A novel extension of hesitant fuzzy sets on UP (BCC)-algebras: neutrosophic hesitant fuzzy sets

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Abstract

In this paper, we introduce the concepts of neutrosophic hesitant fuzzy UP (BCC)-subalgebras, UP (BCC)-ideals, and strong UP (BCC)-ideals of UP (BCC)-algebras. The characteristic neutrosophic hesitant fuzzy UP (BCC)-subalgebras, UP (BCC)-ideals, and strong UP (BCC)-ideals have also been studied. The relationship between neutrosophic hesitant fuzzy UP (BCC)-subalgebras, UP (BCC)-ideals, and strong UP (BCC)-ideals and their level subsets is provided. The Cartesian product of neutrosophic hesitant fuzzy UP (BCC)-subalgebras, UP (BCC)-ideals, and strong UP (BCC)-ideals is also supplied. Finally, we also find the property of the homomorphic pre-image of neutrosophic hesitant fuzzy UP (BCC)-subalgebras, UP (BCC)-ideals, and strong UP (BCC)-ideals.

Keywords: UP (BCC)-algebra; neutrosophic hesitant fuzzy UP (BCC)-subalgebra; neutrosophic hesitant fuzzy UP (BCC)-ideal; neutrosophic hesitant fuzzy strong UP (BCC)-ideal.

1 Introduction

The concept of fuzzy sets was proposed by Zadeh.[15] The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in.[1–3] In 2009-2010, Torra and Narukawa[13,14] introduced the notion of hesitant fuzzy sets (HFSs), which is a function from a reference set to a power set of the unit interval. The notion of HFSs is the other generalization of the notion of fuzzy sets. The HFS theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the notion of HFSs by Torra and Narukawa,[13,14] several pieces of research were conducted on the generalizations of the notion of HFSs and application to many logical algebras such as: in 2012, Zhu et al.[16] introduced the notion of dual HFSs, which is a new extension of fuzzy sets. In 2014, Jun, Ahn and Muhiuddin[7] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song[9] introduced the

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notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. Iampan[6] introduced a new algebraic structure called a UP-algebra, and Mosrijai et al.[11] introduced the notion of HFSs on UP-algebras. The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy UP-ideals play an important role in studying the many logical algebras. The notion of UP-algebras (see[6]) and the concept of BCC-algebras (see[10]) are the same concept, as shown by Jun et al.[8] in 2022. We shall refer to it as BCC rather than UP in this article out of respect for Komori, who initially described it in 1984.

In this article, we have the following research objectives:

(1) to introduce the concepts of neutrosophic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals of BCC-algebras.

(2) to study the properties of the characteristic neutrosophic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals.

(3) to provide a relationship between neutrosophic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals and their level subsets.

(4) to determine the results of the Cartesian product of neutrosophic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals.

(5) to find the properties of the homomorphic pre-image of neutrosophic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals.

2 Preliminaries

The concept of BCC-algebras (see[10]) can be redefined without the condition (6) as follows:

An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *BCC-algebra* (see[5]) if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0) \tag{1}$$
$$(\forall x \in X)(0 \cdot x = x) \tag{2}$$
$$(\forall x \in X)(x \cdot 0 = 0) \tag{3}$$
$$(\forall x, y \in X)(x \cdot y = 0 = y \cdot x \Rightarrow x = y) \tag{4}$$

After this, we assign X instead of a BCC-algebra $(X, \cdot, 0)$ until otherwise specified.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{5}$$

In X, the following assertions are valid (see[6]).

$$(\forall x \in X)(x \leq x) \tag{6}$$
$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{7}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow z \cdot x \leq z \cdot y) \tag{8}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow y \cdot z \leq x \cdot z) \tag{9}$$
$$(\forall x, y, z \in X)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)) \tag{10}$$
$$(\forall x, y \in X)(y \cdot x \leq x \Leftrightarrow x = y \cdot x) \tag{11}$$
$$(\forall x, y \in X)(x \leq y \cdot y) \tag{12}$$
$$(\forall a, x, y, z \in X)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))) \tag{13}$$
$$(\forall a, x, y, z \in X)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z) \tag{14}$$
$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot z) \tag{15}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z \cdot y) \tag{16}$$
$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)) \tag{17}$$
$$(\forall a, x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)) \tag{18}$$

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Definition 2.1. [4,6,12] A nonempty subset S of X is called

(1) a *BCC-subalgebra* of X if

$$(\forall x, y \in S)(x \cdot y \in S), \tag{19}$$

(2) a *BCC-ideal* of X if

$$0 \in S, \tag{20}$$
$$(\forall x, y, z \in X)(x \cdot (y \cdot z), y \in S \Rightarrow x \cdot z \in S), \tag{21}$$

(3) a *BCC-filter* of X if (20) and

$$(\forall x, y, z \in X)(x \cdot y, x \in S \Rightarrow y \in S), \tag{22}$$

(4) a *strong BCC-ideal* of X if (20) and

$$(\forall x, y, z \in X)((z \cdot y) \cdot (z \cdot x), y \in S \Rightarrow x \in S). \tag{23}$$

Definition 2.2. [13] A *hesitant fuzzy set* (HFS) on a reference set X is defined in term of a function h that when applied to X return a subset of $[0, 1]$, that is, $h : X \to \mathcal{P}([0, 1])$.

Definition 2.3. [13] An *neutrosophic hesitant fuzzy set* (NHFS) on a reference set X is defined in the form $\mathcal{N} = (h, k, n)$, where $h, k,$ and n are functions that when applied to X return a subset of $[0, 1]$, that is, $h, k, n : X \to \mathcal{P}([0, 1])$.

Definition 2.4. [11] A HFS h on X is said to be

(1) a *hesitant fuzzy BCC-subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(h(x \cdot y) \supseteq h(x) \cap h(y)) \tag{24}$$

(2) a *hesitant fuzzy BCC-ideal* of X if the following conditions hold:

$$(\forall x \in X)(h(0) \supseteq h(x)) \tag{25}$$
$$(\forall x, y, z \in X)(h(x \cdot y) \supseteq h(x \cdot (y \cdot z)) \cap h(y)) \tag{26}$$

(3) a *hesitant fuzzy BCC-filter* of X if (25) and the following condition hold:

$$(\forall x, y, z \in X)(h(y) \supseteq h(x \cdot y) \cap h(x)) \tag{27}$$

(4) a *hesitant fuzzy strong BCC-ideal* of X if (25) and the following condition hold:

$$(\forall x, y, z \in X)(h(x) \supseteq h((z \cdot y) \cdot (z \cdot x)) \cap h(y)) \tag{28}$$

Definition 2.5. [13] The *complement* of a HFS h in a reference set X is the HFS $\overline{h}$ defined by $\overline{h}(x) = [0, 1] - h(x)$ for all $x \in X$.

Definition 2.6. [13] The *complement* of a NHFS $\mathcal{N} = (h, k, n)$ on a reference set X is the NHFS $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$.

3 Neutrosophic hesitant fuzzy BCC-subalgebras

In this section, we introduce the concepts of neutrosophic hesitant fuzzy UP (BCC)-subalgebras, UP (BCC)-ideals, and strong UP (BCC)-ideals of UP (BCC)-algebras and study their properties.

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Definition 3.1. A NHFS $\mathcal{N} = (h, k, n)$ on X is called a *neutrosophic hesitant fuzzy BCC-subalgebra* of X if it satisfies the following property:

$$(\forall x, y \in X)\left(\begin{array}{c} h(x \cdot y) \supseteq h(x) \cap h(y) \\ k(x \cdot y) \subseteq k(x) \cup k(y) \\ n(x \cdot y) \supseteq n(x) \cap n(y) \end{array}\right) \tag{29}$$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	1	0	3
3	0	0	0	0

Then X is a BCC-algebra. We define a NHFS $\mathcal{N} = (h, k, n)$ on X as follows:

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.2, 1] & [0.2, 0.5] & [0.3, 0.4] & [0.3, 0.4] \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \emptyset & [0.2, 0.8] & [0.2, 0.3] & [0.8, 0.9] \end{pmatrix}$$

$$n = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.1, 1] & [0.1, 0.3] & [0.5, 0.7] & [0.6, 0.7] \end{pmatrix}$$

Then $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X.

Proposition 3.3. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X, then*

$$(\forall x \in X)\left(\begin{array}{c} h(0) \supseteq h(x) \\ k(0) \subseteq k(x) \\ n(0) \supseteq n(x) \end{array}\right). \tag{30}$$

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. For any $x \in X$, we have

$$h(0) = h(x \cdot x) \supseteq h(x) \cap h(x) = h(x),$$
$$k(0) = k(x \cdot x) \subseteq k(x) \cup k(x) = k(x),$$
$$n(0) = n(x \cdot x) \supseteq n(x) \cap n(x) = n(x).$$

□

Definition 3.4. The *characteristic neutrosophic hesitant fuzzy set* (characteristic NHFS) of a subset A of a set X is defined to be the structure $\chi_A = (h_{\chi_A}, k_{\chi_A}, n_{\chi_A})$, where

$$h_{\chi_A}(x) = \begin{cases} [0,1] & \text{if } x \in A \\ \emptyset & \text{otherwise} \end{cases}, k_{\chi_A}(x) = \begin{cases} \emptyset & \text{if } x \in A \\ [0,1] & \text{otherwise} \end{cases}, \text{ and } n_{\chi_A}(x) = \begin{cases} [0,1] & \text{if } x \in A \\ \emptyset & \text{otherwise} \end{cases}.$$

Lemma 3.5. *The constant 0 of X is in a nonempty subset B of X if and only if $h_{\chi_B}(0) \supseteq h_{\chi_B}(x), k_{\chi_B}(0) \subseteq k_{\chi_B}(x)$, and $n_{\chi_B}(0) \supseteq n_{\chi_B}(x)$ for all $x \in X$.*

Proof. If $0 \in B$, then $h_{\chi_B}(0) = [0,1], k_{\chi_B}(0) = \emptyset$, and $n_{\chi_B}(0) = [0,1]$. Thus $h_{\chi_B}(0) = [0,1] \supseteq h_{\chi_B}(x), k_{\chi_B}(0) = \emptyset \subseteq k_{\chi_B}(x)$, and $n_{\chi_B}(0) = [0,1] \supseteq n_{\chi_B}(x)$ for all $x \in X$.

Conversely, assume that $h_{\chi_B}(0) \supseteq h_{\chi_B}(x), k_{\chi_B}(0) \subseteq k_{\chi_B}(x)$, and $n_{\chi_B}(0) \supseteq n_{\chi_B}(x)$ for all $x \in X$. Since B is a nonempty subset of X, we have $a \in B$ for some $a \in X$. Then $h_{\chi_B}(0) \supseteq h_{\chi_B}(a) = [0,1]$, so $h_{\chi_B}(0) = [0,1]$. Hence, $0 \in B$. □

Theorem 3.6. *A nonempty subset S of X is a BCC-subalgebra of X if and only if the characteristic NHFS χ_S is a neutrosophic hesitant fuzzy BCC-subalgebra of X.*

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Proof. Assume that S is a BCC-subalgebra of X. Let $x, y \in X$.

Case 1 : If $x, y \in S$, then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = [0,1]$. Thus $h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0,1]$. Since S is a BCC-subalgebra of X, we have $x \cdot y \in S$ and so $h_{\chi_S}(x \cdot y) = [0,1]$. Thus $h_{\chi_S}(x \cdot y) = [0,1] \supseteq [0,1] = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = \emptyset$ and $k_{\chi_S}(y) = \emptyset$. Thus $k_{\chi_S}(x) \cup k_{\chi_S}(y) = \emptyset$. Since S is a BCC-subalgebra of X, we have $x \cdot y \in S$ and so $k_{\chi_S}(x \cdot y) = \emptyset$. Thus $k_{\chi_S}(x \cdot y) = \emptyset \subseteq \emptyset = k_{\chi_S}(x) \cup k_{\chi_S}(y)$. Also, $n_{\chi_S}(x) = [0,1]$ and $n_{\chi_S}(y) = [0,1]$. Thus $n_{\chi_S}(x) \cap n_{\chi_S}(y) = [0,1]$. Since S is a BCC-subalgebra of X, we have $x \cdot y \in S$ and so $n_{\chi_S}(x \cdot y) = [0,1]$. Thus $n_{\chi_S}(x \cdot y) = [0,1] \supseteq [0,1] = n_{\chi_S}(x) \cap n_{\chi_S}(y)$.

Case 2 : If $x \in S$ and $y \notin S$, then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = \emptyset$. Thus $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = \emptyset$ and $k_{\chi_S}(y) = [0,1]$. Thus $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0,1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0,1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$. Also, $n_{\chi_S}(x) = [0,1]$ and $n_{\chi_S}(y) = \emptyset$. Thus $n_{\chi_S}(x) \cap n_{\chi_S}(y) = \emptyset$. Then $n_{\chi_S}(x \cdot y) \supseteq \emptyset = n_{\chi_S}(x) \cap n_{\chi_S}(y)$.

Case 3 : If $x \notin S$ and $y \in S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = [0,1]$. Thus $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = [0,1]$ and $k_{\chi_S}(y) = \emptyset$. Thus $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0,1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0,1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$. Also, $n_{\chi_S}(x) = \emptyset$ and $n_{\chi_S}(y) = [0,1]$. Thus $n_{\chi_S}(x) \cap n_{\chi_S}(y) = \emptyset$. Then $n_{\chi_S}(x \cdot y) \supseteq \emptyset = n_{\chi_S}(x) \cap n_{\chi_S}(y)$.

Case 4 : If $x \notin S$ and $y \notin S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = \emptyset$. Thus $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = [0,1]$ and $k_{\chi_S}(y) = [0,1]$. Thus $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0,1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0,1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$. Also, $n_{\chi_S}(x) = \emptyset$ and $n_{\chi_S}(y) = \emptyset$. Thus $n_{\chi_S}(x) \cap n_{\chi_S}(y) = \emptyset$. Then $n_{\chi_S}(x \cdot y) \supseteq \emptyset = n_{\chi_S}(x) \cap n_{\chi_S}(y)$.

Hence, χ_S is a neutrosophic hesitant fuzzy BCC-subalgebra of X.

Conversely, assume that χ_S is a neutrosophic hesitant fuzzy BCC-subalgebra of X. Let $x, y \in S$. Then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = [0,1]$. Thus $h_{\chi_S}(x \cdot y) \supseteq h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0,1]$, so $h_{\chi_S}(x \cdot y) = [0,1]$. Hence, $x \cdot y \in S$ and so S is a BCC-subalgebra of X. □

Definition 3.7. A NHFS $\mathcal{N} = (h, k, n)$ on X is called a *neutrosophic hesitant fuzzy BCC-ideal* of X if (30) and the following condition hold:

$$(\forall x, y, z \in X) \begin{pmatrix} h(x \cdot z) \supseteq h(x \cdot (y \cdot z)) \cap h(y) \\ k(x \cdot z) \subseteq k(x \cdot (y \cdot z)) \cup k(y) \\ n(x \cdot z) \supseteq n(x \cdot (y \cdot z)) \cap n(y) \end{pmatrix} \tag{31}$$

Example 3.8. Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	0
3	0	0	2	0

Then X is a BCC-algebra. We define a NHFS $\mathcal{N} = (h, k, n)$ on X as follows:

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.1, 1] & [0.6, 0.9] & [0.6, 0.8] & [0.6, 0.9] \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \{0.5\} & [0.5, 0.7] & [0.5, 0.8] & [0.5, 0.7] \end{pmatrix}$$

$$n = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.5, 0.9] & [0.5, 0.9] & [0.6, 0.9] & [0.55, 0.9] \end{pmatrix}$$

Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-ideal of X.

Definition 3.9. A NHFS $\mathcal{N} = (h, k, n)$ on X is called a *neutrosophic hesitant fuzzy strong BCC-ideal* of X if (30) and the following condition hold:

$$(\forall x, y, z \in X) \begin{pmatrix} h(x) \supseteq h((z \cdot y) \cdot (z \cdot x)) \cap h(y) \\ k(x) \subseteq k((z \cdot y) \cdot (z \cdot x)) \cup k(y) \\ n(x) \supseteq n((z \cdot y) \cdot (z \cdot x)) \cap n(y) \end{pmatrix} \tag{32}$$

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Example 3.10. Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	2	0

Then X is a BCC-algebra. We define a NHFS $\mathcal{N} = (h, k, n)$ on X as follows:

$$(\forall x \in X) \left(\begin{array}{l} h(x) = [0.7, 0.9] \\ k(x) = [0.3, 0.5] \\ n(x) = [0.5, 0.9] \end{array} \right) \tag{33}$$

Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X.

Theorem 3.11. *Every neutrosophic hesitant fuzzy strong BCC-ideal of X is a neutrosophic hesitant fuzzy BCC-ideal.*

Proof. Let $\mathcal{N} = (h, k, n)$ be a neutrosophic hesitant fuzzy strong BCC-ideal of X. Then (30) holds. Let $x, y, z \in X$. Then

$$\begin{array}{rcl} h(x \cdot z) & \supseteq & h((z \cdot y) \cdot (z \cdot (x \cdot z))) \cap h(y) \\ & = & h((z \cdot y) \cdot 0) \cap h(y) \\ & \supseteq & h(0) \cap h(y) \\ & = & h(y) \\ & \supseteq & h(x \cdot (y \cdot z)) \cap h(y), \\ \\ k(x \cdot z) & \subseteq & k((z \cdot y) \cdot (z \cdot (x \cdot z))) \cup k(y) \\ & = & k((z \cdot y) \cdot 0) \cup k(y) \\ & \subseteq & k(0) \cup k(y) \\ & = & k(y) \\ & \subseteq & k(x \cdot (y \cdot z)) \cup k(y), \\ \\ n(x \cdot z) & \supseteq & n((z \cdot y) \cdot (z \cdot (x \cdot z))) \cap n(y) \\ & = & n((z \cdot y) \cdot 0) \cap n(y) \\ & \supseteq & n(0) \cap n(y) \\ & = & n(y) \\ & \supseteq & n(x \cdot (y \cdot z)) \cap n(y). \end{array}$$

Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-ideal of X. □

Remark 3.12. The converse of Theorem 3.11 is not true in general. From Example 3.8, we have $\mathcal{N}$ is a neutrosophic hesitant BCC-ideal of X. Since $h(1) = [0.7, 0.9] \not\supseteq [0.9, 1] = h((2 \cdot 1) \cdot (2 \cdot 1)) \cap h(0)$, $\mathcal{N}$ is not a neutrosophic hesitant strong BCC-ideal of X.

Theorem 3.13. *A NHFS $\mathcal{N} = (h, k, n)$ on X is a neutrosophic hesitant fuzzy strong BCC-ideal of X if and only if $h, k,$ and n are constant HFSs on X.*

Proof. Let $\mathcal{N} = (h, k, n)$ be a neutrosophic hesitant fuzzy strong BCC-ideal of X. For any $x \in X$,

$$\begin{array}{rcl} h(x) & \supseteq & h((x \cdot 0) \cdot (x \cdot x)) \cap h(0) \\ & = & h(0 \cdot 0) \cap h(0) \\ & = & h(0) \cap h(0) \\ & = & h(0) \\ & \supseteq & h(x), \\ \\ k(x) & \subseteq & k((x \cdot 0) \cdot (x \cdot x)) \cup k(0) \\ & = & k(0 \cdot 0) \cup k(0) \\ & = & k(0) \cup k(0) \\ & = & k(0) \\ & \subseteq & k(x), \end{array}$$

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$$\begin{aligned} n(x) &\supseteq n((x\cdot 0)\cdot(x\cdot x))\cap n(0)\\ &= n(0\cdot 0)\cap n(0)\\ &= n(0)\cap n(0)\\ &= n(0)\\ &\supseteq n(x). \end{aligned}$$

Hence, h, k, and n are constant HFSs on X.

Conversely, assume that h, k, and n are constant HFSs on X. Hence obviously, $\mathcal{N}$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X. □

The following two theorems can be proved similarly to Theorem 3.6.

Theorem 3.14. *A nonempty subset S of X is a BCC-ideal of X if and only if the characteristic NHFS χ_S is a neutrosophic hesitant fuzzy BCC-ideal of X.*

Theorem 3.15. *A nonempty subset S of X is a strong BCC-ideal of X if and only if the characteristic NHFS χ_S is a neutrosophic hesitant fuzzy strong BCC-ideal of X.*

Definition 3.16. A NHFS $\mathcal{N} = (h, k, n)$ on X is called a *prime NHFS* on X if it satisfies the following property:

$$(\forall x, y \in X)\begin{pmatrix} h(x\cdot y)\subseteq h(x)\cup h(y)\\ k(x\cdot y)\supseteq k(x)\cap k(y)\\ n(x\cdot y)\subseteq n(x)\cup n(y)\end{pmatrix} \tag{34}$$

Theorem 3.17. *A nonempty subset S of X is a prime subset of X if and only if the characteristic NHFS χ_S is a prime NHFS on X.*

Proof. Assume that S is a prime subset of X and let $x, y \in X$.

Case 1 : If $x\cdot y \in S$, then $h_{\chi_S}(x\cdot y) = [0,1]$. Since S is a prime subset of X, we have $x \in S$ or $y \in S$. Then $h_{\chi_S}(x) = [0,1]$ or $h_{\chi_S}(y) = [0,1]$, so $h_{\chi_S}(x)\cup h_{\chi_S}(y) = [0,1]$. Hence, $h_{\chi_S}(x\cdot y) = [0,1] \subseteq [0,1] = h_{\chi_S}(x)\cup h_{\chi_S}(y)$. Also, $k_{\chi_S}(x\cdot y) = \emptyset \supseteq \emptyset = k_{\chi_S}(x)\cap k_{\chi_S}(y)$ and $n_{\chi_S}(x\cdot y) = [0,1] \subseteq [0,1] = n_{\chi_S}(x)\cup n_{\chi_S}(y)$.

Case 2 : If $x\cdot y \notin S$, then $h_{\chi_S}(x\cdot y) = \emptyset \subseteq h_{\chi_S}(x)\cup h_{\chi_S}(y)$. Also, $k_{\chi_S}(x\cdot y) = [0,1] \supseteq k_{\chi_S}(x)\cap k_{\chi_S}(y)$ and $n_{\chi_S}(x\cdot y) = \emptyset \subseteq n_{\chi_S}(x)\cup n_{\chi_S}(y)$.

Hence, χ_S is a prime NHFS on X.

Conversely, assume that χ_S is a prime NHFS on X. Let $x, y \in X$ be such that $x\cdot y \in S$. Then $h_{\chi_S}(x\cdot y) = [0,1]$, so $[0,1] = h_{\chi_S}(x\cdot y) \subseteq h_{\chi_S}(x)\cup h_{\chi_S}(y)$. Thus $h_{\chi_S}(x)\cup h_{\chi_S}(y) = [0,1]$, so $h_{\chi_S}(x) = [0,1]$ or $h_{\chi_S}(y) = [0,1]$. Hence, $x \in S$ or $y \in S$ and so S is a prime subset of X. □

Theorem 3.18. *Let $\mathcal{N} = (h, k, n)$ be a NHFS on X. Then the following statements are equivalent:*

(1) $\mathcal{N}$ is a prime neutrosophic hesitant fuzzy BCC-subalgebra (resp., prime neutrosophic hesitant fuzzy BCC-ideal, prime neutrosophic hesitant fuzzy strong BCC-ideal) of X,

(2) h, k, and n are constant HFSs on X,

(3) $\mathcal{N}$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X.

Proof. $(1) \Leftrightarrow (2)$: Assume that $\mathcal{N}$ is a prime neutrosophic hesitant fuzzy BCC-subalgebra (resp., prime neutrosophic hesitant fuzzy BCC-ideal, prime neutrosophic hesitant fuzzy strong BCC-ideal) of X. Then $h(0) \supseteq h(x), k(0) \subseteq k(x)$, and $n(0) \supseteq n(x)$ for all $x \in X$. By (6), we have $h(0) = h(x\cdot x) \subseteq h(x)\cup h(x) = h(x), k(0) = k(x\cdot x) \supseteq k(x)\cup k(x) = k(x)$, and $n(0) = n(x\cdot x) \subseteq n(x)\cup n(x) = n(x)$ for all $x \in X$ and so $h(x) = h(0), k(x) = k(0)$, and $n(x) = n(0)$ for all $x \in X$. Hence, h, k, and n are constant HFSs on X.

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Assume that h, k, and n are constant HFSs on X. Hence, we can easily show that $\mathcal{N}$ is a prime neutrosophic hesitant fuzzy BCC-subalgebra (resp., prime neutrosophic hesitant fuzzy BCC-ideal, prime neutrosophic hesitant fuzzy strong BCC-ideal) of X.

$(2) \Leftrightarrow (3)$: It is straightforward by Theorem 3.13. □

Definition 3.19. [4] A nonempty subset S of X is called a *weakly prime subset* of X if it satisfies the following property:

$$(\forall x, y \in X, x \neq y)(x \cdot y \in S \Rightarrow x \in S \text{ or } y \in S)$$

Definition 3.20. [4] A BCC-subalgebra (resp., BCC-ideal, strong BCC-ideal) S of X is called a *weakly prime BCC-subalgebra* (resp., weakly prime BCC-ideal, weakly prime strong BCC-ideal) of X if S is a weakly prime subset of X.

Definition 3.21. A NHFS $\mathcal{N} = (h, k, n)$ on X is called a *weakly prime NHFS* on X if it satisfies the following property:

$$(\forall x, y \in X, x \neq y) \begin{pmatrix} h(x \cdot y) \subseteq h(x) \cup h(y) \\ k(x \cdot y) \supseteq k(x) \cap k(y) \\ n(x \cdot y) \subseteq n(x) \cup n(y) \end{pmatrix} \tag{35}$$

Definition 3.22. A neutrosophic hesitant fuzzy BCC-subalgebra (resp., neutrosophic hesitant fuzzy BCC-ideal, neutrosophic hesitant fuzzy strong BCC-ideal) $\mathcal{N} = (h, k, n)$ of X is called a *weak prime neutrosophic hesitant fuzzy BCC-subalgebra* (resp., weak prime neutrosophic hesitant fuzzy BCC-ideal, weak prime neutrosophic hesitant fuzzy strong BCC-ideal) of X if $\mathcal{N}$ is a weakly prime NHFS on X.

Theorem 3.23. *For BCC-algebras, the notions of weak prime neutrosophic hesitant fuzzy strong BCC-ideals and prime neutrosophic hesitant fuzzy strong BCC-ideals coincide.*

Proof. It is straightforward by Theorem 3.13. □

The following theorem can be proved similarly to Theorem 3.17.

Theorem 3.24. *A nonempty subset S of X is a weakly prime subset of X if and only if the characteristic NHFS χ_S is a weakly prime NHFS on X.*

Theorem 3.25. *A nonempty subset S of X is a weakly prime BCC-subalgebra of X if and only if the characteristic NHFS χ_S is a weakly prime neutrosophic hesitant fuzzy BCC-subalgebra on X.*

Proof. It is straightforward by Theorems 3.6 and 3.24. □

Theorem 3.26. *A nonempty subset S of X is a weakly prime BCC-ideal of X if and only if the characteristic NHFS χ_S is a weakly prime neutrosophic hesitant fuzzy BCC-ideal on X.*

Proof. It is straightforward by Theorems 3.14 and 3.24. □

Theorem 3.27. *A nonempty subset S of X is a weakly prime strong BCC-ideal of X if and only if the characteristic NHFS χ_S is a weakly prime neutrosophic hesitant fuzzy strong BCC-ideal on X.*

Proof. It is straightforward by Theorems 3.15 and 3.24. □

Theorem 3.28. *If a NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X, then the HFSs $h, k, n, \overline{h}, \overline{k}$, and $\overline{n}$ are constant HFSs on X.*

Proof. It is straightforward by Theorem 3.13. □

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Theorem 3.29. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X if and only if the HFSs $h, \overline{k}$, and n are hesitant fuzzy BCC-subalgebras of X.*

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. Then for any $x, y \in X$, we have $h(x \cdot y) \supseteq h(x) \cap h(y)$ and $n(x \cdot y) \supseteq n(x) \cap n(y)$. Hence, h and n are hesitant fuzzy BCC-subalgebras of X. Now for any $x, y \in X$, we have

$$\begin{array}{rcl} \overline{k}(x \cdot y) & = & [0,1] - k(x \cdot y) \\ & \supseteq & [0,1] - (k(x) \cup k(y)) \\ & = & ([0,1] - k(x)) \cap ([0,1] - k(y)) \\ & = & \overline{k}(x) \cap \overline{k}(y). \end{array}$$

Hence, $\overline{k}$ is a hesitant fuzzy BCC-subalgebra of X.

Conversely, assume that $h, \overline{k}$, and n are hesitant fuzzy BCC-subalgebras of X. Then for any $x, y \in X$, we have $h(x \cdot y) \supseteq h(x) \cap h(y)$ and $n(x \cdot y) \supseteq n(x) \cap n(y)$. Now for any $x, y \in X$, we have $\overline{k}(x \cdot y) \supseteq \overline{k}(x) \cap \overline{k}(y)$. Thus $[0, 1] - k(x \cdot y) \supseteq ([0, 1] - k(x)) \cap ([0, 1] - k(y)) = [0, 1] - (k(x) \cup k(y))$, so $k(x \cdot y) \subseteq k(x) \cup k(y)$. Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. □

The following two theorems can be proved similarly to Theorem 3.29.

Theorem 3.30. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X if and only if the HFSs $h, \overline{k}$, and n are hesitant fuzzy BCC-ideals of X.*

Theorem 3.31. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X if and only if the HFSs $h, \overline{k}$, and n are hesitant fuzzy strong BCC-ideals of X.*

Theorem 3.32. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X if and only if the NHFS $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X.*

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. Then for any $x, y \in X$, we have $h(x \cdot y) \supseteq h(x) \cap h(y)$. Thus for any $x, y \in X$, we have

$$\begin{array}{rcl} \overline{h}(x \cdot y) & = & [0,1] - h(x \cdot y) \\ & \subseteq & [0,1] - (h(x) \cap h(y)) \\ & = & ([0,1] - h(x)) \cup ([0,1] - h(y)) \\ & = & \overline{h}(x) \cup \overline{h}(y). \end{array}$$

Now for any $x, y \in X$, we have $k(x \cdot y) \subseteq k(x) \cup k(y)$. Thus for any $x, y \in X$, we have

$$\begin{array}{rcl} \overline{k}(x \cdot y) & = & [0,1] - k(x \cdot y) \\ & \supseteq & [0,1] - (k(x) \cup k(y)) \\ & = & ([0,1] - k(x)) \cap ([0,1] - k(y)) \\ & = & \overline{k}(x) \cap \overline{k}(y). \end{array}$$

Hence $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X.

Conversely, assume that $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. Then for any $x, y \in X$, we have $\overline{k}(x \cdot y) \supseteq \overline{k}(x) \cap \overline{k}(y)$. Thus $[0, 1] - k(x \cdot y) \supseteq ([0, 1] - k(x)) \cap ([0, 1] - k(y)) = [0, 1] - (k(x) \cup k(y))$, so $k(x \cdot y) \subseteq k(x) \cup k(y)$. Now for any $x, y \in X$, we have $\overline{h}(x \cdot y) \subseteq \overline{h}(x) \cup \overline{h}(y)$. Thus $[0, 1] - h(x \cdot y) \subseteq ([0, 1] - h(x)) \cup ([0, 1] - h(y)) = [0, 1] - (h(x) \cap h(y))$, so $h(x \cdot y) \supseteq h(x) \cap h(y)$. Hence, $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. □

The following two theorems can be proved similarly to Theorem 3.32.

Theorem 3.33. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X if and only if the NHFS $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X.*

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Theorem 3.34. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X if and only if the NHFS $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X.*

Definition 3.35. Let $\mathcal{N} = (h, k, n)$ be a NHFS on a set X. The NHFSs $\oplus\mathcal{N}, \otimes\mathcal{N}$, and $\odot\mathcal{N}$ are defined as $\oplus\mathcal{N} = (h, \overline{h}, n), \otimes\mathcal{N} = (\overline{k}, k, n)$, and $\otimes\mathcal{N} = (\overline{k}, k, \overline{k})$.

Theorem 3.36. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X if and only if the NHFSs $\oplus\mathcal{N}, \otimes\mathcal{N}$, and $\otimes\mathcal{N}$ are neutrosophic hesitant fuzzy BCC-subalgebras of X.*

Proof. Assume that $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. Let $x, y \in X$. Then

$$\begin{array}{rcl} \overline{h}(x \cdot y) & = & [0,1] - h(x \cdot y) \\ & \subseteq & [0,1] - (h(x) \cap h(y)) \\ & = & ([0,1] - h(x)) \cup ([0,1] - h(y)) \\ & = & \overline{h}(x) \cup \overline{h}(y). \end{array}$$

Hence, $\oplus\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. Let $x, y \in X$. Then

$$\begin{array}{rcl} \overline{k}(x \cdot y) & = & [0,1] - k(x \cdot y) \\ & \supseteq & [0,1] - (k(x) \cup k(y)) \\ & = & ([0,1] - k(x)) \cap ([0,1] - k(y)) \\ & = & \overline{k}(x) \cap \overline{k}(y). \end{array}$$

Hence, $\otimes\mathcal{N}$ and $\odot\mathcal{N}$ are neutrosophic hesitant fuzzy BCC-subalgebras of X.

Conversely, assume that $\oplus\mathcal{N}, \otimes\mathcal{N}$, and $\odot\mathcal{N}$ are neutrosophic hesitant fuzzy BCC-subalgebras of X. Then for any $x, y \in X$, we have $h(x \cdot y) \supseteq h(x) \cap h(y), k(x \cdot y) \subseteq k(x) \cup k(y)$, and $n(x \cdot y) \supseteq n(x) \cap n(y)$. Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. □

The following two theorems can be proved similarly to Theorem 3.36.

Theorem 3.37. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X if and only if the NHFSs $\oplus\mathcal{N}, \otimes\mathcal{N}$, and $\otimes\mathcal{N}$ are neutrosophic hesitant fuzzy BCC-ideals of X.*

Theorem 3.38. *A NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X if and only if the NHFSs $\oplus\mathcal{N}, \otimes\mathcal{N}$, and $\otimes\mathcal{N}$ are neutrosophic hesitant fuzzy strong BCC-ideals of X.*

Theorem 3.39. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X, then the sets $X_h := \{x \in X \mid h(x) = h(0)\}, X_k := \{x \in X \mid k(x) = k(0)\}$, and $X_n := \{x \in X \mid n(x) = n(0)\}$ are BCC-subalgebras of X.*

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. Let $x, y \in X_h$. Then $h(x) = h(0) = h(y)$ and so $h(x \cdot y) \supseteq h(x) \cap h(y) = h(0)$. By using Proposition 3.3, we have $h(x \cdot y) = h(0)$; hence $x \cdot y \in X_h$. Again, let $x, y \in X_k$. Then $k(x) = k(0) = k(y)$ and so $k(x \cdot y) \subseteq k(x) \cup k(y) = k(0)$. Again, by Proposition 3.3, we have $k(x \cdot y) = k(0)$; hence $x \cdot y \in X_k$. Also, $n(x) = n(0) = n(y)$ and so $n(x \cdot y) \supseteq n(x) \cap n(y) = n(0)$. By using Proposition 3.3, we have $n(x \cdot y) = n(0)$; hence $x \cdot y \in X_n$. Hence, X_h, X_k, and X_n are BCC-subalgebras of X. □

The following two theorems can be proved similarly to Theorem 3.39.

Theorem 3.40. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X, then the sets X_h, X_k, and X_n are BCC-ideals of X.*

Theorem 3.41. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X, then the sets X_h, X_k, and X_n are strong BCC-ideals of X.*

Lemma 3.42. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X, then*

$$(\forall x, y, z, w \in X) \left(x \leq w \cdot (y \cdot z) \Rightarrow \left\{ \begin{array}{l} h(x \cdot z) \supseteq h(w) \cap h(y) \\ k(x \cdot z) \subseteq k(w) \cup k(y) \\ n(x \cdot z) \supseteq n(w) \cap n(y) \end{array} \right. \right). \tag{36}$$

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Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X. Let $x, y, z, w \in X$ be such that $x \leq w \cdot (y \cdot z)$. Then $x \cdot (w \cdot (y \cdot z)) = 0$. Thus

$$\begin{aligned} h(x \cdot z) &\supseteq h(x \cdot (y \cdot z)) \cap h(y) \\ &\supseteq h(x \cdot (w \cdot (y \cdot z))) \cap h(w) \cap h(x) \\ &= h(0) \cap h(w) \cap h(y) \\ &= h(w) \cap h(y), \end{aligned}$$

$$\begin{aligned} k(x \cdot z) &\subseteq k(x \cdot (y \cdot z)) \cup k(y) \\ &\subseteq k(x \cdot (w \cdot (y \cdot z))) \cup k(w) \cup k(x) \\ &= k(0) \cup k(w) \cup k(y) \\ &= k(w) \cup k(y), \end{aligned}$$

$$\begin{aligned} n(x \cdot z) &\supseteq n(x \cdot (y \cdot z)) \cap n(y) \\ &\supseteq n(x \cdot (w \cdot (y \cdot z))) \cap n(w) \cap n(x) \\ &= n(0) \cap n(w) \cap n(y) \\ &= n(w) \cap n(y). \end{aligned}$$

□

Lemma 3.43. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X, then*

$$(\forall x, y, z \in X) \left(x \leq y \cdot z \Rightarrow \begin{cases} h(x \cdot z) \supseteq h(y) \\ k(x \cdot z) \subseteq k(y) \\ n(x \cdot z) \supseteq n(y) \end{cases} \right). \tag{37}$$

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of X. Let $x, y, z \in X$ be such that $x \leq y \cdot z$. By Lemma 3.42, put $w = 0$. Then $x \leq 0 \cdot (y \cdot z)$. Hence,

$$h(x \cdot z) \supseteq h(0) \cap h(y) = h(y),$$

$$k(x \cdot z) \subseteq k(0) \cup k(y) = k(y),$$

$$n(x \cdot z) \supseteq n(0) \cap n(y) = n(y).$$

□

Definition 3.44. Let $h : X \to \mathcal{P}([0, 1])$. For any $\pi \in \mathcal{P}([0, 1])$, the sets $U(h, \pi) = \{x \in X \mid h(x) \supseteq \pi\}$ and $U^+(h, \pi) = \{x \in X \mid h(x) \supset \pi\}$ are called an upper π-level subset and an upper π-strong level subset of h, respectively. The sets $L(h, \pi) = \{x \in X \mid h(x) \subseteq \pi\}$ and $L^-(h, \pi) = \{x \in X \mid h(x) \subset \pi\}$ are called a lower π-level subset and a lower π-strong level subset of h, respectively. The set $E(h, \pi) = \{x \in X \mid h(x) = \pi\}$ is called an equal π-level subset of h. Then $U(h, \pi) = U^+(h, \pi) \cup E(h, \pi)$ and $L(h, \pi) = L^-(h, \pi) \cup E(h, \pi)$.

Theorem 3.45. *A NHFS $\mathcal{N} = (h, k, n)$ on X is a neutrosophic hesitant fuzzy BCC-subalgebra of X if and only if for all $\pi \in \mathcal{P}([0, 1])$, the nonempty subsets $U(h, \pi), L(k, \pi)$, and $U(n, \pi)$ of X are BCC-subalgebras.*

Proof. Assume that $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. Let $\pi \in \mathcal{P}([0, 1])$ be such that $U(h, \pi) \neq \emptyset$ and let $x, y \in U(h, \pi)$. Then $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. By (29), we have $h(x \cdot y) \supseteq h(x) \cap h(y) \supseteq \pi$ and thus $x \cdot y \in U(h, \pi)$. So, $U(h, \pi)$ is a BCC-subalgebra of X. Let $\pi \in \mathcal{P}([0, 1])$ be such that $L(k, \pi) \neq \emptyset$ and let $x, y \in L(k, \pi)$. Then $k(x) \subseteq \pi$ and $k(y) \subseteq \pi$. By (29), we have $k(x \cdot y) \subseteq k(x) \cup k(y) \subseteq \pi$ and thus $x \cdot y \in L(k, \pi)$. So, $L(k, \pi)$ is a BCC-subalgebra of X. Let $\pi \in \mathcal{P}([0, 1])$ be such that $U(n, \pi) \neq \emptyset$ and let $x, y \in U(n, \pi)$. Then $n(x) \supseteq \pi$ and $n(y) \supseteq \pi$. By (29), we have $n(x \cdot y) \supseteq n(x) \cap n(y) \supseteq \pi$ and thus $x \cdot y \in U(n, \pi)$. So, $U(n, \pi)$ is a BCC-subalgebra of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0, 1])$, the nonempty subsets $U(h, \pi), L(k, \pi)$, and $U(n, \pi)$ of X are BCC-subalgebras. Let $x, y \in X$. Choose $\pi = h(x) \cap h(y) \in \mathcal{P}([0, 1])$. Then $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Thus $x, y \in U(h, \pi) \neq \emptyset$. By assumption, $U(h, \pi)$ is a BCC-subalgebra of X and thus $x \cdot y \in U(h, \pi)$. So, $h(x \cdot y) \supseteq \pi = h(x) \cap h(y)$. Let $x, y \in X$. Choose $\pi = k(x) \cup k(y) \in \mathcal{P}([0, 1])$. Then $k(x) \subseteq \pi$ and $k(y) \subseteq \pi$. Thus $x, y \in L(k, \pi) \neq \emptyset$. By assumption, $L(k, \pi)$ is a BCC-subalgebra of X and thus $x \cdot y \in L(k, \pi)$. So, $k(x \cdot y) \subseteq \pi = k(x) \cup k(y)$. Let $x, y \in X$. Choose $\pi = n(x) \cap n(y) \in \mathcal{P}([0, 1])$. Then $n(x) \supseteq \pi$ and $n(y) \supseteq \pi$. Thus $x, y \in U(n, \pi) \neq \emptyset$. By assumption, $U(n, \pi)$ is a BCC-subalgebra of X and thus $x \cdot y \in U(n, \pi)$. So, $n(x \cdot y) \supseteq \pi = n(x) \cap n(y)$. Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. □

The following two theorems can be proved similarly to Theorem 3.45.

Theorem 3.46. *A NHFS $\mathcal{N} = (h, k, n)$ on X is a neutrosophic hesitant fuzzy BCC-ideal of X if and only if for all $\pi \in \mathcal{P}([0, 1])$, the nonempty subsets $U(h, \pi)$, $L(k, \pi)$, and $U(n, \pi)$ of X are BCC-ideals.*

Theorem 3.47. *A NHFS $\mathcal{N} = (h, k, n)$ on X is a neutrosophic hesitant fuzzy strong BCC-ideal of X if and only if for all $\pi \in \mathcal{P}([0, 1])$, the nonempty subsets $U(h, \pi)$, $L(k, \pi)$, and $U(n, \pi)$ of X are strong BCC-ideals.*

Definition 3.48. Let $\{\mathcal{N}_\alpha \mid \alpha \in \Delta\}$ be a family of NHFSs on a reference set X. We define the NHFS $\bigcap_{\alpha\in\Delta} \mathcal{N}_\alpha = (\bigcap_{\alpha\in\Delta} h_\alpha, \bigcup_{\alpha\in\Delta} k_\alpha, \bigcap_{\alpha\in\Delta} n_\alpha)$ by $(\bigcap_{\alpha\in\Delta} h_\alpha)(x) = \bigcap_{\alpha\in\Delta} h_\alpha(x)$, $(\bigcup_{\alpha\in\Delta} k_\alpha)(x) = \bigcup_{\alpha\in\Delta} k_\alpha(x)$, and $(\bigcap_{\alpha\in\Delta} n_\alpha)(x) = \bigcap_{\alpha\in\Delta} n_\alpha(x)$ for all $x \in X$, which is called the *neutrosophic hesitant intersection of NHFSs*.

Proposition 3.49. *If $\{\mathcal{N}_\alpha \mid \alpha \in \Delta\}$ is a family of neutrosophic hesitant fuzzy BCC-subalgebras of X, then $\bigcap_{\alpha\in\Delta} \mathcal{N}_\alpha$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X.*

Proof. Let $\{\mathcal{N}_\alpha \mid \alpha \in \Delta\}$ be a family of neutrosophic hesitant fuzzy BCC-subalgebras of X. Let $x, y \in X$. Then

$$\begin{aligned}
(\bigcap_{\alpha\in\Delta} h_\alpha)(x \cdot y) &= \bigcap_{\alpha\in\Delta} h_\alpha(x \cdot y) \\
&\supseteq \bigcap_{\alpha\in\Delta} (h_\alpha(x) \cap h_\alpha(y)) \\
&= (\bigcap_{\alpha\in\Delta} h_\alpha(x)) \cap (\bigcap_{\alpha\in\Delta} h_\alpha(y)) \\
&= (\bigcap_{\alpha\in\Delta} h_\alpha)(x) \cap (\bigcap_{\alpha\in\Delta} h_\alpha)(y), \\
(\bigcup_{\alpha\in\Delta} k_\alpha)(x \cdot y) &= \bigcup_{\alpha\in\Delta} k_\alpha(x \cdot y) \\
&\subseteq \bigcup_{\alpha\in\Delta} (k_\alpha(x) \cup k_\alpha(y)) \\
&= \bigcup_{\alpha\in\Delta} k_\alpha(x) \cup \bigcup_{\alpha\in\Delta} k_\alpha(y) \\
&= (\bigcup_{\alpha\in\Delta} k_\alpha)(x) \cup (\bigcup_{\alpha\in\Delta} k_\alpha)(y), \\
(\bigcap_{\alpha\in\Delta} n_\alpha)(x \cdot y) &= \bigcap_{\alpha\in\Delta} n_\alpha(x \cdot y) \\
&\supseteq \bigcap_{\alpha\in\Delta} (n_\alpha(x) \cap n_\alpha(y)) \\
&= (\bigcap_{\alpha\in\Delta} n_\alpha(x)) \cap (\bigcap_{\alpha\in\Delta} n_\alpha(y)) \\
&= (\bigcap_{\alpha\in\Delta} n_\alpha)(x) \cap (\bigcap_{\alpha\in\Delta} n_\alpha)(y).
\end{aligned}$$

Hence, $\bigcap_{\alpha\in\Delta} \mathcal{N}_\alpha$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. □

The following two theorems can be proved similarly to Theorem 3.49.

Proposition 3.50. *If $\{\mathcal{N}_\alpha \mid \alpha \in \Delta\}$ is a family of neutrosophic hesitant fuzzy BCC-ideals of X, then $\bigcap_{\alpha\in\Delta} \mathcal{N}_\alpha$ is a neutrosophic hesitant fuzzy BCC-ideal of X.*

Proposition 3.51. *If $\{\mathcal{N}_\alpha \mid \alpha \in \Delta\}$ is a family of neutrosophic hesitant fuzzy strong BCC-ideals of X, then $\bigcap_{\alpha\in\Delta} \mathcal{N}_\alpha$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X.*

Definition 3.52. Let $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ be NHFSs on sets X and Y, respectively. The Cartesian product $A \times B = (h, k, n)$ defined by $h(x, y) = h_A(x) \cap h_B(y)$, $k(x, y) = k_A(x) \cup k_B(y)$, and $n(x, y) = n_A(x) \cap n_B(y)$, where $h : X \times Y \to \mathcal{P}([0, 1])$ and $k : X \times Y \to \mathcal{P}([0, 1])$ for all $x \in X$ and $y \in Y$.

Let $(X, \cdot, 0_X)$ and $(Y, \star, 0_Y)$ be BCC-algebras. Then $(X \times Y, \diamond, (0_X, 0_Y))$ is a BCC-algebra defined by $(x, y) \diamond (u, v) = (x \cdot u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.

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Proposition 3.53. *If $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy BCC-subalgebras of BCC-algebras X and Y, respectively, then the Cartesian product $A \times B$ is a neutrosophic hesitant fuzzy BCC-subalgebra of $X \times Y$.*

Proof. Assume that $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy BCC-subalgebras of BCC-algebras X and Y, respectively. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned} h((x_1,y_1) \diamond (x_2,y_2)) &= h(x_1 \cdot x_2, y_1 \star y_2) \\ &= h_A(x_1 \cdot x_2) \cap h_B(y_1 \star y_2) \\ &\supseteq (h_A(x_1) \cap h_A(x_2)) \cap (h_B(y_1) \cap h_B(y_2)) \\ &= (h_A(x_1) \cap h_B(y_1)) \cap (h_A(x_2) \cap h_B(y_2)) \\ &= h(x_1,y_1) \cap h(x_2,y_2), \end{aligned}$$

$$\begin{aligned} k((x_1,y_1) \diamond (x_2,y_2)) &= k(x_1 \cdot x_2, y_1 \star y_2) \\ &= k_A(x_1 \cdot x_2) \cup k_B(y_1 \star y_2) \\ &\subseteq (k_A(x_1) \cup k_A(x_2)) \cup (k_B(y_1) \cup k_B(y_2)) \\ &= (k_A(x_1) \cup k_B(y_1)) \cup (k_A(x_2) \cup k_B(y_2)) \\ &= k(x_1,y_1) \cup k(x_2,y_2), \end{aligned}$$

$$\begin{aligned} n((x_1,y_1) \diamond (x_2,y_2)) &= n(x_1 \cdot x_2, y_1 \star y_2) \\ &= n_A(x_1 \cdot x_2) \cap n_B(y_1 \star y_2) \\ &\supseteq (n_A(x_1) \cap n_A(x_2)) \cap (n_B(y_1) \cap n_B(y_2)) \\ &= (n_A(x_1) \cap n_B(y_1)) \cap (n_A(x_2) \cap n_B(y_2)) \\ &= n(x_1,y_1) \cap n(x_2,y_2). \end{aligned}$$

Hence, $A \times B$ is a neutrosophic hesitant fuzzy BCC-subalgebra of $X \times Y$. □

Theorem 3.54. *Two NHFSs $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy BCC-subalgebras of BCC-algebras X and Y, respectively if and only if the NHFSs $\oplus(A \times B), \otimes(A \times B)$, and $\odot(A \times B)$ are neutrosophic hesitant fuzzy BCC-subalgebras of $X \times Y$.*

Proof. It follows from Proposition 3.53 and Theorem 3.36. □

The following two propositions and two theorems can be proved similarly to Proposition 3.53 and Theorem 3.54.

Proposition 3.55. *If $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy BCC-ideals of BCC-algebras X and Y, respectively, then the Cartesian product $A \times B$ is a neutrosophic hesitant fuzzy BCC-ideal of $X \times Y$.*

Theorem 3.56. *Two NHFSs $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy BCC-ideals of BCC-algebras X and Y, respectively if and only if the NHFSs $\oplus(A \times B), \otimes(A \times B)$, and $\odot(A \times B)$ are neutrosophic hesitant fuzzy BCC-ideals of $X \times Y$.*

Proposition 3.57. *If $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy strong BCC-ideals of BCC-algebras X and Y, respectively, then the Cartesian product $A \times B$ is a neutrosophic hesitant fuzzy strong BCC-ideal of $X \times Y$.*

Theorem 3.58. *Two NHFSs $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy strong BCC-ideals of BCC-algebras X and Y, respectively if and only if the NHFSs $\oplus(A \times B), \otimes(A \times B)$, and $\odot(A \times B)$ are neutrosophic hesitant fuzzy strong BCC-ideals of $X \times Y$.*

A mapping $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ of BCC-algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCC-algebras, then $f(0_X) = 0_Y$.

Definition 3.59. Let f be a function from a nonempty set X to a nonempty set Y. If $\mathcal{N} = (h, k, n)$ is a NHFS on Y, then the NHFS $f^{-1}(\mathcal{N}) = (h \circ f, k \circ f, n \circ f)$ on X is called the *pre-image of $\mathcal{N}$ under f*.

Theorem 3.60. *Let $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of BCC-algebras. If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f, n \circ f)$ is a neutrosophic hesitant fuzzy BCC-ideal of X.*

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Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-ideal of Y. By assumption, $h(f(0_X))$ $= h(0_X) \supseteq h(y)$ for every $y \in Y$. In particular, $(h \circ f)(0_X) = h(f(0_X)) \supseteq h(f(x)) = (h \circ f)(x)$ for all $x \in X$. Also, $k(f(0_X)) = k(0_Y) \subseteq k(y)$ and $n(f(0_X)) = n(0_Y) \supseteq n(y)$ for every $y \in Y$. In particular, $(k \circ f)(0_X) = k(f(0_X)) \subseteq k(f(x)) = (k \circ f)(x)$ and $(n \circ f)(0_X) = n(f(0_X)) \supseteq n(f(x)) = (n \circ f)(x)$ for all $x \in X$. Let $x, y, z \in X$. Then

$$\begin{aligned}
(h \circ f)(x \cdot z) &= h(f(x \cdot z)) \\
&= h((f(x) \star f(z)) \\
&\supseteq h(f(x) \star (f(y) \star f(z))) \cap h(f(y)) \\
&= h(f(x \cdot (y \cdot z))) \cap h(f(y)) \\
&= (h \circ f)(x \cdot (y \cdot z)) \cap (h \circ f)(y),
\end{aligned}$$

$$\begin{aligned}
(k \circ f)(x \cdot z) &= k(f(x \cdot z)) \\
&= k((f(x) \star f(z)) \\
&\subseteq k(f(x) \star (f(y) \star f(z))) \cup k(f(y)) \\
&= k(f(x \cdot (y \cdot z))) \cup k(f(y)) \\
&= (k \circ f)(x \cdot (y \cdot z)) \cup (k \circ f)(y),
\end{aligned}$$

$$\begin{aligned}
(n \circ f)(x \cdot z) &= n(f(x \cdot z)) \\
&= n((f(x) \star f(z)) \\
&\supseteq n(f(x) \star (f(y) \star f(z))) \cap n(f(y)) \\
&= n(f(x \cdot (y \cdot z))) \cap n(f(y)) \\
&= (n \circ f)(x \cdot (y \cdot z)) \cap (n \circ f)(y).
\end{aligned}$$

Hence $f^{-1}(\mathcal{N})$ is a neutrosophic hesitant fuzzy BCC-ideal of X. □

The following two theorems can be proved similarly to Theorem 3.60.

Theorem 3.61. *Let $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of BCC-algebras. If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f, n \circ f)$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X.*

Theorem 3.62. *Let $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of BCC-algebras. If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy strong BCC-ideal of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f, n \circ f)$ is a neutrosophic hesitant fuzzy strong BCC-ideal of X.*

4 Conclusion

In the present paper, we have introduced the concepts of neutrosophic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals of BCC-algebras. The relationship between neutrosophic hesitant fuzzy BCC-subalgebras (BCC-ideals, strong BCC-ideals) and their level subsets are described. Moreover, the homomorphic pre-images of neutrosophic hesitant fuzzy BCC-subalgebras (BCC-ideals, strong BCC-ideals) in BCC-algebras are also studied, and some related properties are investigated.

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