can assume that $\langle x, a_n \rangle \neq 0$.

Since $(\langle x, a_n \rangle) \in l_p$ and $(a_n / |\langle x, a_n \rangle|^{1/P})$ is an equicontinuous sequence in E ', T is a p-quasi- l_p -nuclear operator. ⊔

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Proposition 2.6 *If p >1 , then every p-quasi-λ-nuclear operator is a p-quasi-nuclear operator.*
Proof: Let $T: E \to F$ be a quasi-λ-nuclear operator. Then for every continuous seminorm q on F there exist a sequence $(\alpha_n) \in \lambda$, and an equicontinuous sequence (a_n)in E' such that

$$q(T(x)) \le \left(\sum_{n}^{\infty} |\alpha_n| |\langle x, a_n\rangle|^P\right)^{1/P}, \quad \forall x \in E$$

$$= \left(\sum_{n}^{\infty} \left|\langle x, |\alpha_n|^{1/P} a_n\rangle\right|^P\right)^{1/P}$$

Since $(|\alpha_n|^{1/P} a_n)$ is an equicontinuous sequence in E', T is p-quasi-nuclear operator. ⊔

Theorem 2.7 *If $\lambda = l_1$ and p >1 , then, then every nuclear operator is p-quasi-λ-nuclear operator.*
Proof: Let $T: E \to F$ be a quasi-λ-nuclear operator. Then there exist a sequence $(\alpha_n) \in \lambda$, an equicontinuous sequence (a_n) in E', a Banach disk B in F, and a bounded sequence (y_n)in F_B such that for all $x \in E$

$$T(x) = \sum_{n=1}^{\infty} \alpha_n \langle x, a_n\rangle y_n$$

Then for all continuous seminorms q on F and $x \in E$ we have

$$q(T(x)) = q\left(\sum_{n=1}^{\infty} \alpha_n \langle x, a_n\rangle y_n\right)$$

$$\le \sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n\rangle| q(y_n)$$

Since (y_n) is a bounded sequence, there exists $M > 0$ such that $q(y_n) \le M$,
and so

$$q(T(x)) =\le \sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n\rangle| M$$

$$= \sum_{n=1}^{\infty} |\alpha_n| |\langle x, M a_n\rangle|$$

Thus, T is a quasi-λ-nuclear operator, and by proposition (2.4) T is a p-quasi-λ-nuclear operator. ⊔

Theorem 2.8 *Let $0 < p < q < \infty$. Then*
1. *Every p-quasi-l_1 -nuclear operator is q-quasi- l_1 -nuclear operator.*
2. *The continuous operator $T: E \to F$ is p-quasi-s-nuclear iff it is q-quasi-s-nuclear.*

Proof:(1) Suppose that $0 < p < q < \infty$ and $T: E \to F$ is a p-quasi-λ-nuclear operator. Then for each continuous seminorm r on F there exists an $(\alpha_n) \in l_1$and an equicontinuous sequence (a_n) in E' such that

$$r(Tx) \le \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n\rangle|^P\right)^{1/P}, \quad \forall x \in E$$

Let $t = \frac{q}{q-p}$, then $\frac{1}{t} = \frac{q-p}{q} = 1 - \frac{p}{q}$, so $\frac{1}{t} + \frac{1}{q/p} = 1$, then we have

$$(r(Tx))^p \le \sum_{n=1}^{\infty} |\alpha_n|^{\frac{1}{t}} \left(|\alpha_n|^{\frac{1}{q/p}} |\langle x, a_n\rangle|^p\right), \quad \forall x \in E$$

By Hölder's inequality we have

$$(r(Tx))^p \le \left(\sum_{n=1}^{\infty} \left(|\alpha_n|^{\frac{1}{t}}\right)^t\right)^{\frac{1}{t}} \left(\sum_{n=1}^{\infty} \left(|\alpha_n|^{\frac{p}{q}}\right)^{\frac{q}{p}} (|\langle x, a_n\rangle|^p)^{\frac{q}{p}}\right)^{\frac{1}{q/p}}, \quad \forall x \in E$$

Now we have

$$r(Tx) \le \left(\sum_{n=1}^{\infty} |\alpha_n|\right)^{1/Pt} \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n\rangle|^q\right)^{1/q} \quad \forall x \in E$$

Let $\beta = \left(\sum_{n=1}^{\infty} |\alpha_n|\right)^{1/Pt}$ and $b_n = \beta a_n$. Then

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$$r(Tx) \leq \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n \rangle|^q \right)^{1/q} \quad \forall x \in E$$

and so T is q-quasi- l_1 -nuclear operator.

(2) The " if " part condition follows from part (1). To prove the only "if part", let $T: E \to F$ be a q-quasi-s-nuclear operator. Then for each continuous seminorm r on F, there exist an $(\alpha_n) \in s$ and an equicontinuous sequence (a_n) in E' such that

$$r(Tx) \leq \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n \rangle|^q \right)^{\frac{1}{q}} \forall x \in E$$

Then by Jeansen's inequality [4], we have

$$r(Tx) \leq \left(\sum_{n=1}^{\infty} |\alpha_n|^{\frac{p}{q}} |\langle x, a_n \rangle|^p \right)^{\frac{1}{p}} \forall x \in E$$

To finish our proof, it is enough to show that $(|\alpha_n|^{p/q}) \in s$. Since $\alpha_n \in s$, then $(|\alpha_n| n^k)$ is bounded for all $k \in \mathbb{N}$. Let h be a natural number such that $h.\frac{p}{q} \geq k + 2$. By hypothesis, there is a positive number M with $|\alpha_n| n^h \leq M \quad \forall n \in \mathbb{N}$. Then

$$(|\alpha_n| n^h)^{\frac{p}{q}} = |\alpha_n|^{\frac{p}{q}} n^{h\frac{p}{q}} \leq M^{\frac{p}{q}} \quad \forall n \in \mathbb{N}$$

Now for a fixed $k \in \mathbb{N}$ we have

$$|\alpha_n|^{\frac{p}{q}} n^{k+2} \leq |\alpha_n|^{\frac{p}{q}} n^{h\frac{p}{q}} \leq M^{\frac{p}{q}} \quad \forall n \in \mathbb{N}$$

Hence $|\alpha_n|^{\frac{p}{q}} n^k \leq M^{\frac{p}{q}} n^{-2} \quad \forall n \in \mathbb{N}$, and so

$$\sum_{n=1}^{\infty} |\alpha_n|^{\frac{p}{q}} n^k \leq M^{\frac{p}{q}} \sum_{n}^{\infty} n^{-2} < \infty$$

Therefore, $(|\alpha_n|^{p/q}) \in s$, hence T is a p-quasi-s-nuclear operator. ⊔

Theorem 2.9 *Let $T: E \to F$ and $S: F \to G$ be any continuous operators. If either T or S is a p-quasi-λ-nuclear operator, then $S \circ T$ is a p-quasi-λ- nuclear operator.*

Proof: Suppose $S: F \to G$ is a p-quasi-λ-nuclear operator. Then for each continuous seminorm q on F there exist $(\alpha_n) \in \lambda$ and an equicontinuous sequence (b_n) in F' such that

$$q(Sy) \leq \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle y, b_n \rangle|^p \right)^{1/p}, \forall y \in F$$

Since $Tx \in F$ for all $x \in E$, we have

$$q(STx) \leq \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle Tx, b_n \rangle|^P \right)^{1/P}$$

$$= \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, T'(b_n) \rangle|^P \right)^{1/P}, \quad \forall x \in E,$$

By [2], $(T'(b_n))$ is an equicontinuous sequence in E', and so $S \circ T$ is a p-quasi-λ-nuclear operator.

Now let $T: E \to F$ be any p-quasi-λ-nuclear operator and let q be any continuous seminorm on G Since $q \circ S$ is a seminorm on F. Then there exist a sequence (α_n) in λ and an equicontinuous sequence(a_n) in E' such that

$$(q \circ S)(Tx) = q(S(T(x))) \leq \left(\sum_{n=1}^{\infty} |\alpha_n| |\langle x, a_n \rangle|^P \right)^{1/P} \quad \forall x \in E,$$

and so $S \circ T$ is a p-quasi-λ-nuclear operator.

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