Some Results About the Behaviour of Non-Linear Third Order Differential Equations

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Abstract

The aim of this paper is to study the asymptotic behaviour of the following non-linear third order differential equations in large scale of time

$[|u^{''}(t)|^{p-1}u^{''}(t)]^{'} + f(t,u(t),u^{'}(t),u^{''}(t) = 0 \quad ; p \geq 1 \quad (1).$

Many results about this behavior will be presented and discussed in terms of theorems, as well as many related examples will be illustrated.

Keywords: non-linear; third order; Laplacian; differential equations

1. Introduction and Preliminaries

In this paper, we concentrate on the asymptotic behaviours of the differential equation $[|u^{''}(t)|^{p-1}u^{''}(t)]^{'} + f(t,u(t),u^{'}(t),u^{''}(t) = 0 \quad ; p \geq 1$ (1), with the following initial conditions $u(t_0) = u_1 \, ; \, u^{'}(t_0) = u_2 \, ; \, u^{''}(t_0) = u_3$, where the necessary and sufficient conditions for the behavior of global solutions to be similar to the curves $a, b, c \in R \, ; a \neq 0 : \; t \to \infty : \; at^2 + bt + c$ will be mentioned and presented.

This problem has been discussed in the literature in [14], where many generalizations of famous inequalities were used for this discussion.

In [7,8,10-20], the sufficient conditions for the existence of such solutions were presented. In addition, for some similar functional differential equations in [1-4].

Definition:

The function $w : [0,\infty) \to [0,\infty)$ is said to be that it is contained in the class H if:

(H_1)The function $w(u)$ is continuous for $u \geq 0$ and positive for $u > 0$.

(H_2)There exists a continuous function $\emptyset$ on $[0,\infty)$ such that:

$u \geq 0 \;\; \alpha > 0 \;\; w(\alpha.u) \leq \emptyset(\alpha).w(u).$

Definition:

If $ut \leq m + \int_{t_0}^{t} \mu(s) g\big(u(s)\big) ds \; ; \; t \geq t_0$, then m is a positive constant and μ u are two real continuous non-negative functions on $[t_0, \infty)$, g is a real continuous positive function on $(0,\infty)$, with $\int_1^{\infty} \frac{dz}{g(z)} = \infty$, then

$u(t) \leq G^{-1}(G(m) + \int_{t_0}^{t} \mu(s)ds)) \; < M \, ; m < M \leq +\infty, \, G(x) = \int_k^x \frac{du}{g(u)} \quad ; \; X \geq K \; \geq 0 \, , G.$

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Results and Discussion

Definition:

The function $[t_0,t_1) \to (-\infty,+\infty);\ t_1 > t_0$ is called a solution for (1) if it holds for all $t \in [t_0,t_1)$.

U(t) is said to have the property (L_1) if:
$u(t) = at^2 + bt + c + o(t^2)\ ;\ \ t \to \infty\ ;\ \ a,b,c \in R$

Theorem:

If we have the following:

(a)The function *f(t,u,v,w)* is continuous on $D = \{(t,u,v,w): t \in [t_0,\infty); u,v,w\ \in R\}$.

(b)There exists continuous functions $h_1,h_2,h_3,g_1,g_2,g_3: R^+ \to R^+$ such that:

$|f(t,u,v,w)| \le h_1(t)g_1\left(\left[\frac{|u|}{t^2}\right]^r\right) + h_2(t)g_2(\left[\frac{|v|}{t}\right]^r),$

$+h_3(t)g_3([|w|]^r)\ \ ;\ \ r > 0,$

(c) The functions $g_1(s), g_2(s), g_3(s)$ for $s > 0$ are positive

If we put $G(x) = \int_{t_0}^{\infty} \frac{ds}{g_1\left(s^{\frac{r}{p}}\right)+g_2\left(s^{\frac{r}{p}}\right)+g_3\left(s^{\frac{r}{p}}\right)}$, *then*

$$G(+\infty) = \int_{t_0}^{\infty} \frac{ds}{g_1\left(s^{\frac{r}{p}}\right) + g_2\left(s^{\frac{r}{p}}\right) + g_3\left(s^{\frac{r}{p}}\right)} = \frac{p}{r}\int_{(t_0^{\frac{r}{p}})}^{\infty} \frac{\tau^{\frac{p}{r}-1}dx}{g_1(x)+g_2(x) + g_3(x)} = +\infty$$

(d) If

$\overline{H}_i = \int_{t_U}^{\infty} h_i(S)ds < \infty\ \ ;\ \ i = 1,2,3$, then every global solution of (1) has the property (L_1).

Proof:

Without affecting the generality we can put $t_0 = 1$, according to Standard existence theorems, the equation (1) has a solution $u(t) \in C^2([1,\infty))$ with the following initial conditions
$u(1) = |u_1|\ ;\ u'(1) = |u_2|\ ;\ u''(1) = |u_3|^p$, so that, we get:

$$|u''(t)|^{p-1}u''(t) = c_3 - \int_1^t f\left(s,u(s),u'(s),u''(s)\right)ds\ \ ;\ c_3 = |u_3|^p$$

$$|u''(t)|^p \le |u''(t)|^{p-1}u''(t)\ \ ;(u''(t))^{p-1} \le |u''(t)|^{p-1}$$

$$\le c_3 + \int_1^t |f\left(s,u(s),u'(s),u''(s)\right)|ds \qquad (2)$$

We put

$$Q(t) = c_3 + \int_1^t |f\left(s,u(s),u'(s),u''(s)\right)|ds \qquad (3)$$

(2) becomes:

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$$(u''(t))^p \leq Q(t) \qquad (4)$$

Thus

$$u''(t) \leq [Q(t)]^{\frac{1}{p}} \qquad (5)$$

$$u'(t) \leq c_2 + \int_1^t [Q(S)]^{\frac{1}{p}} ds \leq c_2 + (t-1)\,[Q(S)]^{\frac{1}{p}}$$

$$\leq t[c_2 + \left(Q(S)\right)^{\frac{1}{p}}\Big] \quad ; \mathrm{t} \in [t_0, \infty)\,;\; c_2 - |u_2|$$

Hence

$$u'(t) \leq t\left[c_2 + Q(S))^{\frac{1}{p}}\right] \quad ; \; c_2 = |u_2| \qquad (6)$$

From (6), we get

$$\mathrm{u}(t) \leq t^2\left[c_1 + Q(S))^{\frac{1}{p}}\right] \quad ; \; c_1 = |u_1| \qquad (7)$$

Thus

$$[\frac{|u'(t)|}{t}]^p \leq \left[c_2 + Q(S))^{\frac{1}{p}}\right]^p \qquad (8)$$

$$[\frac{|u(t)|}{t^2}]^p \leq \left[c_1 + Q(S))^{\frac{1}{p}}\right]^p \qquad (9)$$

It is known that :$(a \;\mid\; b)^p \leq 2^{p-1}(a^p \mid b^p) \quad ; \quad a, b \geq 0$

So,

$$[\frac{|u(t)|}{t^2}]^p \leq 2^{p-1}[c_1^p + Q(t)] = 2^{p-1}c_1^p + 2^{p-1}Q(t)$$

$$\leq 2^{p-1}c_1^p + 2^{p-1}[c_3 + \int_1^t |f\left(s, u(s), u'(s), u''(s)\right)| ds\,]$$

Now, according to (b), we find

$$[\frac{|u(t)|}{t^2}]^p \leq e_1 + \int_1^t H_1(s)g_1\left(\left[\frac{|u|}{s^2}\right]^r\right) ds + \int_1^t H_2(s)g_2\left([\frac{|u'|}{s}]^r\right) ds +$$

$$+ \int_1^t H_3(s)g_3([|u''|])ds \qquad (10)$$

$$e_1 = 2^{p-1}(c_1^p + c_3) \quad , \quad H_i(t) = 2^{p-1}h_i(t); i = 1,2,3$$

and

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$$[\frac{|u'(t)|}{t}]^p \leq e_2 + \int_1^t H_1(s) g_1\left(\left[\frac{|u|}{s^2}\right]^r\right) ds + \int_1^t H_2(s) g_2\left([\frac{|u'|}{s}]^r\right) ds +$$

$$\int_1^t H_3(s) g_3([|u''|]^r) ds \qquad (11)$$

$$e_2 = 2^{p-1}(c_2^p + c_3) \quad , \quad H_i(t) = 2^{p-1} h_i(t); i = 1,2,3$$

(4) implies

$$|u''(t)|^P \leq c_3 \int_1^t h_1(s) g_1\left(\left[\frac{|u|}{s^2}\right]^r\right) ds + \int_1^t h_2(s) g_2\left([\frac{|u'|}{s}]^r\right) ds +$$

$$+ \int_1^t h_3(s) g_3([|u''|]^r) ds \qquad (12)$$

We put

$$A(t) = e_1 + \int_1^t H_1(s) g_1\left(\left[\frac{|u|}{s^2}\right]^r\right) ds + \int_1^t H_2(s) g_2\left([\frac{|u'|}{s}]^r\right) ds +$$

$$+ \int_1^t H_3(s) g_3([|u''|]^r) ds \qquad (13)$$

This implies that:

$$[\frac{|u(t)|}{t^2}]^p \leq A(t) \qquad (14)$$

$$[\frac{|u'(t)|}{t}]^p \leq A(t) \qquad (15)$$

$$|u''(t)|^P \leq A(t) \qquad (16)$$

$$[\frac{|u(t)|}{t^2}]^r \leq [A(t)]^{\frac{r}{p}} \qquad (17)$$

$$[\frac{|u'(t)|}{t}]^r \leq [A(t)]^{\frac{r}{p}} \qquad (18)$$

$$|u''(t)|^r \leq [A(t)]^{\frac{r}{p}} \qquad (19)$$

$$g_1([\frac{|u(t)|}{t^2}]^r) \leq g_1\left([A(t)]^{\frac{r}{p}}\right) \qquad (20)$$

$$g_2([\frac{|u'(t)|}{t}]^r) \leq g_2\left([A(t)]^{\frac{r}{p}}\right) \qquad (21)$$

$$g_3(|u''(t)|^r) \leq g_3\left([A(t)]^{\frac{r}{p}}\right) \qquad (22)$$

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From (13) we get:

$$A(t) = e_1 + \int_1^t H_1(s) g_1\left([A(S)]^{\frac{r}{p}}\right) ds + \int_1^t H_2(s) g_2\left([A(S)]^{\frac{r}{p}}\right) ds +$$

$$+ \int_1^t H_3(s) g_3\left([A(S)]^{\frac{r}{p}}\right) ds$$

It is known that:

$$(H_1 g_1 + H_2 g_2 + H_3 g_3) \leq (H_1 + H_2 + H_3)(g_1 + g_2 + g_3)$$

This implies

$$A(t) < e_1 + \qquad (23)$$

$$+ \int_1^t (H_1(s) + H_2(s) + H_3(s))\ (g_1[A(S)]^{\frac{r}{p}} + g_2[A(S)]^{\frac{r}{p}} + g_3[A(S)]^{\frac{r}{p}} ds$$

By using Bihari's inequality, we get:

$$A(t) < G^{-1}[G(e_1) + 2^{p-1} \int_1^t h_1(s) + h_2(s) + h_3(s))\ ds]$$

$G(w) = \int_1^w \frac{ds}{g_1[(S)]^{\frac{r}{p}} + g_2[(S)]^{\frac{r}{p}} + g_3[(S)]^{\frac{r}{p}}}$, with $G(+0) < 0$, $k = G(e_1) + 2^{p-1} \int_1^t h_1(s) + h_2(s) + h_3(s))\ ds < +\infty$, $A(T) < G^{-1}(w) < +\infty$, *this means that:*

$$G^{-1}[\frac{|u(t)|}{t^2}]^p \leq G^{-1}(K)$$

$$[\frac{|u'(t)|}{t}]^p \leq G^{-1}(K)$$

$$|u''(t)|^P \leq G^{-1}(K)$$

Thus

$$\frac{|u(t)|}{t^2} \leq [G^{-1}(K)]^{\frac{1}{p}}$$

$$\frac{|u'(t)|}{t} \leq [G^{-1}(K)]^{\frac{1}{p}}$$

$$|u''(t)| \leq [G^{-1}(K)]^{\frac{1}{p}}$$

and

$$\int_1^t |f\left(s, u(s), u'(s), u''(s)\right)| ds \ \leq \int_1^t h_1(s) g_1\left(\left[\frac{|u|}{s^2}\right]^r\right) ds + \int_1^t h_2(s) g_2\left([\frac{|u'|}{s}]^r\right) ds +$$

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$$\int_1^t h_3(s)g_3([|u''|]^r)ds$$

$$\leq e_1 + \int_1^t H_1(s)g_1\left(\left[\frac{|u|}{s^2}\right]^r\right)ds + \int_1^t H_2(s)g_2\left([\frac{|u'|}{s}]^r\right)ds +$$

$$+\int_1^t H_3(s)g_3([|u''|]^r)ds = A(t) \leq G^{-1}(w) < +\infty \ ; t \geq 1$$

So that,

$\lim_{t\to\infty} \int_0^t f\left(s, u(s), u'(s), u''(s)\right) ds < \infty$, and there exists $a \in R$ such that

$$\lim_{t\to\infty} u''(t) = a$$

$$\lim_{t\to\infty} \frac{u(t)}{t^2} = \frac{u_1 + \int_1^t u'(s)ds}{t^2} = \lim_{t\to\infty} u''(t) = a$$

For any real constants b,c

$$\lim_{t\to\infty} \{\frac{u(t) - at^2 + c)}{t^2}\} = 0$$

Example:

Consider the equation:

$$u''' + t^{\frac{3}{2}}(u'')^{p-r} + t^{\frac{3}{2}}(\frac{u'}{t})^{p-r} + t^{\frac{3}{2}}(\frac{u}{t^2})^{p-r} = 0 \ ; r > 0, p \geq 1 \quad (*)$$

We have:

$$f(t, u, u', u'') = t^{\frac{3}{2}}(u'')^{p-r} + t^{\frac{3}{2}}(\frac{u'}{t})^{p-r} + t^{\frac{3}{2}}(\frac{u}{t^2})^{p-r}$$

For ≥ 1 :

$$\frac{u'}{t} \leq u' \Rightarrow (\frac{u'}{t})^{p-r} \leq (u')^{p-r}$$

$$\frac{u}{t^2} \leq u \Rightarrow (\frac{u}{t^2})^{p-r} \leq (u)^{p-r}$$

Also,

$$f(t, u, u', u'') \leq t^{\frac{3}{2}}[|u''|^{p-r} + |u'|^{p-r} + |u|^{p-r}]$$

$$h_1(t) = h_2(t) = h_3(t) = t^{\frac{3}{2}} \ ; g_1(u) = g_2(u) = g_3(u) = u^{\frac{p}{r}-1}$$

$$\overline{H}_i = \int_1^\infty t^{\frac{3}{2}}dt = 2 < +\infty \ ; \ i = 1,2,3$$

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$$G(+\infty) = \frac{p}{r} \int_{1}^{\infty} \frac{\tau^{\frac{p}{r}-1} dx}{3\tau^{\frac{p}{r}-1}} = +\infty$$

So that every global solution u(t) has an asymptotic behavior $at^2 + bt + c$.

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