The Neutrosophic Hyperbolic Functions

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Abstract

This paper aims at studying the neutrosophic hyperbolic functions, wherein the neutrosophic hyperbolic functions are defined, also, neutrosophic hyperbolic identities were discussed, in addition to introducing the rules of derivatives and integrals of the neutrosophic hyperbolic functions and inverse neutrosophic hyperbolic functions, also, logarithmic forms of inverse neutrosophic hyperbolic functions were presented.

Keywords: hyperbolic; inverse neutrosophic hyperbolic function; neutrosophic derivatives and integrals.

1. Introduction

As In an attempt to replace the current logics, Smarandache introduced the neutrosophic logic to illustrate a mathematical model of redundancy, uncertainty, contradiction, unknown, ambiguity, undefined, inconsistency, vagueness, imprecision, and incompleteness. Smarandache defined neutrosophic real number [2-4], probabilities according to neutrosophic logic [3-5-12], the neutrosophic statistics [4][6], he has also introduced the concept of integration and differentiation in neutrosophic [1-7]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [8]. Chakraborty utilized pentagonal neutrosophic number in networking problems, and Shortest Path Problems [10-11]. Yaser Alhasan presented several papers in the field of nitro calculus [13-20-9]. Also, Mohamed Abdel-Basset presented a study TOPSIS technique in neutrosophic logic [14]. On the other hand, the neutrosophic sets played a significant role in the field of public life such as health, industry and others [15-16-17]. Recently, there are growing efforts to probe the generalized structures and spaces of neutrosophic such as the modules of refined neutrosophic [18-19].

There are five parts in this artical. An introduction to neutrosophic science is provided in the first section. The second section includes neutrosophic integrals theoreis and derivative ruls. The third section defines the neutrosophic hyperbolic functions along with its derivatives, in addition to inverse neutrosophic hyperbolic functions. The fourth section introduces integrals of the neutrosophic hyperbolic functions and inverse neutrosophic hyperbolic functions. The fifth section concludes the paper.

2. Preliminaries

2.1 The rules of the neutrosophic derivative [10]

1) $\frac{d}{dx}(c + dI) = 0 + 0I$; where c, d are real numbers, while $I =$ indeterminacy.

2) $\frac{d}{dx}[(a + bI)x + c + dI] = a + bI$; where c, d are real numbers, while $I =$ indeterminacy.

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3) $\frac{d}{dx}[(a+bI)x^n] = n(a+bI)x^{n-1}; n\ is$ real number.

4) $\frac{d}{dx}\left[e^{(a+bI)x+c+dI}\right] = (a+bI)e^{(a+bI)x+c+dI}$

2.2 Neutrosophic integration by substitution method [21]

Definition2.2.1

Let $f: D_f \subseteq R \to R_f \cup \{I\}$, to evaluate $\int f(x)dx$

Put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

By substitution, we get:

$$\int f(x)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

Theorme2.2.1:

If $\int f(x,I)dx = \varphi(x,I)\ \ then,$

$$\int f\big((a+bI)x+c+dI\big)\,dx = \left(\frac{1}{a} - \frac{b}{a(a+b)}I\right)\varphi\big((a+bI)x+c+dI\big) + C$$

where C is an indeterminate real constant, $a \neq 0\ , a \neq -b\ and\ b, c, d$ are real numbers, while I = indeterminacy.

3. Neutrosophic hyperbolic functions

Let $f(x,I) = e^{(\alpha+\beta I)x+k+lI}$, then we can express of the neutrosophic function as the following:

$$f(x,I) = e^{(\alpha+\beta I)x+k+lI} = \frac{e^{(\alpha+\beta I)x+k+lI} + e^{-((\alpha+\beta I)x+k+lI)}}{2} + \frac{e^{(\alpha+\beta I)x+k+lI} - e^{-((\alpha+\beta I)x+k+lI)}}{2}$$

The odd function is called the neutrosophic hyperbolic sine of x and the even function is called the neutrosophic hyperbolic cosine of x.

Hence:

$$sinh\big((\alpha+\beta I)x+k+lI\big) = \frac{e^{(\alpha+\beta I)x+k+lI} - e^{-((\alpha+\beta I)x+k+lI)}}{2}$$

$$cosh\big((\alpha+\beta I)x+k+lI\big) = \frac{e^{(\alpha+\beta I)x+k+lI} + e^{-((\alpha+\beta I)x+k+lI)}}{2}$$

For all $x \in (-\infty, +\infty)$, Where $\alpha \neq 0\ and\ \beta, k, l$ are real numbers, while I = indeterminacy.

Then we can write the following definitions:

Definition3.1

The neutrosophic function

$$cosh\big((\alpha+\beta I)x+k+lI\big) = \frac{e^{(\alpha+\beta I)x+k+lI} + e^{-((\alpha+\beta I)x+k+lI)}}{2}$$

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is called neutrosophic hyperbolic cosine.

Definition3.2

The neutrosophic function

$$sinh\big((\alpha+\beta I)x+k+lI\big)=\frac{e^{(\alpha+\beta I)x+k+lI}-e^{-\left((\alpha+\beta I)x+k+lI\right)}}{2}$$

is called neutrosophic hyperbolic sine.

Definition3.3

The neutrosophic function

$$tanh\big((\alpha+\beta I)x+k+lI\big)=\frac{e^{(\alpha+\beta I)x+k+lI}-e^{-\left((\alpha+\beta I)x+k+lI\right)}}{e^{(\alpha+\beta I)x+k+lI}+e^{-\left((\alpha+\beta I)x+k+lI\right)}}$$

is called neutrosophic hyperbolic tangent.

Definition3.4

The neutrosophic function

$$coth\big((\alpha+\beta I)x+k+lI\big)=\frac{e^{(\alpha+\beta I)x+k+lI}+e^{-\left((\alpha+\beta I)x+k+lI\right)}}{e^{(\alpha+\beta I)x+k+lI}-e^{-\left((\alpha+\beta I)x+k+lI\right)}}$$

is called neutrosophic hyperbolic cotangent.

Definition3.5

The neutrosophic function

$$sech\big((\alpha+\beta I)x+k+lI\big)=\frac{1}{cosh\big((\alpha+\beta I)x+k+lI\big)}$$

$$=\frac{2}{e^{(\alpha+\beta I)x+k+lI}+e^{-\left((\alpha+\beta I)x+k+lI\right)}}$$

is called neutrosophic hyperbolic secant.

Definition3.6

The neutrosophic function

$$csch\big((\alpha+\beta I)x+k+lI\big)=\frac{1}{sinh\big((\alpha+\beta I)x+k+lI\big)}$$

$$=\frac{2}{e^{(\alpha+\beta I)x+k+lI}-e^{-\left((\alpha+\beta I)x+k+lI\right)}}$$

is called neutrosophic hyperbolic cosecant.

3.1 Neutrosophic hyperbolic identities

1) $cosh\big((\alpha+\beta I)x+k+lI\big)+sinh\big((\alpha+\beta I)x+k+lI\big)=e^{(\alpha+\beta I)x+k+lI}$

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2) $cosh\big((\alpha+\beta I)x+k+lI\big)-sinh\big((\alpha+\beta I)x+k+lI\big)=e^{-((\alpha+\beta I)x+k+lI)}$

3) $cosh^2\big((\alpha+\beta I)x+k+lI\big)-sinh^2\big((\alpha+\beta I)x+k+lI\big)=1$

4) $1-tanh^2\big((\alpha+\beta I)x+k+lI\big)=sech^2\big((\alpha+\beta I)x+k+lI\big)$

5) $coth^2\big((\alpha+\beta I)x+k+lI\big)-1=csch^2\big((\alpha+\beta I)x+k+lI\big)$

6) $sinh\Big(2\big((\alpha+\beta I)x+k+lI\big)\Big)=2\,sinh\big((\alpha+\beta I)x+k+lI\big)\,cosh\big((\alpha+\beta I)x+k+lI\big)$

7) $cosh\Big(2\big((\alpha+\beta I)x+k+lI\big)\Big)=cosh^2\big((\alpha+\beta I)x+k+lI\big)+sinh^2\big((\alpha+\beta I)x+k+lI\big)$

$$or\quad =2\,cosh^2\big((\alpha+\beta I)x+k+lI\big)-1$$

$$or\quad =2\,sinh^2\big((\alpha+\beta I)x+k+lI\big)+1$$

8) $cosh^2\big((\alpha+\beta I)x+k+lI\big)=\frac{1}{2}(cosh2\big((\alpha+\beta I)x+k+lI\big)+1)$

9) $sinh^2\big((\alpha+\beta I)x+k+lI\big)=\frac{1}{2}(cosh2\big((\alpha+\beta I)x+k+lI\big)-1)$

Note:

We can easily prove of these identities, for example:

Prove (3)

$cosh^2\big((\alpha+\beta I)x+k+lI\big)-sinh^2\big((\alpha+\beta I)x+k+lI\big)$

$$=\left(\frac{e^{(\alpha+\beta I)x+k+lI}+e^{-((\alpha+\beta I)x+k+lI)}}{2}\right)^2-\left(\frac{e^{(\alpha+\beta I)x+k+lI}-e^{-((\alpha+\beta I)x+k+lI)}}{2}\right)^2$$

$$=\frac{e^{2((\alpha+\beta I)x+k+lI)}+2+e^{-2((\alpha+\beta I)x+k+lI)}}{4}-\left(\frac{e^{2((\alpha+\beta I)x+k+lI)}-2+e^{-2((\alpha+\beta I)x+k+lI)}}{2}\right)=\frac{4}{4}=1$$

Prove (4)

$$cosh^2\big((\alpha+\beta I)x+k+lI\big)-sinh^2\big((\alpha+\beta I)x+k+lI\big)=1$$

divide both sides by: $cosh^2\big((\alpha+\beta I)x+k+lI\big)$, we get

$$1-\frac{sinh^2\big((\alpha+\beta I)x+k+lI\big)}{cosh^2\big((\alpha+\beta I)x+k+lI\big)}=\frac{1}{cosh^2\big((\alpha+\beta I)x+k+lI\big)}$$

Hence:

$$1-tanh^2\big((\alpha+\beta I)x+k+lI\big)=sech^2\big((\alpha+\beta I)x+k+lI\big)$$

3.2 Derivatives of the neutrosophic hyperbolic functions

1) $\frac{d}{dx}\big[sinh\big((\alpha+\beta I)x+k+lI\big)\big]=(\alpha+\beta I)\,cosh\big((\alpha+\beta I)x+k+lI\big)$

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2) $\frac{d}{dx}\left[cosh\big((\alpha+\beta I)x+k+lI\big)\right]=(\alpha+\beta I)\ sinh\big((\alpha+\beta I)x+k+lI\big)$

3) $\frac{d}{dx}\left[tanh\big((\alpha+\beta I)x+k+lI\big)\right]=(\alpha+\beta I)\ \ sech^2\big((\alpha+\beta I)x+k+lI\big)$

4) $\frac{d}{dx}\left[coth\big((\alpha+\beta I)x+k+lI\big)\right]=-(\alpha+\beta I)\ \ csch^2\big((\alpha+\beta I)x+k+lI\big)$

5) $\frac{d}{dx}\left[sech\big((\alpha+\beta I)x+k+lI\big)\right]=-(\alpha+\beta I)\ \ sech\big((\alpha+\beta I)x+k+lI\big)\,tanh\big((\alpha+\beta I)x+k+lI\big)$

6) $\frac{d}{dx}\left[csch\big((\alpha+\beta I)x+k+lI\big)\right]=-(\alpha+\beta I)\ \ csch\big((\alpha+\beta I)x+k+lI\big)\,coth\big((\alpha+\beta I)x+k+lI\big)$

Example3.2.1

1) $\frac{d}{dx}\left[sinh\big((3+4I)x+5+I\big)\right]=(3+4I)\ cosh\big((3+4I)x+5+I\big)$

2) $\frac{d}{dx}[tanh(1+10I)x]=(1+10I)\ \ sech^2(1+10I)x$

3) $\frac{d}{dx}\left[csch\big((6+7I)x+1-5I\big)\right]=-(6+7I)\ \ csch\big((6+7I)x+1-5I\big)\,coth\big((6+7I)x+1-5I\big)$

4) $\frac{d}{dx}[cosh(3+9I)x^3]=(9+27I)x^2\ sinh(3+9I)x^3$

5) $\frac{d}{dx}\left[sech\sqrt{(3+5I)x-2I}\right]\ =\frac{(3+5I)x}{2\sqrt{(3+5I)x-2I}}sech\sqrt{(3+5I)x-2I}\ tanh\sqrt{(3+5I)x-2I}$

6) $\frac{d}{dx}\left[ln\big(tanh\big((2+7I)x+5\big)\big)\right]\ =\ \frac{sech^2\big((2+7I)x+5\big)}{tanh\big((2+7I)x+5\big)}$

3.3 Derivatives of the inverse neutrosophic hyperbolic functions

1) $\frac{d}{dx}\left[sinh^{-1}\big((\alpha+\beta I)x+k+lI\big)\right]=\frac{\alpha+\beta I}{\sqrt{1+\big((\alpha+\beta I)x+k+lI\big)^2}}$

2) $\frac{d}{dx}\left[cosh^{-1}\big((\alpha+\beta I)x+k+lI\big)\right]=\frac{\alpha+\beta I}{\sqrt{\big((\alpha+\beta I)x+k+lI\big)^2-1}}\ ;\ x>\frac{1-k-lI}{\alpha+\beta I}$

Where $\alpha\neq 0,\alpha\neq-\beta\ and\ k,l$ are real numbers, while $I=$ indeterminacy.

3) $\frac{d}{dx}\left[tanh^{-1}\big((\alpha+\beta I)x+k+lI\big)\right]=\frac{\alpha+\beta I}{1-\big((\alpha+\beta I)x+k+lI\big)^2}\ ;\ |(\alpha+\beta I)x+k+lI|<1$

Where $\alpha\neq 0,\alpha\neq-\beta\ and\ k,l$ are real numbers, while $I=$ indeterminacy.

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4) $\frac{d}{dx}\left[coth^{-1}\big((\alpha+\beta I)x+k+lI\big)\right]=\frac{\alpha+\beta I}{1-\big((\alpha+\beta I)x+k+lI\big)^2}$; $|(\alpha+\beta I)x+k+lI|>1$

Where $\alpha \neq 0, \alpha \neq -\beta$ and k, l are real numbers, while $I =$ indeterminacy.

5) $\frac{d}{dx}\left[sech^{-1}\big((\alpha+\beta I)x+k+lI\big)\right]=\frac{-\alpha-\beta I}{\big((\alpha+\beta I)x+k+lI\big)\sqrt{1-\big((\alpha+\beta I)x+k+lI\big)^2}}$

; $\frac{-k-lI}{\alpha+\beta I}<x<\frac{1-k-lI}{\alpha+\beta I}$

Where $\alpha \neq 0, \alpha \neq -\beta$ and k, l are real numbers, while $I =$ indeterminacy.

6) $\frac{d}{dx}\left[csch^{-1}\big((\alpha+\beta I)x+k+lI\big)\right]=\frac{-\alpha-\beta I}{\left|\big((\alpha+\beta I)x+k+lI\big)\right|\sqrt{1+\big((\alpha+\beta I)x+k+lI\big)^2}}$

; $x \neq \frac{-k-lI}{\alpha+\beta I}$

Where $\alpha \neq 0, \alpha \neq -\beta$ and k, l are real numbers, while $I =$ indeterminacy.

Example3.3.1

1) $\frac{d}{dx}\left[sinh^{-1}\big((5+3I)x-2+7I\big)\right]=\frac{5+3I}{\sqrt{1+\big((5+3I)x-2+7I\big)^2}}$

2) $\frac{d}{dx}\left[cosh^{-1}\big((10-11I)x+15+51I\big)\right]=\frac{10-11I}{\sqrt{\big((10-11I)x+15+51I\big)^2-1}}$

3) $\frac{d}{dx}\left[tanh^{-1}\left(sin\big((5-3I)x+8-10I\big)\right)\right]=\frac{(5-3I)\,cos\big((5-3I)x+8-10I\big)}{1-sin^2\big((5-3I)x+8-10I\big)}$

$$=\frac{(5-3I)\,cos\big((5-3I)x+8-10I\big)}{cos^2\big((5-3I)x+8-10I\big)}=\frac{5-3I}{cos\big((5-3I)x+8-10I\big)}$$

$$=(5-3I)\ \ sec\big((5-3I)x+8-10I\big)$$

5) $\frac{d}{dx}\left[sech^{-1}\big((7+2I)x+1+5I\big)\right]=\frac{-7-2I}{\big((7+2I)x+1+5I\big)\sqrt{1-\big((7+2I)x+1+5I\big)^2}}$

6) $\frac{d}{dx}\left[csch^{-1}\big((5-3I)x+8-9I\big)\right]=\frac{-5+3I}{|(5-3I)x+8-9I|\sqrt{1+\big((5+3I)x+8-9I\big)^2}}$

4. Integrals of the neutrosophic hyperbolic functions

Let $\alpha \neq 0, \alpha \neq -\beta$ and k, l are real numbers, while $I =$ indeterminacy, then:

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1) $\int cosh\big((\alpha+\beta I)x+k+lI\big)\,dx=\left(\frac{1}{\alpha}-\frac{\beta}{\alpha(\alpha+\beta)}I\right)sinh\big((\alpha+\beta I)x+k+lI\big)+C$

2) $\int sinh\big((\alpha+\beta I)x+k+lI\big)\,dx=\left(\frac{1}{\alpha}-\frac{\beta}{\alpha(\alpha+\beta)}I\right)cosh\big((\alpha+\beta I)x+k+lI\big)+C$

3) $\int sech^2\big((\alpha+\beta I)x+k+lI\big)\,dx=\left(\frac{1}{\alpha}-\frac{\beta}{\alpha(\alpha+\beta)}I\right)tanh\big((\alpha+\beta I)x+k+lI\big)+C$

4) $\int csch^2\big((\alpha+\beta I)x+k+lI\big)\,dx=-\left(\frac{1}{\alpha}-\frac{\beta}{\alpha(\alpha+\beta)}I\right)coth\big((\alpha+\beta I)x+k+lI\big)+C$

5) $\int sech\big((\alpha+\beta I)x+k+lI\big)\,tanh\big((\alpha+\beta I)x+k+lI\big)\,dx$

$$=-\left(\frac{1}{\alpha}-\frac{\beta}{\alpha(\alpha+\beta)}I\right)sech\big((\alpha+\beta I)x+k+lI\big)+C$$

6) $\int csch\big((\alpha+\beta I)x+k+lI\big)\,coth\big((\alpha+\beta I)x+k+lI\big)\,dx$

$$=-\left(\frac{1}{\alpha}-\frac{\beta}{\alpha(\alpha+\beta)}I\right)csch\big((\alpha+\beta I)x+k+lI\big)+C$$

Example4.1

1) $\int cosh^5\big((7+2I)x+8+5I\big)\,sinh\big((7+2I)x+8+5I\big)\,dx$

Solution:

$$u=cosh\big((7+2I)x+8+5I\big)\quad\Longrightarrow\quad du=(7+2I)\,sinh\big((7+2I)x+8+5I\big)dx$$

$$\frac{du}{7+2I}=sinh\big((7+2I)x+8+5I\big)\,dx$$

Then:

$$\int cosh^5\big((7+2I)x+8+5I\big)\,sinh\big((7+2I)x+8+5I\big)\,dx=\int u^5\frac{du}{7+2I}=\frac{1}{7+2I}\,\frac{u^6}{6}+C$$

$$=\left(\frac{1}{42}-\frac{1}{189}I\right)u^6+C=\left(\frac{1}{42}-\frac{1}{189}I\right)cosh^6\big((7+2I)x+8+5I\big)+C$$

2) $\int coth\big((6-13I)x+14+55I\big)\,dx=\int\frac{cosh\big((6-13I)x+14+55I\big)}{sinh\big((6-13I)x+14+55I\big)}dx$

$$u=sinh\big((6-13I)x+14+55I\big)\quad\Longrightarrow\quad du=(6-13I)cosh\big((6-13I)x+14+55I\big)\,dx$$

$$\frac{du}{6-13I}=cosh\big((6-13I)x+14+55I\big)\,dx$$

Then:

$$\int\frac{cosh\big((6-13I)x+14+55I\big)}{sinh\big((6-13I)x+14+55I\big)}dx=\int\frac{1}{u}\,\frac{du}{6-13I}=\frac{1}{6-13I}\int\frac{1}{u}\,du=\left(\frac{1}{6}-\frac{13}{42}I\right)ln|u|+C$$

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$$= \left(\frac{1}{6} - \frac{13}{42}I\right) ln\left|sinh\big((6-13I)x+14+55I\big)\right| + C$$

3) $\int sech\big((5+I)x+7+3I\big)\, tanh\big((5+I)x+7+3I\big)\, dx = \left(-\frac{1}{5}+\frac{1}{30}I\right) sech\big((5+I)x+7+3I\big) + C$

4.1. Logarithmic forms of inverse neutrosophic hyperbolic functions

1) $sinh^{-1}\big((\alpha+\beta I)x+k+lI\big) = ln\left((\alpha+\beta I)x+k+lI+\sqrt{\big((\alpha+\beta I)x+k+lI\big)^2+1}\right)$

2) $cosh^{-1}\big((\alpha+\beta I)x+k+lI\big) = ln\left((\alpha+\beta I)x+k+lI+\sqrt{\big((\alpha+\beta I)x+k+lI\big)^2-1}\right)$

3) $tanh^{-1}\big((\alpha+\beta I)x+k+lI\big) = \frac{1}{2}\ ln\left(\frac{1+(\alpha+\beta I)x+k+lI}{1-(\alpha+\beta I)x+k+lI}\right)$

4) $coth^{-1}\big((\alpha+\beta I)x+k+lI\big) = \frac{1}{2}\ ln\left(\frac{(\alpha+\beta I)x+k+lI+1}{(\alpha+\beta I)x+k+lI-1}\right)$

5) $sech^{-1}\big((\alpha+\beta I)x+k+lI\big) = ln\left(\frac{1+\sqrt{1-\big((\alpha+\beta I)x+k+lI\big)^2}}{(\alpha+\beta I)x+k+lI}\right)$

6) $csch^{-1}\big((\alpha+\beta I)x+k+lI\big) = ln\left(\frac{1}{(\alpha+\beta I)x+k+lI}+\frac{\sqrt{1+\big((\alpha+\beta I)x+k+lI\big)^2}}{|(\alpha+\beta I)x+k+lI|}\right)$

Proof (1)

Let: $y = sinh^{-1}\big((\alpha+\beta I)x+k+lI\big)$, then:

$$(\alpha+\beta I)x+k+lI = sinhy$$

But:

$$sinhy = \frac{e^y-e^{-y}}{2}$$

$$(\alpha+\beta I)x+k+lI = \frac{e^y-e^{-y}}{2}$$

$$e^y-e^{-y} = 2\big((\alpha+\beta I)x+k+lI\big)$$

$$e^y-e^{-y}-2\big((\alpha+\beta I)x+k+lI\big) = 0$$

Multiplying by e^y, we get:

$$e^{2y}-2\big((\alpha+\beta I)x+k+lI\big)e^y-1 = 0$$

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solving by the quadratic formula, we find:

$$\Delta = 4\left((\alpha+\beta I)x+k+lI\right)^2+4$$

$$= 4\left[\left((\alpha+\beta I)x+k+lI\right)^2+1\right]$$

$$e^y = \frac{2\left((\alpha+\beta I)x+k+lI\right) \pm \sqrt{4\left[\left((\alpha+\beta I)x+k+lI\right)^2+1\right]}}{2}$$

$$e^y = (\alpha+\beta I)x+k+lI \pm 2\sqrt{\left((\alpha+\beta I)x+k+lI\right)^2+1}$$

Since $e^y > 0$, then:

$$e^y = (\alpha+\beta I)x+k+lI + 2\sqrt{\left((\alpha+\beta I)x+k+lI\right)^2+1}$$

$$y = ln\left((\alpha+\beta I)x+k+lI + 2\sqrt{\left((\alpha+\beta I)x+k+lI\right)^2+1}\right)$$

Hence:

$$sinh^{-1}\left((\alpha+\beta I)x+k+lI\right) = ln\left((\alpha+\beta I)x+k+lI + 2\sqrt{\left((\alpha+\beta I)x+k+lI\right)^2+1}\right)$$

Note:

Similarly, we can prove the rest of the formula.

4.2 Integrals of the inverse neutrosophic hyperbolic functions

1) $\int \frac{1}{\sqrt{(k+lI)^2+x^2}}dx = sinh^{-1}\left(\frac{x}{k+lI}\right)+C \quad , or: ln\left(x+\sqrt{(k+lI)^2+x^2}\right)+C$

2) $\int \frac{1}{\sqrt{x^2-(k+lI)^2}}dx = cosh^{-1}\left(\frac{x}{k+lI}\right)+C \quad , or: ln\left(x+\sqrt{x^2-(k+lI)^2}\right)+C \quad ; x > k+lI$

3) $\int \frac{1}{(k+lI)^2-x^2}dx = \begin{cases} \frac{1}{k+lI}tanh^{-1}\left(\frac{x}{k+lI}\right)+C & ; |x| < k+lI \\ \frac{1}{k+lI}coth^{-1}\left(\frac{x}{k+lI}\right)+C & ; |x| > k+lI \end{cases}$

$$or: \quad \frac{1}{2(a+bI)}ln\left|\frac{k+lI+x}{k+lI-x}\right|+C \quad ; |x| \neq k+lI$$

4) $\int \frac{1}{x\sqrt{(k+lI)^2-x^2}}dx = \frac{-1}{k+lI}sech^{-1}\left|\frac{x}{k+lI}\right|+C$

$$or: \quad \frac{-1}{k+lI}ln\left(\frac{k+lI+\sqrt{(k+lI)^2-x^2}}{|x|}\right)+C \quad ; 0 < |x| < k+lI$$

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5) $\int \frac{1}{x\sqrt{(k+lI)^2+x^2}}dx = \frac{-1}{k+lI}csch^{-1}\left|\frac{x}{k+lI}\right| + C$

$$or: \quad \frac{-1}{k+lI}ln\left(\frac{k+lI+\sqrt{(k+lI)^2+x^2}}{|x|}\right)+C \qquad ; x \neq 0$$

Where $k \neq 0, k \neq -l$, and:

$$\frac{1}{k+lI} = \frac{1}{k} - \frac{l}{k(k+l)}I$$

for all the previous rules.

Proof (1)

$$\int \frac{1}{\sqrt{(k+lI)^2+x^2}}dx = sinh^{-1}\left(\frac{x}{k+lI}\right)+C \quad , or: ln\left(x+\sqrt{(k+lI)^2+x^2}\right)+C$$

Methode1:

$$\frac{d}{dx}\left[sinh^{-1}\left(\frac{x}{k+lI}\right)+C\right] = \frac{\frac{1}{k+lI}}{\sqrt{1+\left(\frac{x}{k+lI}\right)^2}} = \frac{\frac{1}{k+lI}}{\sqrt{1+\frac{x^2}{(k+lI)^2}}}$$

$$= \frac{\frac{1}{k+lI}}{\sqrt{\frac{(k+lI)^2+x^2}{(k+lI)^2}}} = \frac{\frac{1}{k+lI}}{\frac{1}{k+lI}\sqrt{(k+lI)^2+x^2}}$$

$$= \frac{1}{\sqrt{(k+lI)^2+x^2}}$$

Methode2:

$$\frac{d}{dx}\left[ln\left(x+\sqrt{(k+lI)^2+x^2}\right)+C\right] = \frac{1+\frac{2x}{2\sqrt{(k+lI)^2+x^2}}}{x+\sqrt{(k+lI)^2+x^2}}$$

$$= \frac{\frac{2\sqrt{(k+lI)^2+x^2}+2x}{2\sqrt{(k+lI)^2+x^2}}}{x+\sqrt{(k+lI)^2+x^2}} = \frac{\sqrt{(k+lI)^2+x^2}+x}{\sqrt{(k+lI)^2+x^2}\left(x+\sqrt{(k+lI)^2+x^2}\right)}$$

$$= \frac{1}{\sqrt{(k+lI)^2+x^2}}$$

Methode3:

Let:

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$$y = sinh^{-1}\left(\frac{x}{k+lI}\right) \quad \Rightarrow \quad \frac{x}{k+lI} = sinhy$$

by differentiate implicitly with respect to x , we get:

$$\frac{1}{k+lI} = coshy\ \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{(k+lI)coshy} \qquad (*)$$

Since: $cosh^2y - sinh^2 = 1 \quad \Rightarrow \quad cosh^2y = 1 + sinh^2 \quad \Rightarrow \quad coshy = \sqrt{1+sinh^2}$

By substitution in $(*)$, we get:

$$\frac{dy}{dx} = \frac{1}{(k+lI)\sqrt{1+sinh^2}}$$

But: $\frac{x}{k+lI} = sinhy$, then:

$$\frac{dy}{dx} = \frac{1}{(k+lI)\sqrt{1+\left(\frac{x}{a+bI}\right)^2}} = \frac{1}{(k+lI)\sqrt{\frac{(a+bI)^2+x^2}{(a+bI)^2}}}$$

$$= \frac{1}{(k+lI)\frac{1}{(k+lI)}\sqrt{(a+bI)^2+x^2}}$$

$$= \frac{1}{\sqrt{(k+lI)^2+x^2}}$$

Example4.2.1

1) $\int \frac{1}{\sqrt{(7+I)^2+x^2}}dx = sinh^{-1}\left(\frac{x}{7+I}\right) + C$

$$= sinh^{-1}\left(\frac{1}{7} - \frac{1}{54}I\right)x + C \quad , or: ln\left(x + \sqrt{(7+I)^2+x^2}\right) + C$$

2) $\int \frac{1}{\sqrt{16x^2-(8+12I)^2}}dx = \int \frac{1}{4\sqrt{x^2-(2+3I)^2}}dx$

$$= \frac{1}{4}cosh^{-1}\left(\frac{x}{2+3I}\right) + C$$

$$= \frac{1}{4}cosh^{-1}\left(\frac{1}{2} - \frac{3}{10}I\right)x + C \quad , \quad or:\ \frac{1}{4}ln\left(x + \sqrt{x^2-(2+3I)^2}\right) + C$$

5. Conclusions

This artical is an expansion of the studies on neutrosophic derivatives and integrals that we previously wrote. Calculus is crucial to our world because it makes numerous mathematical operations possible, and which is why we introduced

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the idea of the neutrosophic hyperbolic functions and study the derivatives and integrals of the neutrosophic hyperbolic functions and inverse neutrosophic hyperbolic functions.

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