A Study of Shrinkage Estimators for Reliability Function and Variance Properties

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Abstract
A variety of shrinkage methods for estimating unknown parameters has been considered. We derive and compare the shrinkage estimators for the reliability function of the two-parameter exponential distribution. Simulation experiments are used to study the performances of these estimators.

Keywords: Shrinkage Estimators; Reliability Function; Variance Properties

1. Introduction

In the estimation of an unknown parameter there often exists some form of a prior knowledge about the parameter which one would like to utilize in order to get a better estimate.
Thompson (1968) described a shrinkage technique for estimating the mean of a population. Mehta and Srinivasan (1971) proposed another shrinkage estimator for the mean of a population.Pandey and Singh (1977) and Pandey (1979) described a shrinkage technique for estimating the variance of a normal distribution. Lemmer (1981) consider a shrinkage estimator for the parameter of the binomial distribution.
In this paper we consider the problem of estimating the reliability function R (t) of the two-parameter exponential distribution when the prior information regarding R(t) is available in the form of a guess value . A variety of shrinkage estimators proposed by Thompson (1968), Mehta and Srinivasan (1971), Pandey (1979) and Lemmer (1981) are used for this purpose.
The performances of these estimators are compared through simulation.
Estimators considered
Let the length of life X of a certain system be distributed as
F(X; θ, μ) =$\frac{1}{\theta}$ **exp**(-(X- μ)/ θ), 0≤ μ≤X , θ>0.
The reliability function of this system at time t is defined by
R (t) = exp [- (t - μ) / θ].
Let us consider a random sample of n items of such a system subjected to test and the test terminated as soon as the first r (≤ n)items fail.
Let X = {X(1), X(2), …, X(r)} be the first r- ordered failure times.
It is well known from Epstein and Sobel (1954) and Basu(1964) that
$\hat{\theta}$= $[\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)} - nX_{(1)}]$/(r-1), $r > 1$,
$\hat{\mu}$= $X_{(1)}$ - $\hat{\theta}/ n$,
And
$\hat{R}$ (t)= $\frac{n-1}{n}[1-\frac{t-X_{(1)}}{(r-1)\hat{\theta}}]^{r-2}$, r>1,
are the minimum variance unbiased estimators for the parameters θ, μ and R(t), respectively.
The variances of these estimators (see Lee (1978), p163) are given by
Var ($\hat{\theta}$)= $\frac{\theta^2}{r-1}$, r>1,
Var ($\hat{\mu}$)= $\frac{r\theta^2}{n^2(r-1)}$, r>1,
And

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Var($\hat{R}$ (t))= $\frac{(n-1)^2}{n^2\sqrt{(r-1)}}$ [$\sum_{i=0}^{2r-4}(2r-4i)\frac{i!\sqrt{(r-i+1)}}{n^i}\sum_{m=0}^{i}\frac{(\frac{n(u-t)}{\theta})^m}{m!}$]-$R^2(t)$, r>1,

Firstly, we consider the Thompson (1968)- type estimator for the reliability function R(t):
$\hat{R}_T(t)= R_0(t)+c(\hat{R}(t)-R_0(t))\ 0\le c\le 1$,
Where $R_0(t)$ is the guessed value of R(t).
Thompson (1968) suggested to determine c from

$$\frac{\partial MSE(\hat{R}_T(t))}{\partial c}=0$$

Where MSE $(\hat{R}_T(t))= E\ (\hat{R}_T(t)-R(t))^2$, the mean squared error of $\hat{R}_T(t)$.
It can be shown easily that
MSE $(\hat{R}_T(t))= c^2$var (R^(t))+ $(1-c)^2\ (R(t)-R_0(t))^2$,
And
$C= (R(t)-R_0(t))^2/\ [\left(R(t)-R_0(t)\right)^2$ +var ($\hat{R}$ (t))]

In practice c is estimated by replacing the unknown parameters by their sample estimates.

It follows that
$\hat{R}_T(t)= R_0(t)+\ (\hat{R}(t)-\ R_0(t))^3/\ [(\hat{R}(t)-R_0(t))^2$ +var($\hat{R}$ (t))].
Secondly, we consider the Mehta and Srinivasan (1971)-type estimator.
This is given by
$\hat{R}_M(t)=\hat{R}(t)-K(\hat{R}(t)-R_0(t))\exp[-b(\hat{R}(t)-R_0(t))/var(\hat{R}(t))]$
where K and b are positive constants to be suitably chosen such that $0<K<1$ and $b>0$. No general guidance has been given on how K and b should be chosen.

Substituting unknown parameters by their sample estimates in we obtain
$\hat{R}_M(t)=\ \hat{R}(t)-K\left(\hat{R}(t)-R_0(t)\right)\ \exp[-b(\hat{R}(t)-R_0(t))/\ \widehat{var}\ (\hat{R}(t))]$

It can be verified that the minimum and maximum values of $\hat{R}_M(t)$ is attainable when b tends to 0 and ∞ respectively by a suitable choice of K, 0<K<1. So we take

$$\lim_{b\to 0} MSE(\hat{R}_M(t))=\ (1-K)^2\ var\ \left(\hat{R}(t)\right)+K\left(R(t)-R_0(t)\right)^2$$

And

$$\lim_{b\to\infty} MSE(\hat{R}_M(t)=\ \text{var}(\hat{R}(t)))$$

Hence for 0<K<1, b>0 and $R_0(t)$ tends to $R(t)$ we have
$MSE\hat{R}_M(t)\le MSE\left(\hat{R}(t)\right)$
Thirdly, we consider the Pandey (1979) - type estimator.
This is given by
$\hat{R}_P(t)=a[K\ \hat{R}(t)+(1-K)R_0(t)]\ ,0\le K\le 1$,
with k is a constant specified by the experimenter according to his belief in $\hat{R}_0(t)$ and a is determined from

$$\frac{\partial MSE}{\partial a}\hat{R}_P(t)=0$$

It can be shown easily that
$MSE(\hat{R}_P(t))=aK^2\ \text{var}\ \left(\hat{R}(t)\right)+[(1-aK)R(t)-a(1-K)R_0(t)]^2$
And $a= dR^2(t)/\left[K^2\text{var}\ \left(\hat{R}(t)\right)+dR^2(t)\right]$ where
$d=K+(1-K)\ R_0(t)\ /R(t)$
Replacing the unknown parameters by their sample estimates, we obtain
$\hat{R}_P(t)=\hat{d}^2\hat{R}^3(t)/\left[\hat{d}^2\hat{R}^2(t)+K^2\widehat{var}(\hat{R}(t))\right]$
With
$\hat{d}=K+(1-K)R_0(t)/\hat{R}(t)$
It follows that $MSE\ \hat{R}_P(t)\le MSE\ (\hat{R}(t))$ only when $a=1$ and $R_0(t)$ tends to $R(t)$, it is not clear otherwise.
Finally, we consider the Lemmer (1981)-type estimator.
This is given by
$\hat{R}_L(t)=K\ \hat{R}(t)+(1-K)\ R_0(t),0\le K\le 1$

With K is a constant specified by the experimenter according to his belief in $R_0(t)$ and no general guidance has been given on how K should be chosen.

We note that Thompson and Lemmer estimators are equal when c=k, pandey and Lemmer estimators are equal when a=1.

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2. Comparison of Estimators

Simulation experiments are used to study the performances of the estimators. A random sample of size n from the two-parameter exponential distribution with μ=80 and θ=7 is generated. The vector $X=\{X_1, X_2, \ldots, X_r\}$ of the first r-ordered observations is recorded.

Then the minimum variance unbiased estimators $\hat{\mu}$, $\hat{\theta}$ and $\hat{R}(t)$respectively are computed

For a known constant k between zero and one and for specific values

$R_0(t)$ the quantities $\hat{R}_T(t)$, $\hat{R}_M(t)$, $\hat{R}_P(t)$ and $\hat{R}_L(t)$ are computed.

Monote Carlo experiments are repeated 5000 times. The average of the 500 sample values of each squared error, e.g. $(\hat{R}(t) - R(t))^2$, is taken as an estimate of the corresponding mean squared error which is denoted by MSE(.).

The estimates of the mean squared errors of $R(t)$ and the relative efficiencies, e.g.

$R(\hat{R}_T(t)/\hat{R}(t)) \ = MSE(\hat{R}_T(t)/MSE(\hat{R}(t)))$

are calculated for n=30, r=10,20,30, k=0.20,0.50, b =20, 50.

1. Conclusions

Although the results derived in the following table apply strictly to limited cases, they are suggestive of some general conclusions regarding the relative efficiencies of the various methods.

We note from the table that the MSE of $\hat{R}_T(t)$ is always smaller than that of other estimators.

It is obvious that $\hat{R}_T(t)$, $\hat{R}_M(t)$ and $\hat{R}_L(t)$have smaller means squared error than the minimum variance unbiased estimator of $\hat{R}(t)$.

The mean squared error of $\hat{R}_P(t)$ is always higher than the minimum variance unbiased estimator of $\hat{R}(t)$.

The advantages of $\hat{R}_T(t)$ and $\hat{R}_L(t)$ are most marked when r is small.

Sample size n=30, μ = μ_0 = 80, θ = θ_0= 7, t=85, R(t) = $R_0(t)$= 0 .49

Table1: Relative efficiencies of the various shrinkage estimators of reliability function $R(t)$.

No. of failures	M.V. U.E of $\hat{R}(t)$	R ($\hat{R}_T$(t)/$\hat{R}$(t))	K=0.20, b=20	K=0.50, b=50	K=0.20	K=0.50	K=0.20	K=0.50
			R ($\hat{R}_M$(t)/$\hat{R}$(t))		R ($\hat{R}_P$(t)/$\hat{R}$(t))		R($\hat{R}_L$(t)/$\hat{R}$(t))	
10	0.532	4.23×10^{-5}	4.3×10^{-3}	0.452	2.513	1.152	6.7×10^{-4}	0.189
20	0.476	2.58×10^{-4}	7.8×10^{-2}	0.828	2.89	1.231	7.16×10^{-3}	0.219
30	0.543	3.18×10^{-4}	1.45×10^{-3}	0.934	2.93	1.453	6.32×10^{-3}	0.310

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