A Novel Study on Some Pairwise Sub-Lindelöf Topological Spaces

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Abstract

This paper is dedicated to introduce some novel topological generalizations of pairwise Lindelöf spaces, which will be called sub-Lindelöf spaces, and to present some results. Also, We suggest two new types of pairwise sublindelöf spaces, as well as types of topological mappings, such as p_1-paralindelöf and p_2-paralindelöf mappings.

Keywords: sub-Lindelöf spaces; pairwise sublindelöf spaces; bitopological space

1. Introduction.

Bitopological spaces was suggested in [5]. A set X with two topologies τ_1 and τ_2 is called a bitopological space and is denoted by (X,τ_1,τ).
In a bitopological space (X,τ_1,τ_2), τ_1 is said to be regular with respect to τ_2 [5], if for each point x in X and each τ_1-closed set F with $x \notin F$, there are a τ_1-open set U and a τ_2-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$. (X,τ_1,τ_2) is pairwise regular [5] if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.
A cover Ư of a bitopological space (X,τ_1,τ_2) is called $\tau_1\tau_2$-open cover [1] if Ư $\subseteq \tau_1 \cup \tau_2$. If, in addition , Ư contains at least one non- empty member of τ_1 and at least one non-empty member of τ_2, then Ư is called pairwise open, [1].
A bitopological space (X,τ_1,τ_2) is called pairwise Lindelöf [1] if each pairwise open cover of X has a countable subcover. This concept also was studed by several authers like Hdeib and Fora, [3].
If Ư and Ʊ are $\tau_1\tau_2$-open covers of the bitopological space (X,τ_1,τ_2) , then Ư is called a refinement of Ʊ, [2], if each U ∈ Ư $\cap\, \tau_i$ is contained in some V ∈ Ʊ $\cap\, \tau_i$, i=1,2.
A collection $\mathscr{R}$ of subsets of the bitopological space (X,τ_1,τ_2) is called p_1-locally finite, [2], if $\mathscr{R} \cap \tau_i$ is locally finite in (X,τ_i), i=1,2. Also a collection $\mathscr{R}$ of subsets of the bitopological space (X,τ_1,τ_2) is called p_2-locally finite, [2], if $\mathscr{R} \cap \tau_i$ is locally finite in $(X,\tau j)$ for each $i \neq j$;i,j=1,2.

Hdeib and Fora, [2], initiated a study of pairwise paracompact spaces. A bitopological space (X,τ_1,τ_2) is called p_1-paracompact (p_2-paracompact respectively), if each pairwise open cover of X has a p_1-locally (p_2-locally respectively) finite $\tau_1\tau_2$-open refinement.

A function f: $(X,\tau_1,\tau_2) \rightarrow (Y,\rho_1,\rho_2)$ is called pairwise continuous, [6], if the functions f: $(X,\tau_1) \rightarrow (Y,\rho_1)$ and f: $(X,\tau_2) \rightarrow (Y,\rho_2)$ are continuous.

We shall use p- to denote pairwise, e.g. p-Lindelöf stands for pairwise Lindelöf. The τ_i-closure of a set A will be denoted by clA. and denote the set of real and rational numbers, respectively.

2. Subspaces of Pairwise Sublindelöf Spaces.

Definition 2.1: A bitopological space (Y,τ_1,τ_2) is called pisublindelöf if every p_i-locally finite p-open cover of Y has a countable subcover for i=1,2.

Example 2.2: Let Y = R×R , $\tau_1 = \tau_k \times \tau_k$, $\tau_2 = \tau_l \times \tau_l$ where τ_k is the sorgenfrey line topology and τ_l is the (standered) topology.
The space (Y,τ_1,τ_2) is p_1-sublindelöf, indeed, if it is not p_1- sublindelöf, then there exists a p_1-locally finite p-open cover G ={ Uα :α ∈Λ } of Y such that G has no countable subcover. Obviously G must be an uncountable cover of Y. Now if A = { (x,y) : x,y∈Q } , then A is dense in (Y,τ_1) and dense in (Y,τ_2). Therefore for each $\alpha \in \Lambda$, $U\alpha \cap A \neq \phi$. Since A is countable, some elements of A must be contained in uncountably many members of G. So

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there is an element of A which is contained in uncountably many members of $G \cap \tau_1$ or uncountably many members of $G \cap \tau_2$.

In a similar way we can show that (Y,τ_1,τ_2) is p_2-sublindelöf.

Now the subspace $L = \{ (x,y) : y= -x \}$ is a τ_1-closed subspace of (Y,τ_1,τ_2) which is not p_1-sublindelöf, because $\{\{x\} : x\in L \} \cup \{ [(-2,-1)\times(1,2)] \cap L \}$ is a p-open p1-locally finite cover of L which has no countable subcover. Also, since this subspace is not p_1-sublindelöf then it is not p-Lindelöf and so (Y,τ_1,τ_2) is not p-Lindelöf.

Definition 2.3: Let M be any subset of a bitopological space Y, then it is said to be relatively P_i-sublindelöf if every P_i-locally finite (in Y) p-open (in Y) cover of M has a countable subcover for i=1,2.

Definition 2.4: A subset F of a bitopological space Y is said to be relatively P_i-paracompact if every p-open (in Y) cover of F has a p_1 locally finite (in Y) $\tau_1\tau_2$-open (in Y) refinement for i=1,2.

It is easy to see that every $\tau1$-closed and every τ_2-closed subset of a P_i-sublindelöf space is relatively p_i-sublindelöf for i=1,2.

Definition 2.6 [2] : Let (Y, τ_1, τ_2) be a bitopological space. A subset S of Y is called p-dense if $cl_1S = cl_2S = Y$ and for every popen (in Y) p_1 locally finite (in Y) countable cover $\mho$ of S we have

$cl_1 \cup\{V : V \in \mho \cap \tau_1 \} \subseteq cl_2 \cup \{ V : V \in \mho \cap \tau_1 \}$ or

$cl_2 \cup\{V : V \in \mho \cap \tau_2 \} \subseteq cl_1 \cup \{ V : V \in \mho \cap \tau_2 \}$.

Theorem 2.7: Let (Y,τ_1,τ_2) be a p-regular p_2-paracompact space having a p-dense relatively p_2-sublindelöf subset A. Then Y is p-Lindelöf.

Proof: Let G be any p-open cover of Y. Then there exist $U_0\neq\phi$ and $V_0\neq\phi$ such that $U_0\in G \cap\tau_1$ and $N_0\in G \cap\tau_2$. Since (Y,τ_1,τ_2) is p-regular and p_2-paracompact, G has a p_2-locally finite $\tau_1\tau_2$-open refinement $\mho$ such that for each $T\in \mho \cap \tau_1$; cl_2T is contained in some $U\in G \cap\tau_1$ and for each $T\in \mho\cap\tau_2$; cl_1T is contained in some $U\in G \cap\tau_2$. Now $\mho$ is a p-open cover of A (add U_0 to $\mho$ if $\mho\cap\tau_1=\phi$ and add N_0 to $\mho$ if $\mho\cap\tau_2=\phi$).
Since A is relatively p_2-sublindelöf there exists a countable subcover $\mho_0$ of $\mho$ which covers A. We may assume that $\mho_0$ is p-open. But A is pdense in Y, so we have two cases to consider:

Case 1. $cl_1\cup \{ T : T\in \mho_0\cap\tau_1 \} \subseteq cl_2\cup \{ T : T\in \mho0\cap\tau_1 \}$: In this case we have
$Y = cl_1A$
$\subseteq cl_1\cup \{ T : T\in \mho_0 \}$
$= cl_1\cup \{ T : T\in \mho_0\cap\tau_1\} \cup cl_1\cup\{ T : T\in \mho_0\cap\tau_2 \}$
$\subseteq [cl_2\cup\{ T : T\in \mho_0\cap\tau_1 \}] \cup [cl_1\cup\{ T : T\in \mho_0\cap\tau_2 \}]$
$= [\cup\{ cl_2T : T\in \mho_0\cap\tau_1 \}] \cup [\cup\{ cl_1T : T\in \mho_0\cap\tau_2 \}]$

Case 2. $cl_2\cup \{ T : T\in \mho_0\cap\tau_2 \} \subseteq cl_1\cup \{ T : T\in \mho_0\cap\tau_2 \}$: In this case we have
$Y = cl_2A$
$\subseteq cl_2\cup \{ T : T\in \mho_0 \}$
$= cl2\cup \{ T : T\in \mho0\cap\tau1 \} \cup cl2\cup\{ T : T\in \mho0\cap\tau2 \}$
$\subseteq [cl_2\cup\{ T : T\in \mho_0\cap\tau_1 \}] \cup [cl_1\cup\{ T : T\in \mho_0\cap\tau_2 \}]$
$= [\cup\{ cl_2T : T\in \mho_0\cap\tau_1 \}] \cup [\cup\{ cl_1T : T\in \mho_0\cap\tau_2 \}]$

Now in each case we get
$Y \subseteq [\cup\{ cl_2T : T\in \mho_0\cap\tau_1 \}] \cup [\cup\{ cl_1T : T\in \mho_0\cap\tau_2\}]$
For each $T\in \mho_0\cap\tau_i$ choose one element $U_V\in G$ such that
$CL_jT\subseteq U_T$ ($i\neq j$; i,j=1,2). Then $\{ U_T : T\in \mho_0 \}$ is a countable subcover of G for Y. Hence the result.

3.Pairwise Sublindelöf Mappings.

Definition 3.1: A p-continuous mapping f from a bitopological space (Z,τ_1,τ_2) onto a bitopological space (Y,ρ_1,ρ_2) is called p_i-paralindelöf if for each p_i-locally finite p-open cover G of Z, there is a p_i-locally finite p-open cover N of Y such that for each T in N, $f^{-1}(T)$ is contained in the union of countably many members of G for i=1,2.

Theorem 3.2: Let f be a p-continuous mapping from (X,τ_1,τ_2) onto a p_1-sublindelöf space (Y,ρ_1,ρ_2). Then X is p_1-sublindelöf if and only if f is p_1-paralindelöf.

Proof : Suppose (X,τ_1,τ_2) is p_1-sublindelöf. Let G be a p_1-locally finite open cover of X. Then G has a countable subcover G. Now { Y } is a p_1-locally finite p-open cover of Y such that f^{-1} (Y) is contained in the union of countably many members of G. Hence f is p_1-paralindelöf
Conversely, suppose that $f: X \rightarrow Y$ is p_1-paralindelöf and Y is p_1- sublindelöf. Let U' be a p_1-locally finite p-open cover of X. Then there is a p_1-locally finite p-open cover $\mho$ of Y such that for each V in $\mho$, f^{-1} (V) is a subset of a union of countably many members of G. Since Y is p1 sublindelöf, $\mho$ has acountable subcover $\mho'$. Now $X = \cup\{ f^{-1} (V) : V\in\mho' \}$ and each f^{-1} (V) is covered by a countable subcollection of U'. Consequently X is covered by a countable subcollection of G. Then X is p_1-sublindelöf .

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***Theorem 3.3* :** Let f be a p-continuous mapping from (X,τ_1,τ_2) onto a p_2-sublindelöf space (Y,ρ_1,ρ_2). Then X is p_2-sublindelöf if and only if f is p_2-paralindelöf.

***Theorem 3.4* :** Let f be a p-continuous mapping from a bitopological space X onto a bitopological space Y. Then Y is p_1- sublindelöf if X is so.

***Proof* :** Let X be a p_1-sublindelöf space and N be a p_1-locally finite p-open cover of Y. Since f is a p-continuous mapping, f^{-1} (N) is p1- locally finite p-open cover of X. Since X is p_1-sublindelöf, f^{-1} (N) has a countable subcover, say f^{-1} (M). Therefore M is a countable subcover of N

Similarly we can prove the followig theorem:

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