A Representation of the Generators of the Quotients Group of Sl_2 By Matrices with Special Properties

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Abstract

The problem of the existence and construction of a resolution of singularities is one of the central questions of algebraic geometry. In this paper, we study this problem in connecting with the quotients for Sl_2. It is known that the action of Sl_2 on its Lie algebra is corresponding to the action of SO_3 on $\mathbb{C}^3$. As a result of this action, it will be an invariant ring, which determines the quotients for Sl_2. This paper is devoted to studying the singularity of these quotients. We write this singularity as a matrix with interesting features such as, for example, its quadratic is a zero matrix and its rank is less than or equal to 1. Therefore, in this paper, we reduce the studying of the singularity of the quotients of Sl_2, which is a hard problem, to the studying of a matrix of invariants which is an easy problem.

Keywords: Singularities; Lie algebra; Lie groups; symplectic doubling; quotients

1. Introduction

The problem of the existence and construction of a resolution of singularities is one of the central questions of algebraic geometry. Already in the second half of the $19th$ century, this was a very active field of research and there were many crucial contributions at that time[1]. Singularities arise naturally in many different areas of mathematics and science where the singularity theory connects the applications of mathematics with its most abstract parts. For example, it connects the investigation of optical caustics with simple Lie algebras and regular polyhedral theory. It also connects the hyperbolic partial differential equation wave fronts to knot theory and the theory of the shape of solids to commutative algebra[2]. One of the fundamentals in studying singularities is Lie algebra. Lie algebra is named after the Norwegian mathematician Sophus Lie (1842-1899), who initially used the continuous and discrete symmetries he considered primarily to investigate partial differential equations. To be able to apply these continuous transformation groups (today's Lie groups), he linearized the transformations and investigated the infinitesimal generators[3]. The connection properties of the Lie group can be expressed by commutators of the generators; the commutator algebra of the generators is now called the Lie algebra. Lie groups appear in many places in modern physics. In addition to the classical angular momentum algebra, for example, the internal gauge group of the Standard Model ($SU(3) \times SU(2) \times SU(1)$) is the product of several Lie groups[3]. Larger gauge groups appear in the context of generalized (higher dimensional) gauge theories with which it is hoped to reconcile the Standard Model with the gravitation not yet contained in it. In quantum mechanics, one is interested in groups of unitary operators acting on the vector space of quantum states. Due to the non-commutativity of the operators, these groups are Lie groups. The focus is on continuously generated groups, i.e. those groups whose elements can be generated by repeated infinitesimal transformation. For more details on the theory of Lie algebras and groups and the connections with singularity theory, you can see[1,2,4,5,6] and references therein.

In this paper, we study the problem of singularities which is connecting with the quotients for Sl_2. We take the action of Sl_2 on its Lie algebra, which is corresponding to the action of SO_3 on $\mathbb{C}^3$. As a result of this action, we will have an invariant ring, which determines the quotients for Sl_2. We write this singularity as a matrix from Lie

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algebra of $\mathfrak{sp}_4$ with interesting features, for example, its quadratic is a zero matrix and its rank is less than or equal to 1. Therefore, we reduce the studying of the singularity of the quotients of Sl_2, which is a hard problem, to the studying of matrix of invariants which is an easy problem.

This paper consists of an introduction and four main sections. In the first section, we study the effect of Sl_2, the action of Sl_2 on her Lie algebra and the effects of the mappings $SO_3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ and $SO_3 \times \mathbb{C}^3 \oplus \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \oplus \mathbb{C}^3$. In the second section, we mention the symplectic doubling. The third section is devoted to the invariants and relations of the doubled action. In the last section, we focus on the study of calculation of the desired quotients, that is connected to the problem under consideration in this paper.

2. The Effect of Sl_2

We can formulate the problem of solving singularities as follows:

Given a variety X, we want to find a non-singular variety X' and a birational mapping:

$$\pi : X' \longrightarrow X$$

such that π has an isomorphism

$$X' \backslash \pi^{-1}(SingX) \longrightarrow X \backslash (SingX)$$

and has other good properties. It should actually be in classical topology, that the primitives of compact sets are compact again. Because it is birational, it is also surjective. In this section, we consider the SL_2 action on its Lie Algebra, which we can interpret as the action of the special orthogonal group SO_3 on $\cong \mathbb{C}^3$. Through the invariant and the relations (of this group action) between them, we calculate the invariant ring which determines the quotient according to Sl_2. This resulting quotient has a singular location.

The action of Sl_2 on her Lie algebra[1]

In this subsection, we mention to the 2.1 covering map $\varphi : Sl_2 \longrightarrow SO_3$. One can retract the action of Sl_2, on her Lie Algebra $\mathfrak{s}l_2$ on the action of SO_3 to $\mathbb{C}^3$. To do this, one constructs a mapping

$$\varphi : Sl_2 \longrightarrow SO_3$$

and shows that $g \in Sl_2$, on Sl_2, in the same way on $\mathfrak{s}l_2$ as $\varphi(g) \in SO_3$, on $\mathbb{C}^3$. Let

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in Sl_2,$$

and the following adjoint representation

$$Ad : Sl_2 \longrightarrow Gl(\mathfrak{s}l_2)\,; g \longrightarrow Ad(g)(A) = gAg^{-1}.$$

Then, the matrix for the adjoint representation is

$$\begin{pmatrix} g_{11}^2 & -2g_{11}g_{12} & -g_{12}^2 \\ -g_{11}g_{21} & g_{11}g_{22} + g_{21}g_{12} & g_{12}g_{22} \\ -g_{21}^2 & 2g_{21}g_{22} & g_{22} \end{pmatrix}.$$

This is obtained from the image of the basic elements

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ and\ c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{s}l_2$, under the representation Ad.

The Fundamental Theorem[7]

The mapping $\varphi : Sl_2 \longrightarrow SO_3$, with $\varphi(g) = \alpha \circ Ad(g)$, where $\alpha : SO_3(U) \overset{\cong}{\rightarrow} SO_3$, for the regular symmetric matrix

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is an isomorphism, defined as a 2:1 covering map. Furthermore,

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$$g.a = \varphi(g).e_1, \qquad g.b = \varphi(g).e_2, \quad \text{and} \quad g.c = \varphi(g).e_3$$

hold, by means of Sl_2

$$a \longrightarrow e_1 = (1,0,0),$$

$$b \longrightarrow e_2 = (0,1,0),$$

$$c \longrightarrow e_3 = (0,0,1)$$

if one indentifies with $\mathbb{C}^3$.

Remark:

(1) The invariant rings of the two actions

$$Sl_2 \times \mathfrak{s}l_2 \longrightarrow \mathfrak{s}l_2$$

$$So_3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$$

agree and so do the quotients.

(2) For the simultaneous action on several copies also:

$$\mathbb{C}[(\mathbb{C}^3)^{\oplus k}]^{SO_3} = \mathbb{C}[(\mathfrak{s}l_2)^{\oplus k}]^{Sl_2}.$$

We apply this to the cases $k = 1$ and $k = 2$ with their duplications and analyze the effects of SO_3 on $\mathbb{C}^3$ and $\mathbb{C}^3 \oplus \mathbb{C}^3$, for which we use the fundamental Theorem to determine the quotients $\mathfrak{s}l_2{}^2 \,//\, Sl_2$, $\mathfrak{s}l_2{}^4 \,//\, Sl_2$ or the symplectic reduction $\mathfrak{s}l_2{}^2 \,///\, Sl_2$, $\mathfrak{s}l_2{}^4 \,///\, Sl_2$.

The effects of $SO_3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ and $SO_3 \times \mathbb{C}^3 \oplus \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \oplus \mathbb{C}^3$:

We consider the two operations

$$v_1: SO_3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \ ; \ (g, x) \longrightarrow gx$$

$$v_2: SO_3 \times \mathbb{C}^3 \oplus \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \oplus \mathbb{C}^3 \ ; \ (g, x, y) \longrightarrow (gx, gy)$$

and look for the invariants of the simple action. We already know from the $1st$ fundamental theorem[8,9] for SO_m that all invariants look like this:

(1) $\left(x^{(i)}\right)^t Q x^{(j)}, i \leq j \, ; i, j \in \{1,2, \dots, n\}$

(2) $det\left(x^{(i_1)} | \dots | x^{(i_m)}\right); \ i_1, \dots, i_m \in \{1,2, \dots, n\} \, ; \ i_1 \leq i_2 \leq \cdots \leq i_m,$

here are fewer than $m = 3$ vectors in the two cases, so the second kind of invariant does not occur in this action. The first action has a single invariant, namely $x^t Qx$ while the second effect has the following three invariants $x^t Qx, x^t Qy, y^t Qy$. These invariants appear in the list of the Schwartz, since they do not satisfy any realities according to the $2nd$ fundamental theorem[8,9] of SO_3.

Symplectic Doubling

The action

$$SO_3 \times (\mathbb{C}^m)^{\oplus n} \longrightarrow (\mathbb{C}^m)^{\oplus n}$$

is self-dual, because of the self-duality of the So_m-representation. Hence, the symplectic doubling is as follows:

$$SO_m \times (\mathbb{C}^m)^{\oplus 2n} \longrightarrow (\mathbb{C}^m)^{\oplus 2n}$$

$$(g, x^{(1)}, \dots, x^{(2n)}) \longrightarrow (gx^{(1)}, \dots, g\, x^{(2n)}).$$

In these cases, the symplectic doubling of μ_1 is just μ_2, so we do not have to reanalyse the symplectic doubling of μ_1, since the three invariants have no relations. The symplectic doubling of μ_2 is

$$v_4: SO_3 \times (\mathbb{C}^3)^{\oplus 4} \longrightarrow (\mathbb{C}^3)^{\oplus 4}$$

$$(g, u, v, w, k) \longrightarrow (gu, gv, gw, gk)$$

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3. Invariants and relations of the doubled action

We obtain according to the fundamental theorem for SO_m [8,9]

1. $\binom{2n+1}{2}$ invariants of the form $\left(x^{(i)}\right)^t Qx^{(j)}$ $, i \leq j; i,j \in \{1,2,\dots,n\}$.

2. $\binom{2n}{m}$ invariants of the form

$det\left(x^{(i_1)} | \dots | x^{(i_m)}\right);\ i_1, \dots, i_m \in \{1,2,\dots,n\};\ i_1 \leq i_2 \leq \cdots \leq i_m$.

for this case $n = 2$ we have the following variants:

1. $u^tQu, u^tQv, u^tQw, u^tQk, u^tQv, u^tQw, v^tQk, w^tQw, w^tQk, k^tQk$.
2. $T_1 = \det(v|w|k),\ T_2 = \det(u|w|k),\ T_3 = \det(u|u|k),\ T_4 = \det(u|v|w)$.

Let $M = X^t\, QX$ and $T = (T_1|T_2|T_3|T_4)$ the assigned matrices of the insartants of the first form or the second form, where $X = (u|v|w|k)$.

Furthermore, we can immediately give the relations between these invariants according to the fundamental theorem for SO_m [8,9]

(1) $det\ M = 0$, there is only one relation between the invariants of the first form

(2) $(-1)^i\,(-1)^j\, T_i.T_j - M_j^i = 0$, for $1 \leq i \leq j \leq 4$ or the matrix equation

$$T.T^t - adj(M) = 0$$

There are ten relations between the invariants of the second form, where M_j^i is the matrix resulting from M given the $i-th$ row and $j-th$ column deletes and $adj(M)$ denotes the adjunct of M.

(3) $\det(u|v|w)\, k^tQl - \det(u|v|w)\, w^tQl + \det(u|w|k)\, v^tQl - u^tQl$

Or $T.M = 0$ with $l \in \{u,v,w,k\}$, there are only four relations of third form and second form.

(4) Momentum mapping: We get from the pairing

$$\langle\cdot,\cdot\rangle:\ \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C} \qquad (1)$$

an SO_m - invarient pairing in matrix notation as

$$\langle\cdot,\cdot\rangle : (\mathbb{C}^m)^{\oplus n} \times (\mathbb{C}^m)^{\oplus n} \longrightarrow \mathbb{C} \qquad (2)$$

$$\langle X, Y\rangle = tr(Y^tQX)$$

with

$$X = \left(x^{(1)}|\dots|x^{(n)}\right)$$

$$Y = \left(x^{(n+1)}|\dots|x^{(2n)}\right)$$

With helping of (1), (2) and according to the lemma about the momentum mapping of symplectic doubling [10], we get for the momentum mapping of the action of SO_m, on $(\mathbb{C}^m)^{\oplus 2n}$

$$\mu : (\mathbb{C}^m)^{\oplus 2n} \longrightarrow \mathfrak{so}_m{}^*$$

$$(X,Y) \longrightarrow \left(A \longrightarrow trce(Y^tQAX)\right)$$

By virtue of the identification between $\mathfrak{so}_m{}^*$ and $\mathfrak{so}_m$, where

$$\mathfrak{so}_m \overset{\cong}{\to} \mathfrak{so}_m{}^*\ ;\ A \longrightarrow (B \longrightarrow tr(AB))$$

$$\mathfrak{so}_m{}^* \overset{\cong}{\to} \mathfrak{so}_m\ ; (B \longrightarrow tr(AB)) \longrightarrow \frac{1}{2}(A - QA^tQ)$$

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form the tuple $\left(X^{(1)}, \ldots, X^{(2n)}\right)$ in $\mathfrak{so}_m$ instead of in dual space. We calculate the modified impulse map as follows:

$$tr(Y^t QAX) = tr(AXY^t Q).$$

Since the trace of a matrix is the trace of its transpose, then

$$tr(AXY^t Q) = tr(Q^t Y X^t A^t).$$

Because $A^t = -QAQ^{-1}$ and $Q^t = Q^{-1}$, we can continue to calculate:

$$tr(Q^t Y X^t A^t) = -tr(Q^t Y X^t QAQ^{-1}) = tr(-YX^t Q).$$

Hence

$$\mu(x,y)(A) = \frac{1}{2}(XY^t - YX^t)Q.$$

But the element

$$\frac{1}{2}Q\left((XY^t - YX^t)Q\right)^t Q = \frac{1}{2}QQ^t(YX^t - XY^t)Q = -\frac{1}{2}(XY^t - YX^t)Q$$

lies in $\mathfrak{so}_m$, then we can write μ as follows:

$$\mu : (\mathbb{C}^m)^{\oplus 2n} \longrightarrow \mathfrak{so}_m$$

$$(X,Y) \longrightarrow \frac{1}{2}(XY^t - YX^t)Q.$$

Furthermore, the momentum map can be written as:

$$\mu : (\mathbb{C}^m)^{\oplus 2n} \longrightarrow \mathfrak{so}_m$$

$$(X,Y) \longrightarrow \frac{1}{2}XJX^t Q.$$

Since

$$XJX^t = (U,V)\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} U^t \\ V^t \end{pmatrix} = (-V,U)\begin{pmatrix} U^t \\ V^t \end{pmatrix} = -VU^t + UV^t.$$

Finally, we would like to calculate the momentum mapping for our special cases in coordinates, including specifying their vanishing ideal explicitly. For the doubling of v_1 is $X = x$ and $Y = y$, so we have

$$xy^t - yx^t = \begin{pmatrix} 0 & x_1y_2 - x_2y_1 & x_1y_3 - x_3y_1 \\ x_1y_2 - x_1y_2 & 0 & x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 & x_3y_2 - x_2y_3 & 0 \end{pmatrix}.$$

Since the multiplication by $1/2$ is irrelevant for ideal generators and multiplication by Q only swaps the order of the entries, the ideal corresponding to the fiber $\mu^{-1}(0)$ is

$$I_\mu = (x_1y_2 - x_2y_1,\ x_1y_3 - x_3y_1,\ x_2y_3 - x_3y_2).$$

In the second case v_2 is $X = (u|v), Y = (w,k)$, and thus we have:

$$(U,V)\begin{pmatrix} w^t \\ k^t \end{pmatrix} - (w,k)\begin{pmatrix} u^t \\ v^t \end{pmatrix} =$$

$$\begin{pmatrix} 0 & u_1w_2 + v_1k_2 - u_2w_1 - v_2k_1 & u_1w_3 + v_1k_3 - u_3w_1 - v_3k_1 \\ u_2w_1 + v_2k_1 - u_1w_2 - v_1k_2 & 0 & u_2w_3 + v_2k_3 - u_3w_2 - v_3k_2 \\ u_3w_1 + v_3k_1 - u_1w_3 - v_1k_3 & u_3w_2 + v_3k_2 - u_2w_3 - v_2k_3 & 0 \end{pmatrix}.$$

Since the multiplication by $1/2$ is irrelevant for the ideal generator and the multiplication by Q only swaps the order of the entries, the ideal corresponding to the fiber $\mu^{-1}(0)$ is

$$I_\mu = (u_1w_2 + v_1k_2 - u_2w_1 - v_2k_1\ , u_1w_3 + v_1k_3 - u_3w_1 - v_3k_1\ , u_2w_3 + v_2k_3 - u_3w_2 - v_3k_2\ )$$

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4. Calculation of the quotients

Using the singular program introduced in[3] we show that the invariants of the doubling μ^{-1} modulo I_μ, namely

$$i_1 = x^t Qx, i_2 = x^t Qy,\ i_3 = y^t Qy,$$

realize only one relation, which is $i_2^2 - i_1 i_3 = 0$. Therefore, the quotient $\mu^{-1}(0) // \mathfrak{so}_3$ has exactly one isolated A_1 singularity in 0. Since the A_1 singularity can be solved symplectic, carries $\mu^{-1}(0) // \mathfrak{so}_3$ structure, although the zero fiber of the momentum image is not a complete intersection.

Next, we calculate the relations of the 10 invariants with the help of the singular program in[3]

$$i_1 = u^t Qu, i_2 = u^t Qv, i_3 = u^t Qw, i_4 = u^t Qk, i_5 = v^t Qv, i_6 = v^t Qw,$$

$$i_7 = v^t Qk, i_8 = w^t Qk, i_9 = w^t Qk, i_{10} = k^t Qu.$$

Under the 4 determinants - invariants $T_1, \dots, T_4$ modulo I_μ. The results are that the 4 determinant invariants modulo the ideal are superfluous and the other 10 invariants suffice 6 relations

$$i_4 i_8 - i_3 k_9 + i_7 i_9 - i_6 i_{10} = 0,$$

$$i_4 i_6 + i_7^2 - i_9 i_2 - i_5 i_{10} = 0,$$

$$i_3 i_6 + i_6 k_7 - i_8 i_2 - i_5 i_9 = 0,$$

$$i_3 i_4 + i_4 k_7 - i_1 i_9 - i_2 i_{10} = 0,$$

$$i_3^2 + i_4 k_6 - i_1 i_8 - i_2 i_9 = 0,$$

$$i_2 i_3 + i_4 k_5 - i_1 i_6 - i_2 i_7 = 0.$$

Furthermore, the singular location of this quotient agrees exactly with the quotient $\mu^{-1}(0)_{Sing} // Sl_2$, i.e. $\mu^{-1}(0) // Sl_2$, with the Null fiber $\mu^{-1}(0)$ is a symplectic variety. This quotient is the variety

$$Z = \{B \in \mathfrak{sp}_4;\ B^2 = 0\},$$

where $B \in \mathfrak{sp}_4$, of the form

$$B = \begin{pmatrix} i_3 & i_2 & -i_1 & i_4 \\ -i_9 & -i_7 & i_4 & -i_{10} \\ i_8 & i_6 & -i_3 & i_9 \\ i_6 & i_5 & -i_2 & i_7 \end{pmatrix}$$

and the singular locus of this variety is

$$Z_{Sing} = \{B \in \mathfrak{sp}_4; B^2 = 0; rkB \le 1\}.$$

5. Conclusions

In this paper, the problem of singularities connecting with the quotients for Sl_2 was studied. We take the action of Sl_2 on its Lie algebra, which is corresponding to the action of SO_3 on $\mathbb{C}^3$. As a result of this action, we will have an invariant ring, which determines the quotients for Sl_2. We write singularity that is concluded by the quotients for Sl_2, as a matrix from Lie algebra of $\mathfrak{sp}_4$ with some features, for example, its quadratic is a zero matrix and its rank is less than or equal to 1. The resolution of the singularities of the concluded matrix will be the focus of our future work.

References

[1] Gert M, Greuel L, Narváez MS, Xambó D. Singularities, Algebraic Geometry, Commutative Algebra, and Related Topics. Festschrift for Antonio Campillo on the Occasion of his 65th Birthday. Springer, 2018.

[2] Pommaret JF. Partial Differential Equations and Group Theory: New Perspectives for Applications. Kluwer Academic Publishers, Dordrecht, 1994.

Doi: https://doi.org/10.54216/GJMSA.040202

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[3] Amayri M. Zur Deformation einer symplektischen Singularität und ihre symplektische Auflösung. Master thesis, 2009.
[4] Brian HC. Lie groups, Lie algebras, and Representations: An Elementary Introduction. 2nd ed. Graduate Texts in Mathematics. Vol. 222, Springer; 2015
[5] Erdmann K., Wildon M.J.: Introduction to Lie Algebras. Springer Undergraduate Mathematics Series 2006.
[6] Totaro B. Lie Group, Lie Algebra and their Representations, Lecture notes, 2010.
[7] Becker T. On the existence of symplectic resolutions of symplectic reductions. Mathematische Zeitschrift. 2010. Vol. 265, 343–363
[8] Kraft H, Procesi C. Classical Invariant Theory, Preliminary Version, July 1996.
[9] Amayri M. Zur Deformation einer symplektischen Singularität und ihre symplektische Auflösung. Master thesis, 2009
[10] Kaledin D. Symplectic singularities from the Poisson point of view. Journal für die reine und angewandte Mathematik. 2006 Nov. 20. https://doi.org/10.1515/CRELLE.2006.089

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