On The Bäcklund Transformations for Cosgrove's Equation

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Abstract

In this paper we study Bäcklund transformations (BTs) for Cosgrove's equation F-XVIII. We use the generalization of Fokas and Ablowitz method to derive BT between F-XVIII and new fourth-order ordinary differential equations of Painlevé type. Moreover we derive auto-BT and give special solutions for F-XVIII.

Keywords: Cosgrove's equation; Backlund Transformation; Differential equation

1. Introduction

The second-order ODE class was studied by Painlev´e and his school. They found six new functions which are called now Painlev´e transcendent PI-PVI [14]. Higher order first-degree and higher order higher degree equations of Painlev´e-type were investigated by Chazy [2], Bureau [1], Exton [7], Martynov[16], Cosgrove [3,4,5], Kudryashov [15], Clarkson, Joshi and Pickering [6], Mu˜gan and Jrad [17].

We need to study equations that define new transcendental functions to learn more about them. Today BTs are universally recognized as an important property and there is much interest in their derivation. Fokas and Ablowitz [8] used the transformation of the form

$$u = \frac{v' + a_0 v^2 + b_0 v + c_0}{a_1 v^2 + b_1 v + c_1} \qquad (1)$$

Where ai, bi, ci, i= 0,1 are functions of z only, to study the transformation properties of the six Painlev´e equations (PI,. . . ,PVI). It is used to obtain second-order ODE of first and second degree, it is also used in [18, 19] to derive all second-order second-degree equations of Painlev´e-type related to PI-PVI.

Gordoa and Pickering [13] generalize the method of Fokas and Ablowitz so that it can be applied to ODEs of order greater than two. The generalized method was applied to fourth-order Cosgrove's equation F-VI, F-V and F-XVII [10, 11, 12] respectively.

In this article, we will apply the method to Cosgrove's equation F-XVIII. The method for fourth-order ODEs can be summarized as follows. Consider a fourth-order ODE of the form

$$v^{(4)} = F(z, v, v', v'', v''') \qquad (2)$$

and rewrite the BT (1) as

$$v' = Av^2 + Bv + C \qquad (3)$$

where A =a1u-a0 , B =b1u-b0 , C = c1u-c0.

Differentiating equation (3) three times and using equation (2) to replace v(4) and equation (3) and it's derivatives to replace v',v",v''', we get an equation of the form

$$\sum_{i=0}^{n} \phi_i v^i = 0 \qquad (4)$$

where φi , i = 0, 1, 2, . . . , n, are polynomials in z, u,u',u",u'''. Our aim to seek for a lower degree polynomial in v. This can be done by setting leading order
coefficients to zero or by making a possible factorization

$$\sum_{i=0}^{n} \phi_i v^i = \left(\sum_{i=0}^{n-m} g_i v^i\right) = \left(\sum_{i=0}^{m} D_i v^i\right) \qquad (5)$$

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where gi 's are functions of z only, $g_{n-m} = 1$, $D_i's$, ' are functions of z, u, . . . ,u'''. Then solve for, $D_i's, i = 0, \ldots, m$ and set the remaining coefficients of vi to zero to find gi , A, B and C such that we obtain a consistent solution. Eliminating v between (3) and (4), one finds a fourth-order ODE for u.

As an application of this method we will consider here Cosgrove's equation F-XVIII.

Cosgrove's Equation (F-XVIII): Consider the Cosgrove's Equation (F-XVIII)

$$v^{(4)} = -5v'v'' + 5v^2v'' + 5v(v')^2 - v^5 + zv - \frac{\alpha}{2} \quad (6)$$

Where α is constant parameter. This equation defines a fourth-order Painlev´e transcendent and multi-parameter particular solutions involving Painlev´e-I transcendents and hyperelliptic functions also there are mappings between this equation and Fif-I and Fif-II [3], it is, however, well known to be the self- similar reduction of the Modified Sawada-Kotera equation found by Fordy and Gibbous [9].

Applying the method that have explained in the introduction to (6), we find that (4) reads

$$\sum_{i=0}^{5} \phi_i v^i = 0 \quad (7)$$

Where

$$\varphi_5 = 24A^4 + 10A^3 - 15A^2 + 1,$$
$$\varphi_4 = (36A^2 + 5A - 5)A' + 5AB(12A^2 + 5A - 5),$$
$$\varphi_3 = 8AA'' + 6(A')^2 + (52A + 5)BA' + 5(4A^2 + A - 1)B' + 20AC(2A^2 + A - 1) + 10B^2(5A^2 + 2A - 1)$$
$$\varphi_2 = A''' + 7BA'' + (17B^2 + 38AC + 9B' + 5C)A' + 5AB'' + 5(5A + 1)BB' + 5(2A^2 + A - 1)C' + 15B(A^2C + AB^2) + 5B(6AC - 3C + B^2) \quad (8)$$
$$\varphi_1 = 6CA'' + (22BC + 6C')A' + B''' + 4BB'' + 3B'^2 + (16AC + 6B^2 + 5C)B' + 2AC'' + 5B(2A + 1)C' + 2AC(8AC + 11B + 5C) + B^2(B^2 + 10C) - 5C^2 - z,$$
$$\varphi_0 = C''' + BC'' + (5C + 6AC + B^2 + 3B')C' + (6A' + 8AB + 5B)C^2 + (5BB' + 3B'' + B^3)C + \frac{\alpha}{2}$$

We will consider the reduction of (7) to polynomials in v of degrees less than or equal to five. There are five cases to be considered: (1) $\varphi_j = 0, j = 5,4,3,2$and $\varphi_1 \neq 0$, (2) $\varphi_j = 0, j = 5, 4, 3$ and $\varphi_2 \neq 0$ (3) $\varphi_j = 0, j = 5,4$ and $\varphi_3 \neq 0$ (4) $\varphi_5 = 0$and $\varphi_4 \neq 0$, and (5) $\varphi_5 \neq 0$. **It** turns out that **it** is not possible to choose *A*, *B*, and *C* so that the case (2) is satisfied. Thus we have only four cases.

Case (I): $\varphi_j = 0, j = 5,4,3,2$ and $\varphi_1 \neq 0$.

Setting $\phi_5 = 0$, we find that $A = -1, A = \frac{1}{2}, A = -\frac{1}{4}$or $A = \frac{1}{3}$. It turns out that when $A = -\frac{1}{4}$ or $A = \frac{1}{3}$, we can not choose B and C so that $\phi_j = 0, j = 4,3,2$ Thus we have two subcases to be considered in this case.

Case (I.1): $A = -1$

In this case, φ_4 reads $\varphi_4 = 10B$. Thus we have to choose $B = 0$ in order to make $\varphi_4 = 0$. Without loss of generality we choose $C = u$. With these choices for *A*, *B* and *C*, φ_2 and φ_3 are identically zeros, our BT is

$$(u^2 - 2u'' - z)v + (u''' - uu' + \frac{\alpha}{2}) = 0 \quad (9)$$

and

$$v' = u - v^2 \quad (10)$$

Eliminating v between (9) and (10) yields the following fourth-order first- degree equation for u

$$(u^2 - 2u'' - z)u^{(4)} = -(u''')^2 + (2uu' - 1)u''' - 6u(u'')^2 + (5u^3 - 2(u')^2 - 3zu)u'' - z(u')^2 + uu' - u^5 + 2zu^3 - z^2u + \frac{1}{2}\alpha(\frac{1}{2}\alpha - 1) \quad (11)$$

The BT (9) - (10) provides an auto-BT for equation (6). We will use the fact that equation (11) is invariant under the transformation $\alpha \to -(\alpha - 2)$. Note that the BT (9)-(10) defines a mapping between solutions *v* of (6) and solutions *u* of (11). Using the transformation above of α in the BT (9)-(10) yields an alternative BT

$$u = v^2 + v' \quad (12)$$

And

$$v = \frac{u''' - uu' - \frac{1}{2}(\alpha - 2)}{u^2 - 2u'' - z} \quad (13)$$

between solutions *v* of (6) for parameter $-(\alpha - 2)$ and solutions *u* of (11).

Thus given a solution *u* of (11) we can obtain solutions *v* and $\bar{v}$ of (6)

$$v = \frac{u''' - uu' - \frac{1}{2}\alpha}{u^2 - 2u'' - z} \quad (14)$$

And

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$$\bar{v} = \frac{u''' - uu' - \frac{1}{2}\bar{\alpha}}{u^2 - 2u'' - z} \quad (15)$$

for parameters α and $\bar{\alpha} = -(\alpha - 2)$ respectively. Eliminating u between (14) and (15), we obtain the following auto-BT for (6)

$$\bar{v} = v + \frac{\alpha - 1}{2v''' + 4vv'' + 3(v')^2 - 2v^2v' - v^4 + z}, \quad \bar{\alpha} = -(\alpha - 2) \quad (16)$$

provided that the denominator is not zero. If the denominator is zero and, $\alpha = 1$ then we have special solutions of (6) satisfied by the equation

$$2v''' + 4vv'' + 3(v')^2 - 2v^2v' - v^4 + z = 0 \quad (17)$$

Equation (17) is the special case $g(z) = 0, \gamma = -\frac{1}{24}$ of Chazy-XI equation (with $N = 7$) [4]

$$y''' = -2yy'' - 2(y')^2 + \frac{24}{N^2-1}[y' + y^2 - g(z)]^2 + g''(z) + \frac{1}{4}(N^2 - 1)(\gamma + \delta) \quad (18)$$

The solution of (17) is given by

$$v' + v^2 = 12w, \quad (19)$$

where w is a solution of the first Painlev´e equation

$$w'' = 6w^2 - \frac{1}{24}z \quad (20)$$

Case (I.2): $A = \frac{1}{2}$

In this case, again $\emptyset_4 = 0$ only if $B=0$. Without loss of generality we can assume that $C=u$. Again $\emptyset_3 = \emptyset_2 = 0$ identically and the BT becomes

$$v' = \frac{1}{2}v^2 + u \quad (21)$$

And

$$(u'' + 4u^2 - z)v + u''' + 8uu' + \frac{1}{2}\alpha = 0 \quad (22)$$

The equation for u reads

$$\begin{aligned}(u'' + 4u^2 - z)u^{(4)} &= \frac{1}{2}(u''')^2 + (8uu' - 1)u''' - 9u(u'')^2 \\ &-2(20^3 + 4(u')^2 + 5zu)u'' + 8z(u')^2 \\ &-8uu' - 16u^5 + 8zu^3 - z^2u + 2[-\tfrac{1}{4}\alpha(\tfrac{1}{4}\alpha + 1)].\end{aligned} \quad (23)$$

Equation (23) is invariant under the transformation $\alpha \to -(\alpha + 4)$. Thus, given a solution u of (23) we can obtain two solutions v and $\bar{v}$ of (6). Following the same method as in case (I.1) we obtain the following auto BT for (6).

$$\bar{v} = v - \frac{\alpha+2}{v''' - vv'' + 3(v')^2 - 4v^2v' + v^4 - z}, \bar{\alpha} = -(\alpha + 4) \quad (24)$$

provided the denominator is not zero.

The transformation (24) breaks down when the denominator is zero and $\alpha = -2$. Therefore we have proved that if $\alpha = -2$, then equation (6) admits special solutions characterized by the third-order equation

$$v''' - vv'' + 3(v')^2 - 4v^2v' + v^4 - z = 0 \quad (25)$$

The change of variable $v = -2y$ transforms (25) into the special case of Chazy-XI equation (18) (with N=2). $g(z) = 0, \gamma = -\frac{2}{3}, \delta = 0$

The solution of (25) is given by:

$$v' + v^2 = \frac{2}{3}w, \quad (26)$$

where w is a solutions of the first Painlev´e equation

$$w'' = 6w^2 - \frac{2}{3}z \quad (27)$$

Case (II): $\varphi_j = 0, j = 5,4 \; and \; \varphi_3 \neq 0$

In order to make $\varphi_j = 0, \; j = 5,4$ such that $\varphi_3 \neq 0$, we have only the following two choices for A, B and C: $A = -\frac{1}{4}$, $B=0$, $C=u$ and $A = \frac{1}{3}$, $B=0$, $C=u$.

Case (II.1): $A = -\frac{1}{4}$, $B=0$, $C=u$

In this case, the BT reads

$$v' = -\frac{1}{4}v^2 + u \quad (28)$$

And

$$\frac{45}{8}uv^3 - \frac{45}{8}u'v^2 - \frac{1}{2}(u'' + 13u^2 + 2z)v + \frac{1}{2}(2u''' + 7uu' + \alpha) = 0 \quad (29)$$

Eliminating v between (28) and (29), we obtain a third-degree fourth-order equation for u.

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Case (II.2): $A = -\frac{1}{3}$, $B=0$, $C=u$

In this case, the BT reads

$$v' = -\frac{1}{3}v^2 + u, \tag{30}$$

And

$$\frac{80}{27}uv^3 + \frac{20}{27}u'v^2 - \frac{1}{9}(6u'' + u^2 - 9z)v - \frac{1}{2}(2u''' + 14uu' + \alpha) = 0 \tag{31}$$

Eliminating v between (30) and (31), we obtain a third-degree fourth-order equation for u.

Case (III): $\varphi_5 = 0$ and $\varphi_4 \neq 0$ In this case, we have to consider four subcases:
(1) $A = -1$, (2) $A = \frac{1}{2}$, (3) $A = -\frac{1}{4}$ and (4) $A = \frac{1}{3}$.

Case (III.1): $A = -1$

Since $\varphi_4 \neq 0$, we have $B \neq 0$. Thus we have two cases: $b_1 = 0$ and $b_1 \neq 0$.

Case (III.1-a): $b_1 = 0$

In this case, without loss of generality we assume that $C = u$. One can easily see that there is no factorization. Thus we have the BT

$$v' = -v^2 + b_0 v + u, \tag{32}$$

And

$$10b_0v^4 + 10(b_0' + 2b_0^2)v^3 + 5(-b_0'' - 47b_0b_0' - 2b_0^3 + 3b_0u)v^2 - [2u + 5b_0u' - u^2 + (12b_0^2 + 11b_0')u - b_0''' - 4b_0b_0'' - 3(b_0')^2 - 6b_0^2b_0' - b_0^4 + z]v + u''' + b_0u'' + (3b_0' + b_0^2 - u)u'' - 3b_0u^2 + (3b_0'' + 5b_0b_0' + b_0^3)u + \frac{\alpha}{2} = 0, \tag{33}$$

where $b_0 \neq 0$ is an arbitrary function. Eliminating v between (32) and (33) yields a fourth-degree fourth-order equation for u.

Case (III.1-b): $b_1 \neq 0$

In this case, we can assume that $B=u$ and we can look for a possible factorization of (7)

$$\sum_{i=0}^{4} \emptyset_i v^i = (v - g)(10uv^3 + Dv^2 + Ev + F) = 0 \tag{34}$$

where g is a function of z only. We can easily find D, E, F, as functions of u, u', u'', u''', z and then we are left with the coefficient of $v0$ which we ask to vanish identically. In this way we find that $g = -c_1$, $c_0 = c_1^2 - c_1'$ and that g defines a second solution of (6). Thus we have the BT

$$u = \frac{(v-g)'}{(v-g)} + (v + g) \tag{35}$$

$$10uv^3 + Dv^2 + Ev + F = 0, \tag{36}$$

Where

$$D = 10u' + 20u^2 + 10gu,$$

$$E = -5u'' - 10(2u - g)u' - 10u^3 + 5gu^2 + 5(5g^2 + 3g')u,$$

$$F = u''' + 4u - 3g)u'' + 3(u')^2 + (6u^2 - 4gu - g^2 - 7g')u' + u^4 + 2gu^3 - (6g^2 - 7g')u^2 + (-3g'' + 3gg' + 23g^3)u + g^4 - 3(g')^2 + 2g^2g' - 4gg'' - 2g''' \tag{37}$$

Eliminating v between (35) and (36), we get a fourth-order ODE in u of degree three.

Case (III.2) : $A = \frac{1}{2}$

Again in this case $\varphi_4 \neq 0$ implies that $B \neq 0$ and hence we have the two cases $b_1 = 0$ or $b_1 \neq 0$.

Case (III.2-a): $b_1 = 0$

There is no possible factorization and we may assume that $C=u$. The BT is

$$v' = (1/2)v^2 + b_0 v = u, \tag{38}$$

And

$$\frac{5}{4}b_0v^4 + \frac{5}{2}(b_0' + 5b_0^2)v^3 + \frac{5}{2}(b_0'' + 7b_0b_0' + 5b_0^3 + 6b_0u)v^2 + (u'' + 10b_0u' + 4u^2 + 21b_0^2u + 13b_0'u + b_0''' + 4b\,b_0'' + 3(b_0')^2 + 6b_0^2b_0' + b_0^4 - z)v + u''' + b_0u'' + (3b_0' + b_0^2 + 3u)u' + 4b_0u^2 + (3b_0'' + 5b_0b_0' + b_0^3)u = 0 \tag{39}$$

The elimination of v between (38) and (39) gives a fourth order equation for u of degree four.

Case (III.2-b): $b_1 \neq 0$

Without loss of generality we can assume that $B = u$ and use the factorization of the following form

$$\sum_{i=0}^{4} \emptyset_i v^i = (v - g)\left(\frac{5}{4}uv^3 + Dv^2 + Ev + F\right) \tag{40}$$

from which we can find D, E and F .

Then we are left with the coefficients of v^0 that we ask to vanish identically, we find that $g = -c_1, c_0 = -(c_1' + \frac{1}{2}c_1^2)$ and g defines a second solution of(6). The BT in this case is

$$u = \frac{(v-g)'}{(v-g)} - \frac{1}{2}(v + g) \tag{41}$$

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$$\frac{5}{4}uv^3 + Dv^2 + Ev + F = 0 \qquad (42)$$

Where

$$D = \frac{5}{2}(5u^2 + u' + \frac{1}{2}gu),$$

$$E = -\frac{5}{2}u'' + \frac{5}{2}(7u - 1)u' + \frac{25}{2}u^3 - \frac{5}{2}gu^2 - 5(-3g' + \frac{5}{4}g^2)u,$$

$$F = u''' + \left(4u + \frac{3}{2}g\right)u'' + 3(u')^2 + (6u^2 - \frac{11}{2}gu - 4g^2 + 11g')u' + u^4$$

$$-\frac{17}{2}gu^3 - (7g^2 - 11g')u^2 - 3(gg' - 3g - \frac{3}{4}g^3)u + 3(g')^2 + g^4 - 4g'g^2 + g''' - gg'' - z. \qquad (43)$$

Eliminating v between (41) and (42), we have a third-degree fourth order ODE for u.

Case (III.3): $A = -\frac{1}{4}$

We will precede as in the two cases III.1 and III.2 above. For $b_1 = 0$ we have an equation for v of fourth degree. If $b_1 \neq 0$, then without loss of generality we can assume that $B = u$ and using a proper factorization

$$\sum_{i=0}^{4} \emptyset_i v^i = (v - g)\left(\frac{55}{8}uv^3 + Dv^2 + Ev + F\right) \qquad (44)$$

from which we can find D, E and F .

Then we are left with the coefficient of v^0 which we ask to vanish identically. To achieve this goal we have to choose $g = -c_1$, $c_0 = \frac{1}{4}c_1^2 - c_1'$

where g defines a second solution of (6).

Now the BT reads

$$u = \frac{(v-g)'}{(v-g)} + \frac{1}{4}(v + g), \qquad (45)$$

And

$$\frac{55}{8}uv^3 + Dv^2 + Ev + F = 0 \qquad (46)$$

Where

$$D = \frac{45}{32}g^2 + \frac{5}{4}gu + \frac{45}{8}g' - \frac{35}{4}u^2 - 5u',$$

$$E = -\frac{105}{32}g'u - \frac{55}{16}g^2u + 10gu^2 + \frac{5}{8}gu' + \frac{45}{16}gg'$$

$$+\frac{45}{32}g^3 - \frac{45}{8}g'' + \frac{5}{4}u^3 - \frac{5}{4}uu' - \frac{5}{4}u'',$$

$$F = u''' + \left(4u + \frac{3}{4}g\right)u'' + 3(u')^2 + (6u^2 - \frac{19}{2}gu + 2g' + \frac{7}{8}g^2)u' \qquad (47)$$

$$+u^4 - \frac{13}{4}gu^3 + (\frac{37}{8}g^2 + 2g')u^2 + 9gg' + 3g'' - \frac{3}{16}g^3)u$$

$$-(\frac{7}{16}g^2g' + \frac{27}{4}(g')^2 + \frac{1}{2}g''' - g^4 + \frac{47}{8}gg'' + z).$$

Elimination of v between (45) and (46) gives a fourth order ODE for u of degree three.

Case (III.4): $A = \frac{1}{3}$

Then $\emptyset_4 = -\frac{10}{3}B$. Following the steps in the previous case with $b_1 = 0$we have an equation for v of fourth degree. For $b_1 \neq 0$, without loss of generality we can assume that $B = u$ and make a possible factorization

$$\sum_{i=0}^{4} \emptyset_i v^i = (v - g)\left(-\frac{10}{8}uv^3 + Dv^2 + Ev + F\right) \qquad (48)$$

from which we can easily find D, E and F and we are left with the coefficient of v^0 which is identically zero. We find that $g = -c_1$, $c_0 = -(c_1' + \frac{1}{3}c_1^2)$ and that g defines a second solution of (6).

Our BT is

$$u = \frac{(v-g)'}{(v-g)} - \frac{1}{3}(v + g), \qquad (49)$$

And

$$-\frac{10}{3}uv^3 + Dv^2 + Ev + F = 0 \qquad (50)$$

Where

$$D = \frac{20}{9}u^2 - \frac{10}{9}u' - \frac{10}{27}gu + \frac{80}{27}(-g' + \frac{1}{3}g^2),$$

$$E = \frac{5}{3}u'' + (\frac{40}{3}u + \frac{10}{9}g)u' + 10u^3 + \frac{5}{4}gu^2 + (\frac{65}{27}g^2 - \frac{10}{3}g + \frac{35}{9}g')u$$

$$-\frac{20}{9}g'' - \frac{40}{27}gg' + \frac{80}{81}g^3, \qquad (51)$$

$$F = u''' + (4u + g)u'' + 3(u')^2 + (6u^2 - \frac{16}{3}gu + 9g' - \frac{7}{3}g^2)u'$$

$$+u^4 - \frac{22}{3}gu^3 - (\frac{46}{9}g^2 - \frac{17}{3}g')u^2 + (-9g'' + \frac{1}{9}g^3 - 3gg')u$$

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$$+\frac{2}{3}g''' - \frac{8}{3}gg'' - \frac{1}{3}(g')^2 - \frac{14}{9}g^2g' + g^4.$$

Elimination of *v* between (49) and (50) provides another fourth order ODE for *u* of degree three.

Case (V): $\varphi_5 \neq 0$

We can make a possible factorization of equation (7) such that

$$\sum_{i=0}^{5} \emptyset_i v^i = (v^2 - 2gv + h)(\emptyset_5 v^3 + Dv^2 + Ev + F) \qquad (52)$$

we can easily find *D*, *E*, *F* and then we are left with the coefficients of v^0 and which are identically zero.

If $a_1 = a_0 = 0$ then $b_1 = c_1 = 0$ so there is no possible factorization. This is also the case when $a_1 = 0, a_0 \neq 0$ so we are left with the case $a_1 \neq 0$ Without loss of generality we can assume that *A* = *u* then we find that $b_1 = -2g, h = c_1$ and $c_0 = \frac{1}{2}(b_0b_1 - b_0')$ we also find that

$$-b_0(b_1^2 - 4c_1) + \frac{1}{2}(b_1^2 - 4c_1)' = 0 \qquad (53)$$

We will consider two cases $b_1^2 - 4c_1 = 0$ and $b_1^2 - 4c_1 \neq 0$

Case (V.1): $c_1 = \frac{b_1^2}{4}$

We find that *g* defines a second solution of (6) and b_0 satisfies the ODE

$$-b_0''' - 4b_0b_0'' - 3(b_0')^2 - 6b_0^2b_0' - b_0^4 + 5(g^2 - g')(b_0' + b_0^2) + 5(2gg' - g)b_0 - 5g^4 + 10gg'' + 5(g')^2 + z = 0 \qquad (54)$$

In this case our BT is defined by

$$u = \frac{(v-g)' + b_0(v-g)}{(v-g)^2}, \qquad (55)$$

And

$$\emptyset_5 v^3 + Dv^2 + Ev + F = 0 \qquad (56)$$

(we will not give the explicit forms of the coefficients here).

Solving for *v* from equation (56) and equation (55) we can find a fourth order ODE for *u* of degree three.

Case (V.2): $b_1^2 - 4c_1 \neq 0$

Solving equation (53) for b_0, $b_0 = \frac{(b_1^2-4c_1)'}{2(b_1^2-4c_1)}$ and by making the change of variables $b_1 = -(w + r)$ and $c_1 = wr, (w \neq r)$ we find that w and r satisfy equation (6). The BT in this case is

$$u = \frac{(r'v - rv') + (w'r - wr') + (v'w - vw')}{(v-w)(v-r)(w-r)} \qquad (57)$$

And

$$\varphi_5 v^3 + D v^2 + Ev + F = 0 \qquad (58)$$

again we will not give the explicit form of the coefficients. Elimination of *v* between equation (57) and equation (58) we have a fourth order ODE for *u* of degree three.

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