On The Topolpgical Properties of Pairwise Compactness in Intuitionistic Double Topological Spaces

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Abstract

The concept of intuitionistic topological space was introduced by Coker. The aim of this paper is to discuss the relation between bi-topological spaces and double-topological spaces and give a notion of pairwise compact for double topological spaces.

Keywords: Bi-topological space; double space; intutionistic subset

1. Introduction

The concept of a fuzzy topology introduced by Change[2] , after the introduction of fuzzy sets by Zadeh. Later this concept was extended to intuitionistic fuzzy topological spaces by Coker in [4] .

In [5] Coker studied continuity, connectedness, compactness and separation axioms in intuitionistic fuzzy topological spaces. In this paper, we follow the suggestion of J.G. Garcia and S.E. Rodabaugh [7] that (double fuzzy set) is a more appropriate name than (intuitionistic fuzzy set), and therefore adopt the term (double-set) for the intuitionistic set, and (double-topology) for the intuitionistic topology of Dogan Coker, (this issue), we denote by **Dbl-Top** the construct (concrete texture over set) whose objects are pairs (X,τ) where τ is a double-topology on X .In section three, we discuss making use of this relation between bitopological spaces and double- topological spaces, we generalize a notion of compactness for double-topological space in section four.

2. Preliminaries

Throughout the paper by X we denote a non-empty set . In this section we shall present various fundamental definitions and propositions. The following definition is obviously inspired by Atanassov [1].

Definition. [3] A double-set (DS in brief) A is an object having the form $A =< X, A_1, A_2 >$, Where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \emptyset$. The set A_1 is called the set of members of A , while A_2 is called the set of non- members of A . throughout the remainder of this paper we use the simpler$A = (A_1, A_2)$, for a double-set.

Remark. Every subset A of X is may obviously be regarded as a double-set having the form $A' = (A, A^c)$, where $A^c = X - A$ is the complement of A in X .

we recall several relations and operations between DS s as follows:

.Definition. [3] Let the DS's A and B on X be the form $A = (A_1, A_2)$, $B = (B_1, B_2)$, respectively. Furthermore, let $\{A_j: j \in J\}$ be an arbitrary family of DS's in X, where $A_j = (A_j^{(1)}, A_j^{(2)})$ then

(a) $A \subseteq B$ *if and only if* $A_1 \subseteq B_1$ *and* $A_2 \supseteq B_2$;

(b) A=B *if and only if* $A \subseteq B$ *and* $B \subseteq A$;

(c) $\bar{A} = (A_2, A_1)$ denotes the complement of A;

(d) $\cap A_j = (\cap A_j^{(1)}, \cup A_j^{(2)})$;

(e) $\cup A_j = (\cup A_j^{(1)} \cup A_j^{(2)})$;

(f) $[]A$=(A_1, A_1^c);

(g) $\langle\rangle A$ =(A_2^c, A_2);

(h) $\underset{\sim}{\emptyset}= (\emptyset, X)$ and $\underset{\sim}{X}= (X, \emptyset)$

In this paper we require the following:

(i) $()A$=$(A_1, \emptyset)$and $)(A$=$(\emptyset, A_2)$.

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Now, we recall the image and preimage of DS's under a function .

.Definition. [3,8] Let $x \in X$ be a fixed element in X. Then:

(a) The DS given by $\underset{\sim}{x} = (\{x\}, \{x\}^c)$ is called a double point (DP in brief X) .

(b)The DS $\underset{\approx}{x} = (\emptyset, \{x\}^c)$ is called a vanishing double-point (VDP in brief X) .

Definition. [3,8]

(a) Let $x_\sim$ be a DP in X and $A = (A_1, A_2)$ be a DS in X. Then $\underset{\sim}{x} \in A$ iff $x \in A_1$

Let $x_\approx$ be a VDP in X and $A = (A_1, A_2)$ a DS in X . Then $\underset{\approx}{x} \in A$ iff $x \in A_1$

$x \notin A_2$.

It is clear that $\underset{\sim}{x} \in A \Leftrightarrow \underset{\sim}{x} \subseteq A$ and that $\underset{\approx}{x} \in A \Leftrightarrow \underset{\approx}{x} \subseteq A$.

.Definition. [5] A double-topology (DT in brief) on a set X is a family of DS's in X satisfying the following axioms:

T1: $\emptyset, X \in \tau$,

$T2$: $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$,

$T3$: $\cup G_j \in \tau$, for any arbitrary family $\{G_j : j \in J\} \subseteq \tau$.

In this case the pair (X ,τ) is called a double-topological space (DTS in brief), and any DS in τ is known as a double open set (DOS in brief). The complement $\bar{A}$ of a DOS A in a DTS is called a double closed set (DCS in brief) in X.

Definition. [5] Let (X , τ) be an DTS and $A = (A_1, A_2)$be a DS in X.

Then the interior and closure of A are defined by:

$\mathrm{int}(A) = \cup\{G : G \ is \ a \ DOS \ in \ X \ and \ G \subseteq A\}$,

$cl(A) = \{H : H \ is \ a \ DCS \ in \ X \ and \ A \subseteq H\}$, respectively.

It is clear that cl(A) is a DCS in and int(A) a DOS in X . Moreover, A is a DCS in X iff cl(A) = A , and A is a DOS in X iff int (A) = A.

Example. [5] Any topological space (X , τ_0) gives rise to a DT of the form $\tau = \{A' : A \in \tau_0\}$ by identifying a subset A in X with its counterpart $A' = (A , A^c)$, as in Remark 2.2.

3- The Constructs Dbl-Top and Bitop:

We begin recalling the following result which associates a bitopology with a double topology.

.Proposition. [5] Let (X ,τ) be a DTS.

(a) $\tau_1 = \{A_1 : \exists A_2 \subseteq X \ with \ A = (A_1, A_2) \in \tau\}$ is a topology on X.

(b) $\tau_2^* = \{A_2 : \exists A_1 \subseteq X \ with \ A = (A_1, A_2) \in \tau\}$ is the family of closed sets of the topology $\tau_2 = \{A_2^c : \exists A_1 \subseteq X \ with \ A = (A_1, A_2) \in \tau\}$ on X.

(c) Using (a) and (b) we may conclude that (X , τ_1, τ_2) is a bitopological space.

Proposition. Let *(X,u,v)* be a bitopological space. Then the family $\{(U, V^c) : U \in u, V \in v, U \subseteq V\}$ is a double topology on X.

Proof. The condition $U \subseteq V$ ensures that $U \subseteq V^c = \emptyset$, while the given family contains $\emptyset_\sim$ because$\emptyset \in u, v$, and it contains $X_\sim$ because $X \in u, v$ Finally this family is closed under finite intersections and arbitrary unions by Definition 2.3 (d,e) and the corresponding properties of the topologies u and v .

Definition. Let *(X,u,v)* be a bitopological space. Then we set

$$\tau_{uv} = \{(U, V^c) : U \in u, V \in v, U \subseteq V\}$$

and call this the double topology on X associated with *(X,u,v).*

Proposition. If *(X,u,v)* is a bitopological space and τ_{uv} the corresponding DT on X, then $(\tau_{uv})_1 = u \ and \ (\tau_{uv})_2 = v.$

Proof. $U \in u$ implies $(U, \emptyset) \in \tau_{uv}$ since $U \subseteq X \in v, sou \subseteq (\tau_{uv})_1$. Conversely, take $U \in (\tau_{uv})_1$. Then $(U, B) \in \tau_{uv}$ for some $B \subseteq X$. Now $U \in u$, hence $(\tau_{uv})_1 \subseteq u$, and the first equality is proved. □

The proof of the second equality may be obtained in a similar way, and we omit the details.

4- Piarwise Compact in Double- Topological Spaces .

In this section we define double compact set and we use the link between bitopological space and double topological space to established some theorems.

Definition. By an double open cover of a subset A of a double topological space (X, τ), We mean a collection $C = \{G_j : j \in J\}$ of double open subsets of X such that $A \subset \cup\{G_j : j \in J\}$ then we say that C covers A. In particular, a collection C is said to be an open cover of the space X iff $X = \cup\{(G_j^1, G_j^2) : j \in J\}$ of double open subsets of X.

Definition. A double-set A of DTS in (X ,τ) is said to be double compact set iff for every double open cover has double finite sub cover, that is iff for every collection $\{G_j : j \in J\}$of DOS's for which $A \subset \cup\{G_j : j \in J\}$ for $A = (A_1, A_2)$ such that $(A_1, A_2) \cup \ldots \cup (G_{jn}^1, G_{jn}^2)$.

.Definition. Let (X , τ) be double topological space and let Y be a double subset of X. The τ - double relative topology for Y is the collection τ_Y given by $\tau_Y = \{G \cap Y : G \in \tau\}$.The double topological space (Y, τ_Y) is called double subspace of (X ,τ).

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Proposition. Let Y be a subspace of double topological spaces X and let $A \subset Y$, then A is double compact set relative to X iff A is double compact set relative to Y.

Proof : Let A be double compact set relative to X and let $\{V_j : j \in J\}$ be a collection of DS's, double open relative to Y . Which covers A so that

$(A_1, A_2) \subset \{(V_j^1, V_j^2): j \in J\}$ then there exists G_j double open set's relative to X such that $V_j = Y \cap G_j$, for every $j \in J$. It follows that $(A_1, A_2) \subset \{(G_j^1, G_j^2): j \in J\}$ so that $\{G_j : j \in J\}$ is open cover of A relative to X . Since A is double compact set relative to X , there exist finitely many indices $j_1, \dots j_n$such that $(A_1, A_2) \subset (G_{j1}^1, G_{j1}^2) \cup \dots \cup (G_{jn}^1, G_{jn}^2)$

Since $A \subset Y$ we have

$(A_1, A_2) \subset Y \cap \{(G_{j1}^1, G_{j1}^2) \cup \dots \cup (G_{jn}^1, G_{jn}^2)\} = (Y \cap (G_{j1}^1, G_{j1}^2)) \cup \dots \cup (Y \cap (G_{jn}^1, G_{jn}^2))$

Since $Y \cap G_{ji} = V_{ji} (i = 1,2, \dots, n)$ we obtain

$(A_1, A_2) \subset (V_{j1}^1, V_{j1}^2) \cup \dots \cup (V_{jn}^1, V_{jn}^2)$

this shows that A is double compact set relative to Y .

Conversely, let A be double compact set relative to Y and let $\{G_j : j \in J\}$ a collection of DOS's of X which cover A , so that

$(A_1, A_2) \subset \{(G_j^1, G_j^2): j \in J\} \dots .(1)$

Hence $A \subset Y$,(1) implies that $A \subset Y \cap [\cup\{(G_j^1, G_j^2): j \in J\}]=$ $\cup\{Y \cap (G_j^1, G_j^2): j \in J\}$, hence $Y \cap (G_j^1, G_j^2)$ is double open relative to Y , the collection $\{Y \cap G_j : j \in J\}$ is double open cover of A relative to Y . Since A is double compact relative to Y we must have.

$(A_1, A_2) \subset (Y \cap (G_{j1}^1, G_{j1}^2)) \cup \dots \cup (Y \cap (G_{jn}^1, G_{jn}^2)) \dots .(2)$

Some choice of finitely many indices $j_1, \dots j_n$, but (2) implies that $(A_1, A_2) \subset (G_{j1}^1, G_{j1}^2) \cup \dots \cup (G_{jn}^1, G_{jn}^2)$ it follows that A is double compact relative to X .

Proposition. In DTS (X, τ), double close subsets of compact sets are double compact set.

Proof : Let Y be a double compact subset of a topological space X and let F be a double subset of Y, double closed relative to X . To show that F is double compact. let $C = \{G_j : j \in J\}$ be an open cover of F then the collection $D = \{G_j\} \cup \{X - F\}$ forms an open cover of Y. Since Y compact, there is a finite sub collection D' *of* D which covers Y, and hence covers F. If X-F is member of D' we may remove it from D' and still retain an open finite cover of F. Hence F is double compact.

Definition. A collection C of double set's said to have the double finite intersection property (DIFP) or to be finitely common iff the intersection of members of each finite subcollection of C is non-empty .

4.7. Proposition. Double topological space (X, τ) is double compact iff every collection of double closed subset s of X has a nonempty intersection

Proof : Let X be double compact set and let $\{F_j = (F_j^1, F_j^2): j \in J$ be collection of double closed set's of X with FIP and suppose if possible $\cap F_j : j \in J = \cap\{(F_j^1, F_j^2): j \in J\} = \underset{\sim}{\emptyset} = (\emptyset, X) \Rightarrow (\cap F_j^1, \cup F_j^2) = (\emptyset, X)$ then $[\cap\{F_j : j \in J\}]' = [\cap\{(F_j^1, F_j^2): j \in J\}]' = \underset{\sim}{X} = (\emptyset, X)$ or $\cup\{\overline{F_j} : j \in J\} = \cup\{(F_j^2, F_j^1): j \in J\} = \underset{\sim}{X} \Rightarrow (\cap F_j^2, \cup F_j^1) = (\emptyset, X) \Rightarrow \cup\{F_j^2 : j \in J\} = X$

This means that $\{\overline{F_j} : j \in J\}$ is a double open cover of X . Since F_j's are DCS's. Since X is double compact set . We have

$\cup\{\overline{F_{ji}} : i = 1,2, \dots, n\} = \underset{\sim}{X} \Rightarrow [\cap\{F_{ji} : i = 1,2, \dots, n\}]' = X_\sim$

Which implies that

$\cap\{F_{ji} : i = 1,2, \dots, n\} = \emptyset_\sim$ and this contradicts that FIP of F . Hence we must have $\cap\{F_j : j \in J\} \neq \emptyset$.

Conversely let every collection of DCS's of X with the FIP have non-empty intersection and let $C = \{G_j : j \in J\} = (G_j^1, G_j^2): j \in J$ be a double open cover of X so that

$\underset{\sim}{X} = \cup\{(G_j^1, G_j^2): j \in J\} \Rightarrow \cup\{G_j^1 : j \in J\} = X$

Hence taking complements

⇒Thus

$\{(G_j^2, G_j^1): j \in J\}$ is a collection of DCS's with empty intersection and so by hypothesis this collection does not have the FIP, hence there

exists a finite number of $G_{ji}, i = 1,2, \dots, n$ such that

$\underset{\sim}{\emptyset} = \cap\{(G_{ji}^2, G_{ji}^1): i = 1,2, \dots, n\} = [\cup\{(G_{ji}^1, G_{ji}^2): i = 1,2, \dots, n\}]'$

$\Rightarrow \underset{\sim}{X} = \cup\{(G_{ji}^1, G_{ji}^2): i = 1,2, \dots, n\}$, hence X is double compact.

4.8. Definition. [6] A cover H of bitopological space (X,u,v) is pairwise open if $H \subset u \cup v$ *with* $H \cap u$containing a non-empty set and with $H \cap v$ containing a non-empty set .

4.9. Definition. Let A be piarwise open subsets of a topological space X and let

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$C = \{G_j : j \in J\}$ be a collection of pairwise open subsets of X such that
$A \subset \cup\{G_j : j \in J\}$. We then say that C pairwise covers A. By a pairwise sub cover of a pairwise open cover C of A, we mean a pairwise open sub collection *C`* of C such that *C`* pairwise covers A. A pairwise open cover of A is said to be finite if it consists of finite number of pairwise open sets.

Definition. The DTS (X,τ) is called pairwise compact if every pairwise open cover of X has a finite subcover.

Proposition. If *(X,u,v)* is pairwise compact then (X, τ_{uv}) is pairwise compact .

Proof : Let H be pairwise open cover of X and such that H $\cap\, u \neq \emptyset$ and H $\cap\, v \neq \emptyset$ and let A, B subset's in X ,such that $A \in u$, $B \in v$, since X is pairwise compact then $\{G_j : j \in J\}$, $\{H_j : j \in J\}$are respectively an open cover of A , B such that $A \subset \{G_j : j \in J\}$, $B \subset \{H_j : j \in J\}$, there exists a finite sub cover such that $A \subset G_{j1} \cup \ldots \cup G_{jn}$, $B \subset H_{j1} \cup \ldots \cup H_{jn}$, take $U = (A, \emptyset) \in \tau_{uv}$, $V = (\emptyset, B^c) \in \tau_{uv}$ then $U = (A, \emptyset) \subset (G_{j1}, \emptyset) \cup \ldots \cup (G_{jn}, \emptyset)$ and $V = (\emptyset, B^c) \subset \cap(\emptyset, H_j^c) = (\cap\, \emptyset, \cup\, H_j^c)$
so that $H_{j1}^c \cup \ldots \cup H_{jn}^c \subset B^c$then τ_{uv} is pairwise compact.

This suggests the following definition for general double topologies.

Proposition. If (X , τ) is pairwise compact then (X, τ_1, τ_2) is pairwise compact.

Proof : Let A be subset in X , since (X , τ) is pairwise compact then for open cover $\{G_j : j \in J\}$ of $A = (C, D) \in \tau$, there exist a finite open sub cover such that $(C, D) \subset (G_{j1}^1, G_{j1}^2) \cup \ldots \cup (G_{jn}^1, G_{jn}^2)$ by property $\cup A = (\cup A_1 , \cap A_2)$
$C \subset G_{j1}^1 \cup \ldots \cup G_{jn}^1 , G_{j1}^2 \cap \ldots \cap G_{jn}^2 \subset D$ So that $(G_{j1}^2)^c \cup \ldots \cup (G_{jn}^2)^c \subset D^c$
$\therefore\ (C, D^c) \subset \cup\{(G_j^1, (G_j^2)^c) : j \in J\}$
$\therefore\ (X, \tau_1, \tau_2)$ is pairwise compact.

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