On Some Applications of Gamma Function

Josef Al Jumayel
Faculty Of Science, Beirut Arab University, Beirut, Lebanon

Email: Josefjumayel113@gmail.com

Abstract
In this paper we study the gamma function and its representation for complex variable, using either a series or an appropriate integral and its applications in solving some types of Integral Equations and their relation to the Riemann zeta function.

Keywords: gamma function; de gamma function; incomplete gamma function.

1. Introduction

One of the most important special functions that are of great importance in mathematical analysis is the gamma function and can be defined by a complex variable using either a series or an appropriate integral. This function helps in finding a solution to some integral and differential equations and physical, mechanical and other applications. it is known as a mean integration and this integration cannot be calculated by the elementary functions, including the contour Hankel integral.

The first to use the symbol gamma (G) for this function was the French mathematician legender in 1839 ad and because of its great importance it was studied by such outstanding mathematicians as (Dives, Guderman, Liouville, Weierstrass, Hermite), a full historical perspective of the gamma function can be found in [2-3]..

We will show the gamma function and some of its properties by treating it as a function of real variables and a function of a complex variable , which is given in different formulas, it is an analytical function of z, and has multiple applications in the Cartesian and complex plane, and the gamma function has uses by solving some of the Integral Equations, including the Hankel contour integration, according to the Bisel function, for an integer n.

Definition:

General factor is defined as follows:

$$z\,! = \int_{0}^{+\infty} e^{-t}\, t^{z} dt \quad ; \; z \neq -1, -2, -3, \dots \qquad (1)$$

Gamma function is defined as follows:

$$\Gamma(\mathrm{z}) = \int_{0}^{+\infty} e^{-t}\, t^{z-1} dt = (z-1)! \;\; ; \; z \neq -1, -2, -3, \dots \qquad (2)$$

$$\Gamma(\mathrm{z}) = \frac{\Gamma(\mathrm{z}+1)}{z} \quad ; z < 0 \;\; , \Gamma(\mathrm{z}+1) = z! \;\; ; z = 1,2,3, \dots \qquad (3)$$

Digamma Function is defined as follows:

$$\Psi(z) = \frac{d \ln \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \;\; ; z \neq -1, -2, -3, \dots \qquad (4)$$

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Incomplete Gamma function is defined as follows:

$$\Gamma(z) = \gamma(z,x) + \Gamma(z,x) \qquad (5)$$

$$\gamma(z,x) = \int_0^x e^{-t}\, t^{z-1} dt \quad ; x > 0 \qquad (6)$$

$$\Gamma(z,x) = \int_0^{+\infty} e^{-t}\, t^{z-1} dt \quad ; x > 0 \qquad (7)$$

Hankel's contour integral is defined as follows:

:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_c u^{-z} e^u du \qquad (8)$$

Main Results

We will study the Gamma function which is defined with complex variable.

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}\ ; re(z) < 0 \qquad (10)$$

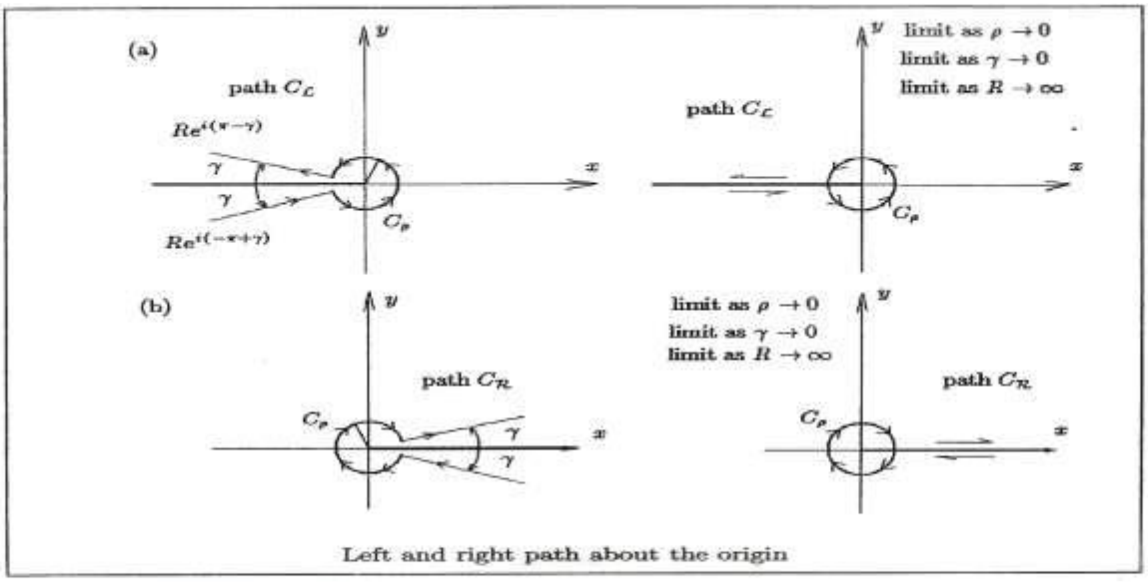
$$\Gamma_{(z)} = \int_0^\infty e^{-t} t^{z-1} dt \quad ; \ Re\{z\} > 0 \qquad (11)$$

Its integral is:

$$\int_0^z e^{-t} t^{z-1} dt \quad ; \ Re\{\alpha\} > 0 \qquad (12)$$

This function is analytical with respect to z except z=0,-1,-2,…

Which are simple poles, now by taking the function $t^{z-1} = e^{(z-1)\log t}$, and we need some conditions to make the function with only one value



Left and right path about the origin

Theorem:

The Gamma function in (2) can be written in the following formula

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$$\Gamma_{(z)} = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)(Z+2)\ldots(z+n)} \qquad (12)$$

Proof:

For this goal, we write (2) as follows:

$$\Gamma_{(z)} = \int_0^1 e^{-t} t^{z-1} dt + \int_0^\infty e^{-t} t^{z-1} dt \qquad ; \ Re\{\alpha\} > 0 \qquad (13)$$

The second integral goes to zero. The first one can be written as follows:

$$\int_0^1 e^{-t} t^{z-1} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(z+n)} \qquad (14)$$

The residues of $\Gamma_{(z)}$ at the simple pole z=-n are:

$$Res\left[\Gamma_{(z),-n}\right] = \frac{(-1)^n}{n!} \qquad ; n = 0,1,2,\ldots \qquad (15)$$

$$\int_0^n (1-\frac{t}{n})^n \quad t^{z-1} dt = \frac{n!\,n^z}{z(z+1)(Z+2)\ldots(z+n)} \qquad (16)$$

By using

$$\lim_{n\to\infty} (1-\frac{t}{n})^n = e^t \qquad (17)$$

We get:

$$\Gamma_{(z)} = \lim_{n\to\infty} \int_0^n (1-\frac{t}{n})^n \quad t^{z-1} dt = \int_0^\infty e^{-t} t^{z-1} dt = \frac{n!\,n^z}{z(z+1)(Z+2)\ldots(z+n)} \qquad (18)$$

Now, we will study some of Gamma function properties. We start with the special case of integer z=n is greater than 1. We can use the formula

$$\Gamma_{(z)} = \int_0^\infty e^{-t} t^{z-1} dt \qquad (19)$$

Thus,

$$\Gamma(\mathrm{n}) = [-t^{n-1}e^{-t}]_{t=0}^{\infty} + (n-1)\int_0^\infty e^{-t} t^{n-2} dt \qquad (20)$$

$$\Gamma(\mathrm{n}) = (n+1)\Gamma(\mathrm{n}-1) \qquad (21)$$

Now, we can write:

$$\Gamma(\mathrm{n}-1) = (n-2)\Gamma(\mathrm{n}-2)$$

$$\Gamma(\mathrm{n}-1) = (n-2)\Gamma(\mathrm{n}-2) \qquad (22)$$

$$\Gamma(\mathrm{n}) = n(n-1)(\mathrm{n}-2)(\mathrm{n}-3)\ldots 3.2.1 \qquad (23)$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1 \qquad (24)$$

$$\Gamma(\mathrm{n}) = (n-1)!\,; \quad \Gamma(\mathrm{n}-1) = n! \qquad (25)$$

Example:

We will use the proved properties to solve the following integral equation:

$I = \int_0^\infty x^{2n} e^{-x^2} \ dx.$

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Solution.

We put $x^2 = t$, then $dx = \frac{dt}{2\sqrt{t}}$, we get

$$I = \frac{1}{2}\int_0^{\infty} t^n e^{-t}\frac{dt}{\sqrt{t}} = dx$$

thus

$$I = \frac{1}{2}\,\Gamma\left(\mathrm{n} + \frac{1}{2}\right)$$

By using

$\Gamma\left(\mathrm{n} + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}.\sqrt{\pi}$, *we get*

$$I = \frac{1}{2}.\frac{(2n-1)!!}{2^n}.\sqrt{\pi} = \frac{(2n-1)!!}{2^n}.\sqrt{\pi}$$

where

$$(2n-1)!! = (2n-1)(2n-3)\ldots 3.1$$

Example:

Consider the equation

$\boldsymbol{I = \int_0^{\infty} \frac{x^{s-1}}{e^x-1}dx}$; $\boldsymbol{s > 0,}$ we have:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

So that

$$\frac{1}{1-e^{-x}} = \sum_{n=0}^{\infty} x^{-nx}$$

$I = \int_0^{\infty} \boldsymbol{\frac{x^{s-1}}{1-e^x}dx} = \int_0^{\infty} \boldsymbol{x^{s-1}} e^{-x}\sum_{n=0}^{\infty} x^{-nx}\,dx = \int_0^{\infty} \boldsymbol{x^{s-1}}\sum_{n=1}^{\infty} x^{-(n+1)x}\,dx$, *we put* t=nx, thus $dx = \frac{dt}{n}$, and

$$n = 0 \Rightarrow t = 0 \quad ; x = +\infty \Rightarrow t = +\infty$$

This means that:

$$I = \int_0^{\infty}\sum_{n=1}^{\infty}\frac{t^{s-1}}{n}e^{-t}\,\frac{dt}{n} = \sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^{\infty} t^{s-1}e^{-t}\;dt = \sum_{n=1}^{\infty}\frac{1}{n^s} = \Gamma(\mathrm{s}) = \Gamma(\mathrm{s}).\zeta(s)$$

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