On Some Properties of Fuzzy Spectrums of Implicative BCI Algebras

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Abstract

An implicative BCI algebra is a non empty set X with a special element O and a binary operation * with many clear conditions.In this work, we study the topological space $spec_F(X)$ and present some properties especially the compactness and connection. Also, we prove that it is a Hausdorff space and regular.

Keywords: Implicative; BCI algebra; Fuzzy spectrums; topological space

Introduction and preliminaries

Algebraic fuzzy structures are very famous and diverse. In the classical algebra we find many structures such that implicative algebras and BCI/BCL algebras, with many interesting fuzzy substructures, especially fuzzy ideals and prime fuzzy ideals.

In this work, we study some topological properties of fuzzy prime spectrums in implicative BCI algebras, where we prove many interesting results concern the topological properties of the space Spec(X).

An implicative BCI algebra is a non empty set X with a special element O and a binary operation * with the following conditions:

1. $x * 0 = x$
2. $x * (x * y) = y * (y * x)$
3. $\big((x * y) * (x * z)\big) * (z * y) = 0$
4. $x * (y * x) = 0$

For all $x, y, z \in X$.

X is called bounded if there exists another special element 1 such that: $x * 1 = 0$; $\forall\, x \in X$.

We put $e(x) = 1 * x$, hence we get the following binary operation:

$$x \wedge y = x * (x * y);\ \forall\, x, y \in X.$$

Also, we have $(x \wedge y) * x = (x \wedge y) * y = 0$ and $x \wedge e(x) = 0$.

Definition [3]:

Let X be an implicative BCI algebra, and I is a nonempty subset of X, then:

[1] I is called ideal of X if $0 \in I$ and
$x * y \in I \text{ and } y \in I \text{ imply } x \in I\ ;\ \forall x, y \in X.$

[2] I is called prime ideal if:
$x \wedge y \in I \text{ and } y \in I \text{ imply } x \in I \text{ or } y \in I\ ;\ \forall x, y \in X.$

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Definition [5]:

Let A be a nonempty set, a fuzzy subset on A is a function μ from A to [0,1] . Also, if μ $,ɲ$ are two fuzzy subsets of A, hence μ ⊆ ⇔ $\mu(x) \leq ɲ(x) forall\ x \in X$.

A fuzzy point x_t is a fuzzy subset of A defined as follows:

$$x_t(y) = \begin{cases} t \ ; y = x \\ 0 \ ; other\ wise \end{cases}, \text{ for all } y \in X.$$

Definition:

Let X be an implicative algebra, and I is an ideal of X. The characteristic function of I is denoted by x_t, which is defined as follows:

$$x_t(x) = \begin{cases} 1 \ ; x \in I \\ 0 \ ; \ x \notin \ I \end{cases}, for\ all\ x \in X.$$

Definition [4]:

Let X be an implicative BCI algebra, and μ is a fuzzy subset of X.

[1] μ is called fuzzy ideal of X if:

$i. \mu(0) \geq \mu(x) \quad ; \quad \forall x \in X$.

$ii. \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \ \ \forall x, y \in X$.

μ is called prime if:

for any two fuzzy ideals θ, σ in X with $\theta . \sigma \subseteq \mu$, implies $\theta \subseteq \mu$ or $\sigma \subseteq \mu$.

Fuzzy Prime Spectrums of Implicative BCI – Algebras .

Let $spec_F(X)$ be the set of all fuzzy prime ideals of an implicative BCI algebra X, we define:

$D(ɲ) = \{\mu \in spec_F(X) \ ; \ ɲ\ \mu\}$, $V(ɲ) = \{\mu \in spec_F(X) \ ; \ ɲ \subseteq \mu\}$.

The set$\tau_X = \{D(ɲ) \ ;$ ɲ is prime $fuzzy\ ideal\ of\ X \}$ is a topology on $c_F(X)$, thus $(spec_F(X), \tau_X)$ is a topological space.

Theorem:

Let X be an implicative BCI bounded algebra, and let $\beta_1, \beta_2 \in]0,1], \beta = \min\{\beta_1, \beta_2\}$, then $D\left(x_{\beta_1}\right) \cap D\left(x_{\beta_2}\right) = D((x \wedge y)_\beta)$ for all $y \in X$.

Proof:

Let $\mu \in D\left(x_{\beta_1}\right) \cap D\left(x_{\beta_2}\right)$, then: $\mu \in D\left(x_{\beta_1}\right) and\ \mu \in D\left(x_{\beta_2}\right)$, thus:

$\mu(x) < \beta_1 \leq \mu(0) and\ \mu(y) < \beta_2\ \mu(y) \neq \mu(0)$, and since μ is a prime fuzzy ideal, we get $\mu(x) = \mu(y) = \mu(1)$, also,

$\mu(x \wedge y) \leq \max\{\mu(x), \mu(y)\} = \mu(1) = \min\{\mu(x), \mu(y)\} < \min\{\beta_1, \beta_2\} = \beta$, thus

$\mu \in D((x \wedge y)_\beta)$.

On the other hand, let $\mu \in D((x \wedge y)_\beta)$, this implies:

$\mu(x) \leq \mu(x \wedge y) < \beta \leq \beta_1 and\ \mu(y) \leq \mu(x \wedge y) < \beta \leq \beta_2$, and

$\mu \in D\left(x_{\beta_1}\right) and\ \mu \in D\left(x_{\beta_2}\right)$, thus $\mu \in D\left(x_{\beta_1}\right) \cap D\left(x_{\beta_2}\right)$.

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Let X_a be a bounded BCI implicative algebra, and $spec_F(X_a)$ be the fuzzy spectrum of X, with the following condition:

For all μ $\in spec_F(X_a)\alpha$ and μ ∈ [0,1[, then μ$(1) = \alpha$. We can prove the following topological properties of the space $spec_F(X_a)$.

Theorem:

$spec_F(X_a)$ is a Hausdorff space.

Proof:

Let μ, σ *be two elements of* $spec_F(X_a)$, *we prove that* $\mu_0 \neq \sigma_0$.

We assume that: $\mu_0 = \sigma_0$ then for any $\in X$, we find that:

If $x \in \mu_0 = \sigma_0$, then μ$(x) = \mu(0) = 1 = \sigma(0) = \sigma(x)$. Also, if

μ$(x) = \mu(1) = \alpha = \sigma(1) = \sigma(x)$, which is a contradiction with $\mu \neq \sigma$.

So that, $\mu_0 \neq \sigma_0$, and this means that there exists $x \in \mu_0$ with $x \notin \sigma_0$, thus

μ$(x) = \mu(0)$, and μ$\big(e(x)\big) \neq \mu(0)$, then $\mu\big(e(x)\big) = \mu(1)$ *and* $\sigma(x) = \sigma(1)$.

For $t_1 \in]$α$, 1]$, we find that $\big(e(x)\big)_{t_1}\big)\big(e(x)\big) = t_1 > \alpha = \mu\big(e(x)\big)$, thus $\mu \in D\big(\big(e(x)\big)_{t_1}\big)$.

Also for $t_2 \in]$α$, 1]$, we get $\big(x_{t_2}\big)(x) = t_2 > \alpha = \sigma(x)$, thus $\sigma \in D\big(x_{t_2}\big)$.

As well as, for $t = \min\{t_1, t_2\}$, we find $D\big(\big(e(x)\big)_{t_1}\big) \cap D\big(x_{t_2}\big) = D\big((e(x) \wedge x)_t\big) = D(0_t) = \emptyset$.

Theorem:

$spec_F(X_a)$ is regular space.

Proof:

Let V(ɲ) be a closed subset of $spec_F(X_a)$, let μ be a prime fuzzy ideal which is not contained in V(ɲ). Since $spec_F(X_a)$ is a Hausdorff space, then for any θ from μ V(ɲ) , there exists a neighbourhood W_θ of θ and a neighbourhood U_μ of such that: $U_\mu \cap W_\theta = \emptyset$.

So, we can get that the covering set $\{W_\theta\}_{\theta \in V(\eta)}$ of $V(\eta)$, is a partial finite covering $\{W_{\theta_1}, W_{\theta_2}, \dots, W_{\theta_n}\}$ of $V(\eta)$, hence $V(\eta) \subseteq W_{\theta_1} \cup W_{\theta_2} \cup \dots \cup W_{\theta_n}$ with $U_{i\mu} \cap W_{\vartheta i} = \emptyset$; $i = 1,2, \dots, n$.

We put $W = W_{\theta_1} \cup W_{\theta_2} \cup \dots \cup W_{\theta_n}$, $U = U_{1\mu} \cap W_{2\mu} \cap \dots \cap W_{n\mu}$.

Every open subset W in $spec_F(X_a)$ contains V(ɲ) , also, U is an open subset in $spec_F(X_a)$ contains μ . On the other hand, we have the same thing about $U \cap W$, this means that is $spec_F(X_a)$regular.

Theorem:

$spec_F(X_a)$ is normal.

Proof:

Let $V(\eta), V(\mu)$ be two disconnected and closed subsets of $c_F(X_a)$, since $spec_F(X_a)$ is regular, then for any $\theta \in V(\mu)$ there exists a neighbourhood W_ϑ of $V(\mu)$ which is compact. We have seen that the open covering $\{U_\theta\}_{\theta \in V(\mu)}$ of $V(\mu)$ is a partial finite one.

We put $W = W_{\theta_1} \cup W_{\theta_2} \cup \dots \cup W_{\theta_n}$, $U = U_{\theta_1} \cap U_{\theta_2} \cap \dots \cap U_{\theta_n}$.

We have U and W are open subsets of $c_F(X_a)$, and $V(\mu) \subseteq U$ و $V(\eta) \subseteq W$ and $W \cap U$ too.

Theorem:

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$spec_F(X_a)$ Is completely regular.

Proof:

Let $V(\mathcal{n})$ be a closed subset of $spec_F(X_a)$, and μ be a prime fuzzy ideal which is not in $(\mathcal{n})$, the set $\{\mu\}$ is closed in $spec_F(X_a)$, that is because it is T_1 space.

Also, $spec_F(X_a)$ is regular, hence there exists a continuous mapping $f: spec_F(X_a) \to [0,1]$ such that $f(V(\mathcal{n})) = \{1\}$ and $f(\{\mu\}) = \{0\}$.

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