On The Fuzzy Semi Sub-Modules of Fuzzy Modules

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Abstract
In this paper, we introduce the concept of fuzzy semi-essential (large) submodule, and we study a necessary and sufficient condition for a fuzzy submodule of a fuzzy module to be a fuzzy semi-essential (large) submodule, also fuzzy images and fuzzy inverse-images of generalized fuzzy semi-essential (large) submodule are studied.
Index Terms

Keywords: Fuzzy set; Fuzzy module; Fuzzy semi submodule.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [9]. It was first applied to the theory of groups by Rosenfeld in 1971 [8] . Since then, many authors introduced fuzzy subring and fuzzy ideals [5],[2]The concept of fuzzy module was introduced by Negoita and Relescu in 1975 [10]. Since then several authors have studied fuzzy modules. For examples see ([4], [5], [3], [7]). we recall the definitions of fuzzy module, fuzzy submodule with some properties of them, which are needed later. In this paper we give the definition of fuzzy semi-essential (large) submodule of fuzzy module and we study some of its propositions.

Definition 1.1 [9]:
Let S be a non-empty set and I be the closed interval [0, 1] of the real line (real numbers). A fuzzy set A of S (a fuzzy subset of S) is a function from S into I.

Definition 1.2 [9]:
A fuzzy set A of a set S is called a fuzzy constant if A (x) = t for all $x \in S$ where $t \in [0,1]$.

Definition 1.3 [10]:
Let $x_t: S \to [0,1]$ be a fuzzy set of S, where $\in S$, $t \in [0,1]$defined by: $x_t(y) = \begin{cases} t & if\ x = y \\ 0 & if\ x \neq y \end{cases}$, for all $y \in S$.
x_t is called a fuzzy singleton or fuzzy point in S.

Proposition 1.4[7]:
Let a_t and b_k be two fuzzy singletons of a set S. If $a_t = b_k$, then a = b and t = k , where t, k ∈ [0,1].

Definition 1.5 [11]:
Let A and B be two fuzzy sets in S, then:
(1) A=B if and only if A (x) = B (x), for all $x \in S$.
(2) $A \subseteq B$ if and only if $A(x) \leq B(X)$, for all $x \in S$. If $A \subset B$ and there exists $x \in S$ such that $A(x) < B(X)$, then: A is called a proper fuzzy subset of B and written $A \subset B$.

Definition 1.6 [10]:
Let A and B be two fuzzy sets in S, then:
(1) $(A \cap B)(x) = min\{A(x), B(x)\}$, for all $x \in S$.
(2) $(A \cup B)(x) = max\{A(x), B(x)\}$, for all $x \in S$.
$A \cap B$ and $A \cup B$ are fuzzy sets in S, In general. if {Aα, α ∈ Λ} is a family of fuzzy sets in S, then
$\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)(x) = inf\{A_\alpha(x), \alpha \in \Lambda\}$, $\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)(x) = sup\{A_\alpha(x), \alpha \in \Lambda\}$ for all $x \in S$.
which are also fuzzy sets in S.

Definition 1.7 [4], [5]:
Let A be a fuzzy set in S, for all $t \in [0,1]$, the set $A_t = \{ x \in S, A(x) \geq t\}$ is called a level subset of A. Note that, At is a subset of S in the ordinary sense.

Doi: https://doi.org/10.54216/GJMSA.050104
Received: January 03, 2023 Accepted: April 02, 2023

Remark 1.8 [9]:

The following properties of level subsets hold for each $t \in [0,1]$.

(1) $(A \cap B)_t = A_t \cap B_t$,

(2) $(A \cup B)_t = A_t \cup B_t$ and

(3) A = B if and only if $A_t = B_t$, for all $t \in [0,1]$.

Now, we give the definitions of image and inverse image of fuzzy sets.

Definition 1.9 [9]:

Let f be a mapping from a set M into a set N, let A be a fuzzy set in M and B be a fuzzy set in N. The image of A denoted by f (A) is the fuzzy set in N defined by:

$$f(A)(y) = \begin{cases} sup\{A(z) | z \in f^{-1}(y)\} if\ f^{-1}(y) \neq \emptyset, forall\ y \in N \\ 0 \quad otherwise \end{cases}$$

where $f^{-1}(y) = \{x \in M, f(x) = y\}$.

And the inverse image of B, denoted by $f^{-1}(B)$ is the fuzzy set in M defined by: $f^{-1}(B)(x) = B(f(x))$, for all $x \in M$.

Definition 1.10 [2]:

Let f be a function from a set M into a set N . A fuzzy subset A of M is called f-invariant if A(x) = A(y) whenever f(x) = f(y) , where $x, y \in M$.

Proposition 1.11 [2]:

If f is a function defined on a set M, A_1 and A_2 are fuzzy subsets of M, B_1 and B_2 are fuzzy subsets of f(M). The following are true:

(1) $A_1 \subseteq f^{-1}(f(A_1))$.

(2) $A_1 = f^{-1}(f(A_1))$, whenever A_1 is f-invariant.

(3) $f(f^{-1}(B_1)) = B_1$.

(4) $If\ A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.

(5) $If\ B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

Proposition 1.12[7]:

Let f be a function from a set M into a set N. If B_1 and B_2 are fuzzy subsets of N, then $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

2-Fuzzy Modules and Fuzzy Submodules

In this section, we recall the definitions of fuzzy modules and fuzzy submodules with some fundamental results about them.

Definition 2.1 [8], [5]: Let M be an R-module. A fuzzy set X of M is called fuzzy module of an R-module M if:

(1) $X(x - y) \geq \min\{X(x), X(y)\}$, for all $x, y \in M$.

(2) $X(rx) \geq X(x)$, for all $x \in M$ and $r \in R$.

(3) $X(0) = 1$.

Definition 2.2 [3]: Let X and A be two fuzzy modules of an R-module M. A is called a fuzzy submodule of X if $A \subseteq X$.

Proposition 2.3 [6]: Let A be a fuzzy set of an R-module M. Then the level subset A_t, $t \in [0,1]$ is a submodule of M if and only if A is a fuzzy submodule of X where X is a fuzzy module of an R-module M.

Now, we shall give some properties of fuzzy submodules, which will be used in the next section.

Definition 2.4 [4]: If A is a fuzzy module of an R-module M, then the submodule At of M is called the level submodule of M where $t \in [0,1]$.

Proposition 2.5 [7]:

Let X be a fuzzy module of an R-module M. Let $\{A_\alpha, \alpha \in \Lambda\}$ be a family of fuzzy submodules of X, then

(1) $\underset{\alpha \in \Lambda}{\cap} A_\alpha$ is a fuzzy submodule of X

(2) If $\{A_\alpha, \alpha \in \Lambda\}$ is a chain, then $\underset{\alpha \in \Lambda}{\cup} A_\alpha$ is a fuzzy submodule of X.

Definition 2.6 [10]:

Let A and B be two fuzzy subset of an R-module M. Then $(A + B)(x) = sup\{min\,\{A(a), B(b), x = a + b\}$, $a, b \in M$, for all $x \in M$. A + B is a fuzzy subset of M.

Proposition 2.7 [10]:

Let A and B be two fuzzy submodules of a fuzzy module X, then A + B is a fuzzy submodule of X.

Let X and Y be two fuzzy modules of R-modules M_1 and M_2 respectively. $f: X \to Y$ is called a fuzzy homomorphism if $f: M_1 \to M_2$ is homomorphism and Y (f (x)) = X (x), for each $x \in M_1$.

Proposition 2.9[7]: Let X and Y be twofuzzy modules of R-modules M1 and M2 respectively. Let $f: X \to Y$ be a fuzzy homomorphism.

If A and B are two fuzzy submodules of X and Y respectively, then:

(1) f(A) is a fuzzy submodule of Y. [13].

(2) $f^{-1}(B)$ is a fuzzy submodule of X. [13].

(3) $f(A \cap A_1) = f(A) \cap f(A_1)$ is a fuzzy submodule of Y, A_1 is a fuzzy submodule of X. [13].

Doi: https://doi.org/10.54216/GJMSA.050104

Received: January 03, 2023 Accepted: April 02, 2023

Definition 2.10 [4]:
Let X be a fuzzy module. A proper fuzzy submodule A of X is called a prime fuzzy submodule whenever $(r_t a_k) \subseteq A$ for fuzzy singleton r_t of R and $a_k \subseteq X$ we have either $r_t \subseteq (A: X)$ or $a_k \subseteq A$ where: $(A:_R X) = \{r_t: r_t X \subseteq A, r_t \text{fuzzy singleton of R}\}$.

3-A fuzzy semi-essential submodule of a fuzzy module

In this section we give the definition of a fuzzy semi-essential submodule of a fuzzy module and we study its properties about the image or inverse image of any fuzzy semi-essential submodule and we study anther properties about it.

Definition 3.1:-[1] A non-zero R submodule N of an R-module M is called a semi-essential submodule of M if $N \cap p = (0)$ for all a prime submodule P of M then P=(0).

Definition 3.2 Let A be a fuzzy module of an R-module M if B a fuzzy submodule of A then B is called a fuzzy semi-essential submodule of A if for all fuzzy prime submodule P of A and $B \cap p = 0_1$ then P=0_1.

Example 3.3 Let A be a fuzzy module of a Z-module Z^2 with scalar multiplication (a,b)r=(ar,br) defined as:

$$A((a,b)) = \begin{cases} 1 & if\ a+b \in Z_e \\ 1/2 & if\ a+b \in Z_o \end{cases}$$

$$B((a,b)) = \begin{cases} 1 & if\ a,b \in Z_e \\ 1/3 & if\ a,b \in Z_o \\ 1/4 & otherwise \end{cases}$$

To prove B is a fuzzy semi-essential submodule of A, Let P be a fuzzy prime submodule of A such that $B \cap p = ((0,0))_1$

It is means that

$$min\ \{B(a,b), P\{(a,b)\} = \begin{cases} 1 & If (a,b) = (0,0) \\ 0 & If (a,b) \neq (0,0) \end{cases} \text{ for all } (a,b) \in Z^2$$

Now $If (a,b) = (0,0) \Rightarrow P((a,b)) = 1$ (since P is a fuzzy submodule)

$If (a,b) \neq (0,0) \Rightarrow P(a,b) = 0$

(since $B((a,b)) \neq 0$ for all $(a,b) \in Z^2$

Therefore $p = (0,0)_1$.

Hence B is a fuzzy semi-essential submodule of A Now we recall the definition of a semi-essential submodule.

Proposition 3.4 Let A be a fuzzy module of an R-module M . A fuzzy submodule B of A is a fuzzy semi-essential submodule if B_t is semi essential submodule of A for all $t \in (0,1]$.

Proof:-

B is a fuzzy submodule of A (by proposition 2.3)

suppose U is a prime fuzzy submodule of A and $B \cap U = O_1$ this implies $(B \cap U)_t = B_t \cap U_t = (0_1)_t = \{0\}$ for all $t \in (0,1]$ (by proposition 1.8) $U_t = (0_1)_t$ (since B_t is a semi-essential submodule of A)

then $U = O_1$ (by proposition 1.8)

Therefore, B is a fuzzy semi-essential submodule of A.

Proposition 3.5 Let X be a fuzzy module of an R-module M if A and B are two fuzzy semi-essential submodule of X then $A \cup B$ is a fuzzy semi-essential submodule of X whenever $A \subseteq B$ or $B \subseteq A$

Proof: If $B \subseteq A$, $A \cup B$ is fuzzy submodule of X(by proposition 2.5)

If $(A \cup B) \cap U = 0_1$Where U is a prime fuzzy submodule of X $(A \cup B) \cap U = (A \cap U)$ (since $B \subseteq A$).

$= 0_1$ (since A is fuzzy semi-essential submodule of X)

This implies $U = O_1$, hence AUB is a fuzzy semi-essential submodule of X. Similarly if $A \subseteq B$.

Proposition 3.6

Let A be a fuzzy submodule of a module M and B be a fuzzy semi-essential submodule of A then the set $B^* = \{X \in M: B(x) > 0\}$ is a semi-essential submodule of M.

Proof: $B^* = \{X \in M: B(x) > 0\} = U_t Bt$ where $t \in (0,1]$ (by proposition 3.4, 3.5).

B^* is semi-essential submodule of M .

Proposition3.7

Let $\mathcal{F}$ be a bijective function from a fuzzy module A into a fuzzy module $B(\mathcal{F}: A \to B)$. If N be a fuzzy semi-essential submodule of A then $\mathcal{T}(N)$ is fuzzy semi-essential submodule of B.

Proof :

$\mathcal{F}(N)$ is submodule of B (by proposition 2.9).

Now, Suppose that U be a prime fuzzy submodule of B such that $\mathcal{F}(N) \cap U = 0_1$

$\mathcal{F}^{-1}(\mathcal{F}(N) \cap U) = \mathcal{F}^{-1}(0_1)$

$\mathcal{F}^{-1}(\mathcal{F}(N)) \cap \mathcal{F}^{-1}(U) = O_1$ (by proposition 1.11 and $\mathcal{F}$ is a bijective).

$N \cap \mathcal{F}^{-1}(U) = O_1$(since $\mathcal{F}$ is a bijective)

$\mathcal{F}^{-1}(U) = O_1$(since N is a fuzzy semi-essential submodule and $\mathcal{F}^{-1}(U)$ is submodule of A)

$\mathcal{F}(\mathcal{F}^{-1}(U)) = \mathcal{F}(O_1)$ (by proposition 1.11 and $\mathcal{F}$ is a bijective). Thus $U = O_1$

Then $\mathcal{F}(N)$ is fuzzy semi-essential submodule of B.

Doi: https://doi.org/10.54216/GJMSA.050104
Received: January 03, 2023 Accepted: April 02, 2023

Proposition3.8

Let $\mathcal{F}$ be a bijective function from a fuzzy module A into a fuzzy module $B(\mathcal{F}: A \rightarrow B)$ if M be a fuzzy semi-essential submodule of B then $\mathcal{F}^{-1}(M)$is fuzzy semi-essential submodule of A.

Proof:

$\mathcal{F}^{-1}(M)$ is a fuzzy submodule of A (by proposition 2.9).

Suppose Ube a prime fuzzy submodule of A such that $\mathcal{F}^{-1}(M) \cap U = O_1$

$$\mathcal{F}(\mathcal{F}^{-1}(M) \cap U) = \mathcal{F}(O_1)$$

$\mathcal{F}(\mathcal{F}^{-1}(M)) \cap \mathcal{F}(U) = \mathcal{F}(O_1)$ (by proposition 1.11).

$M \cap \mathcal{F}(U) = O_1$ (since M is fuzzy semi-essential of A).

$\mathcal{F}(U) = O_1$

$\mathcal{F}^{-1}(\mathcal{F}(U)) = \mathcal{F}^{-1}(O_1)$

$U = O_1$(by proposition 1.11 and $\mathcal{F}$ is a bijective).

Then $\mathcal{F}^{-1}(M)$ is fuzzy semi-essential submodule of A.

Proposition 3.9

Let A and B be two fuzzy semi-essential submodule of a fuzzy module X of an R-module M then A+B is also fuzzy semi-essential submodule of X.

Proof:

A+B is a fuzzy submodule of X (by proposition 2.7). Since A and B are fuzzy semi-essential submodule of X that mean (i.e.)

If $A \cap U = O_1 \Rightarrow U = O_1$

If $B \cap U = O_1 \Rightarrow U = O_1$, For all U is a prime fuzzy submodule of X

Now, if $(A + B) \cap U = O_1$ For all a prime fuzzy submodule U of X

Then $(A \cap U) + (B \cap U) = O_1$ (Distribution law).

Therefore $(A \cap U) = O_1$ and $(B \cap U) = O_1$ implies $U = O_1$ (since both A and B are semi-essential submodule of X), Yields A+B is a fuzzy semi-essential submodule of X .

Proposition 3.10 Let A,B,M and N are fuzzy module of R-module X such that $A \subseteq B \subseteq M \subseteq N$ and A is fuzzy semi-essential submodule of N then B is fuzzy semi-essential submodule of M .

Proof : Let U be a fuzzy submodule of

M and $B \cap U = O_1$then $A \cap U = O_1$(Since $A \subseteq B$).

Since A is fuzzy semi-essential submodule of N and $(U \subseteq M \subseteq N)$ Then $U = O_1$

Therefore,

B is fuzzy semi-essential submodule of M.

Proposition3.11

Let X be a fuzzy module of an R-module M and let N_1 and N_2 are two fuzzy submodules of X such that N_1 is a fuzzy submodule of N_2 then N_1 is a fuzzy semi-essential submodule of X then N_2 is a fuzzy semi-essential submodule of X.

Proof: Suppose that U be a fuzzy prime submodule of X,

Such that $U \cap N_2 = O_1$

$U \cap N_1 = O_1$(since $N_1 \subseteq N_2$).

$U = O_1$(N_1 is a fuzzy semi-essential submodule of X).

Then N_2 is a fuzzy semi-essential submodule of X.

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Doi: https://doi.org/10.54216/GJMSA.050104

Received: January 03, 2023 Accepted: April 02, 2023