# Results on Completely Semi Prime Ideals in Near Rings

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**Abstract**

The goal of this paper is to study the notions of completely semi prime ideal with respect to an element x (x-C.S.P.I) of a near ring and the completely semi prime ideal near ring with respect to an element x (x-C.S.P.I ), where the direct images and endomorphisms will be represented and discussed.

**Keywords:** Near ring; semi ideal; prime ideal; endomorphism

## 1. Introduction and basic concepts

N.J. Groenewald have show several commutability theorems for completely semi prime ideal of near ring [4] .near-rings, completely semi prime ideal, Factor near ring. In this section we give some basic concepts that we need in the second section.

A left near ring is a set N together with two binary operations "+" and "." such that

a. (N,+) is a group (not necessarily abelian)
b. (N, .) is a semigroup.
c. (n1 + n2) . n3 = n1 . n3 + n2 . n3

*For all* $1, n2, n3, \in N$ ;

**Definition:**

Let N be a near-ring. A normal subgroup I of (N,+) is called a left ideal of N if

i. $IN \subseteq I$.
ii. $\forall\, n, n_1 \in N$ and for all $i \in I$,
iii. $n(n_1 + i) - n.n_1 \in I$.

**Definition:**

Let $\{Nj\}_{j\in J}$ be a family of near rings , J is an index set and $\prod_{j\in J} Nj = \{(x_j): x_j \in N_j \; for\ all\ j \in J\}$ be the directed product of $N_j$ with the component wise defined operations '+' and '.' , is called the direct product near ring of the near rings $N_j$

**Definition:**

Let $N_1$ and $N_2$ be two near-rings. The mapping : $f: N_1 \to N_2$ is called a near-ring homomorphism if for all $m, n$?$N_1$

f(m + n) = f(m) + f(n) and f(m. n) = f(m) f(n).

**Theorem:**

Let $f: N_1 \to N_2$ is homomorphism

(1) If I is ideal of a near ring $N_1$ then f(I) is ideal of a near ring $N_2$.
(2) If J is ideal of a near ring $N_2$ then $f^{-1}(J)$is ideal of a near ring $N_1$.

**Definition:**

An ideal I of N is called completely semi prime ideal(C.S.P.I) of a near ring. If $x^2 \in I$ imples $x \in I$ for any $x \in N$.

**Definition:**

Let I be an ideal of a near ring N. Then I its called completely prime ideal of N if $\forall\, x, y \in N, x.y \in$ I implies $x \in I$ or $y \in I$ , denoted by C.P.I of N.

Doi: https://doi.org/10.54216/GJMSA.040204

Received: November 26, 2022 Accepted: March 28, 2023

**Main results**

**Definition:**

let N be a near ring and $x \in N$, I is called completely semi prime ideal with respect to the element x denoted by (x-C.S.P.I) or ( x- completely semi prime ideal )of N if for all $y \in N$, if x.$y^2$ ∈ I implies y ∈ I.

**Example:**

Considers N={0,a,b,c} be a near ring with addition and multiplication defined by the following tables.

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | a | a | a |
| b | 0 | a | b | c |
| c | a | 0 | c | b |

$I$={0,a}is an ideal of N.

Then

- $I$ is b-completely semi prime ideal of the near ring N .
- $I$ is not 0-completely semi prime ideal of the near ring N since $0.b^2 \in I$ *but* $b \notin I$
- $I$ is not a-completely semi prime ideal in near ring since $a.b^2 \in I$ *but* $b \notin I$

**Proposition:**

Let $\{I_j\}_{j\in J}$ be a family of x-C.S.P.I of a near ring N for all $\in J$ , $x \in N$. Then

$$\bigcap_{j\in J} I_j$$

is a x-C.S.P.I

Proof

Let $y \in N$ such that $x.y^2 \in \bigcap_{j\in J} I_j$,

this implies $x.y^2 \in I_j, \forall j \in J \Rightarrow y \in I_j, \forall j \in J$ [since each $I_j$ is a x-C.S.P.I $\forall j \in J$] $\Rightarrow y \in \bigcap_{j\in J} I_j$

$\Rightarrow \bigcap_{j\in J} I_j$ is a x-C.S.P.I of N

**Example:**

Considers N={0,1,2,3} be a near ring with addition and multiplication defined by the following tables

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|

Doi: https://doi.org/10.54216/GJMSA.040204
Received: November 26, 2022 Accepted: March 28, 2023

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 2 | 3 |

The ideals $I_1 = \{0,2\}$ and $I_2 = \{0,2,3\}$ are 1-completely semi prime ideal of a near ring of N.
*$I_1 . I_2 = \{0\}$ is not 1-completely semi prime ideal of the near ring N [Since $1.2^2 = 1.0 = 0 \in I_1 . I_2$ but $2 \notin I_1 . I_2$]

**Proposition:**

Let N be a Boolean near ring and I be (C.P.I) of N. Then I is x-C.S.P.I for all $x \notin I$ and $x \in N$.

Proof

Let $y \in N$ such that $x.y^2 \in I$ .Then $x.y \in I$ [since N is Boolean near ring ($y^2 = y$). Then $y \in I$ [ since I is C.P.I of N and $x \notin I$ ]. this implies I is x-C.S.P.I.

**Proposition:**

let N be a near ring with multiplicative identity $e'$ then I is $e'$- C.S.P.I of the near ring N if and only if it is a C.S.P.I of N.

proof

$\Rightarrow$

let I be an $e'$- C.S.P.I of N and $y \in N$ such that $y^2 \in I$, $y^2 = e'.y^2 \in I$ then $y \in I$ [since I is $e'$ - C.S.P.I] .This implies I is C.S.P.I.

$\Leftarrow$

To prove I is $e'$ - C.S.P.I. Let I be a C.S.P.I of N and $y \in N$ such that $e'.y^2 \in I$
$e'.y^2 = y^2 \in I$
This implies $y \in I$
[since I is C.S.P.I of N ]
Then we have I is $e'$ - C.S.P.I of N

**Example:**

Let N={0,a,b,c} be a near ring with addition and multiplication defined as

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c |
| b | 0 | a | b | c |
| c | 0 | 0 | c | c |

This ideal I={0,1} is a completely semi prime ideal of the near ring N , but it is not a-completely semi prime ideal of a near ring N. Since $a.b^2 \in I$ but $b \notin I$.

**Proposition :**

If N is non zero near ring and I={0} then I is not 0-.C.S.P.I of the near ring N.

Proof

Suppose I is 0-C.S.P.I of N, since $N \neq \{0\}$. Then there exist $y \in N$ such that $y \neq 0$.
$0.y^2 = 0 \in I \Longrightarrow y \in I$ [since I is $0 - \mathrm{C.S.P.I}$] $\Longrightarrow y = 0$
And this contradiction [since $y \neq 0$ ] $\Longrightarrow$ I is not $0 - \mathrm{C.S.O.I}$ of N.

Doi: https://doi.org/10.54216/GJMSA.040204

Received: November 26, 2022 Accepted: March 28, 2023

**Proposition:**

let I be nontrivial ideal of the near ring N then I is not 0-.C.S.P.I of N.

Proof

Suppose I is 0-.C.S.P.I of N and let $y \in N \Longrightarrow 0.y^2 = 0 \in I \Longrightarrow y \in I$ [since I is $0-$ C. S. P. I of N] $\Longrightarrow N \subseteq I$

This contradiction [since $I \subset N$] $\Longrightarrow$ I is not $0-$ C. S. P. I of N

**Theorem:**

Let $N_1$ and $N_2$ be two near ring, $f: N_1 \to N_2$ be epimomorphism and I be x- C.S.P.I of $N_1$ such that ker $f \subseteq I$. Then f(I) is f(x)-C.S.P.I of $N_2$.

Proof

Let I be x-C.S.P.I of $N_1$ $f(I) = \{f(i): i \in I\}$

Is an ideal of $N_2$. To proof f(I) is a f(x)-C.S.P.I of $N_2$.

Let $c \in N_2$ such that

$f(x).c^2 \in f(I)$

$\Longrightarrow f(x).c^2 = f(x).(f(y))^2$

$= f(x).f(y^2) = f(x.y^2) \in f(I),$

Where $c = f(y), y \in N_1$

[since f is an epimoorphism]

$\Longrightarrow x.y^2 \in I \Longrightarrow y \in I$ [since I is $x -$ C. S. P. I of $N_1$]

$\Longrightarrow c = f(y) \in f(I) \Longrightarrow f(I)$ is a f(x)-C.S.P.I of $N_2$

**Theorem:**

Let $N_1$ and $N_2$be two near ring.

$f: N_1 \to N_2$ be epiomomorphism and J be a y- C.S.P.I of $N_2$. Then $f^{-1}$(J) is a x-C.S.P.I of $N_1$ where y= f(x).

Proof

$f^{-1}(J) = \{x \in N_1: f(x) \in J$ is an ideal of the near ring $N_1$

Let $z \in N_1$such that

$x.z^2 \in f^{-1}(J) \Longrightarrow f(x.z^2) \in J$

$f(x.z^2) = f(x).f(z^2)$

$= f(x).(f(z))^2$

$= y.(f(z))^2 \in J$

$\Longrightarrow f(z) \in J$

[since J is y- C.S.P.I of $N_2$]

$\Longrightarrow z \in f^{-1}(J)$

$\to f^{-1}(J)$ is x-C.S.P.I of $N_1$

**Definition:**

The near ring N is called x-completely semi prime ideal near ring denoted by (x-C.S.P.I near ring ), if every ideal of a near ring N is x- C.S.P.I of N.

**Example:**

Consider the near ring in example (2.2)

The ideals of N are $I_1 = \{0, a\}$, $I_2 = N$, $I_3 = \{0\}$ are b- C.S.P.I of N since $\forall y \in N$ $b.y^2 \in I_i$ implies $y \in I_i$, $y, b \in N$ and $i \in \{1,2,3\}$

Then N is b- C.S.P.I near ring.

**Theorem:**

Let $\{N_j\}_{j\in J}$ be a family of a near rings, $x_j \in N_j$ and $I_j$ be $x_j$-C.S.P.I for all $j \in J$.

Then $\prod_{i\in J} I_j$ is $(x_j)$- C.S.P.I of the direct product near ring $\prod_{i\in J} N_j$.

Proof

Let $(y_j) \in \prod_{i\in J} N_j$ such that

$(x_j)(y_j)^2 \in \prod_{i\in J} I_j$

$\Longrightarrow (x_j)(y_j)^2 = (x_j y_j{}^2) \in \prod_{i\in J} I_j$

$\Longrightarrow x_j y_j{}^2 \in I_j$ for all $i \in J$

$\Longrightarrow y_j \in I_j$[since *ce* each $I_j$ is $x_j$-*C.S.P.I*]

$\Longrightarrow (y_j) \in \prod_{i\in J} I_j$

$\Longrightarrow \prod_{i\in J} I_j$ is $(x_j) - C.S.P.I$

**Corollary:**

- C.S.P.I $x_j$ be a family of $\{N_j\}$ $j \in J$ Let. Then $j \in J$ for all $x_j \in N_j$ near rings where is $(x_j) - C.S.P.I.\prod_{i\in J} N_j$ the product near ring

Proof

Let I be an ideal of the product near ring $\prod_{i\in J} N_j \Longrightarrow$ there exist a family of ideals $\{I_j\}_{j\in J}$ such that $I = \prod_{j\in J} I_j$ and each $I_j$ is an ideal of a near ring $N_j$, for all $i \in J \Longrightarrow$ each $I_j$

Doi: https://doi.org/10.54216/GJMSA.040204
Received: November 26, 2022 Accepted: March 28, 2023

$x_j$ -C.S.P.I of $N_j$, for all $i \in J$. [since $N_j$ is $x_j$ -C.S.P.I , for all $i \in J$ ].
Now by proposition (2.16)
We have $\prod_{j\in J} I_j = I$ is $x_j$ -C.S.P.I Of the product near ring $\Longrightarrow \prod_{i\in J} N_j$ is a $x_j$ -C.S.P.I near ring

**Corollary:**

Let I be an ideal of the x- C.S.P.I near ring N. Then the factor near ring $N/_I$ is x+I - C.S.P.I ring

Proof

The natural homomorphism I $\mathrm{nat_I}: N \to N/_I$ which is defined by $\mathrm{nat_I(a) = a + I}$, for all $a \in N$
Is an epimorphism. Now let Jbe an ideal of the factor near ring $N/_I$.Then by theorem (2.11) we have $nat_I^{-1}(J)$ is an ideal of the near ring $\mathrm{N.} \Longrightarrow nat_I^{-1}(J)$ is a x-C.S.P.I of N [since N is a x-C.S.P.I near ring. By theorem (2-12) we have $nat(nat_I^{-1}(J))$ is $\mathrm{nat_I}$ x-C.S.P.I of $\mathrm{N}/_\mathrm{I} \Longrightarrow$ J is $\mathrm{x + I -}$ C.S.P.I of factor near ring .Then $\mathrm{N}/_\mathrm{I}$ is $\mathrm{x + I -}$ C.S.P.I ring.

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Doi: https://doi.org/10.54216/GJMSA.040204
Received: November 26, 2022 Accepted: March 28, 2023