On Some Results About the Asymptotic Behaviour for Some Non-Homogeneous and Non-Linear Differential Equations with A Laplacian $p \geq 1$

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Abstract

In this paper, we study the global existence and the asymptotic behaviour for the solutions of the following non-linear and non-homogeneous differential equation with order 3 and with a Laplacian $p \geq 1$

$$[|u''(t)|^{p-1}u''(t)]' + f\big(t, u(u)\big) = e(t)\,; p \geq 1 \qquad (1)$$

Where $t \to +\infty$. Also, we get the solutions of (1), which the type of the asymptotic behaviour of the global solutions is $at^2 + bt + c; t \to +\infty, a, b, c \in R\,; a \neq 0$.

Keywords: non-linear differential equation; asymptotic behavior; global existence.

Introduction

No one can deny the effective and distinctive role played by the asymptotic behavior of solutions of nonlinear differential equations in the fields of mechanics and in solving many scientific , electronic and physical issues , where many problems of limit values of Mathematical Physics with a Laplacian constant p≥1 lead to asymptotic integrals and thus to solutions that have an asymptotic behavior of a certain form when t→+∞, moreover, the asymptotic behavior of solutions of nonlinear differential equations is an important tool in many branches of mathematical analysis (especially dependent analysis) and Applied Mathematics . Notably, this constant is attributed to the French mathematician Pierre Laplace.

In this paper, we will consider one of the most important nonlinear and inhomogeneous differential equations of the third order with a Laplace constant p≥1, which does not contain derivatives of the form

:$u'(t)\,, u''(t)$:

$$u(t_0) = |u_1|\,;\ u'(t_0) = |u_2|\,; u''(t_0) = |u_3|^p$$

Where new theorems were presented supported by appropriate applied examples on non-linear and inhomogeneous differential equations using the integral permutations of Gronwall [5] and Behari [9], we thus obtained comprehensive solutions (continuous and extendable over the entire real axis) that have asymptotic behavior

$\mathrm{at}^2 + \mathrm{bt} + \mathrm{c}\ \mathrm{t} \to +\infty,\ a, b, c \in R\,; a \neq 0$

Main Results

Definition:

$$\text{if} \lim_{t\to\infty} \frac{f(t)}{g(t)} = 0, \text{then } f(t) = o(g(t))\ .$$

Definition:

We say that the function u(t) has the property (L_1) if:

Doi : https://doi.org/10.54216/GJMSA.030202
Received: October 25, 2022 Accepted: January 24, 2023

$u(t) = at^2 + bt + +c + o(t^2); t \to \infty ; a, b, c \in R$,

$p \geq 1 ; r > 0 ; t_0 = 1 ; p \geq r > 0.$

Theorem:

If the following conditions are true:

[1] The function f(t,u) is continuous on $D = \{(t, u): t \geq 1, u \in R\}$,
[2] The second derivative$f_{uu}(t, u)$ is existed and positive on D,
[3] The inequality $|f(t, u) - e(s)| \leq f_{uu}(t, 0). |u(t)|^r + e(s)$ is true on D.

$$\int_1^{\infty} t^{2R} f_{uu}(t, 0) dt < \infty \quad , \quad k = \int_1^{\infty} e(s)\, ds < \infty \quad \text{(iv)}$$

Then every global solution of (1) has the property (L_1).

Proof:

According to [6], the equation (1) has a unique solution $u(t) \in C^2([1, \infty))$

With the following conditions:
$u(1) = |u_1|; u'(1) = |u_2| ; u''(1) = |u_3|^P$.

Thus:

$$|u''(t)|^{p-1} u''(t) = c_3 - \int_1^t [f(s, u(s)) - e(s)] ds \; ; \; c_3 = |u_3|^P$$

$$(u''(t))^p \leq |u''(t)|^p \leq c_3 + \int_1^t |f(s, u(s)) - e(s)| ds \quad (2)$$

Put: $Q(t) = c_3 + \int_1^t |f(s, u(s)) - e(s)| ds \quad (3)$

Hence, $(u''(t))^p \leq Q(t) \quad (4)$

Hence, $u''(t) \leq [Q(t)]^{\frac{1}{p}} \quad (5)$

$$u'(t) \leq c_2 + \int_1^t [Q(s)]^{\frac{1}{p}} ds \leq c_2 + (t-1) [Q(t)]^{\frac{1}{p}}$$

$$\leq t[c_2 + (Q(t))^{\frac{1}{p}} \; ; t \in [1, \infty) \; ; \; c_2 = |u_2|$$

Thus, $u'(t) \leq t[c_2 + \left(Q(t)\right)^{\frac{1}{p}}\Big] \quad c_2 = |u_2| \quad (6)$

$$u(t) \leq t^2[c_1 + \left(Q(t)\right)^{\frac{1}{p}}\Big] \; ; \; c_1 = |u_1| \quad (7)$$

$\therefore [\frac{|u'(t)|}{t}]^p \leq [c_2 + \left(Q(t)\right)^{\frac{1}{p}}\Big]^p \quad (8)$

$$[\frac{|u(t)}{t^2}]^p \leq [c_1 + \left(Q(t)\right)^{\frac{1}{p}}\Big]^p \quad (9)$$

$\therefore (a + b)^p \leq 2^{p-1}(a^p + b^p) ; \; a, b \geq 0$ since,

$$[\frac{|u(t)}{t^2}]^p \leq 2^{p-1}[c_1^p + Q(t)] = 2^{p-1} c_1^p + 2^{p-1} Q(t)$$

$$\leq 2^{p-1} c_1^p + 2^{p-1} [c_3 + \int_1^t |f(s, u(s)) - e(s)| ds]$$

Doi : https://doi.org/10.54216/GJMSA.030202
Received: October 25, 2022 Accepted: January 24, 2023

$$[\frac{|u(t)}{t^2}]^p \le e_1 + 2^{p-1}\int_1^t f_{uu}(s,0).|u(s)|^r\, ds$$

$$= e_1 + 2^{p-1}\int_1^t s^{2r}.f_{uu}(s,0).[\frac{|u(s)|}{s^2}]^r ds \qquad (10)$$

$.e_1 = 2^{p-1}(c_1^p + c_3 + K)$

$$[\frac{|u(t)}{t^2}]^p \le e_1.\exp\left(\int_1^t s^{2r}.f_{uu}(s,0)ds\right) \qquad (11)$$

$$A = e_1.\exp\left(2^{p-1}\int_1^\infty s^{2r}.f_{uu}(s,0)ds\right) \qquad (12)$$

Thus,

$$[\frac{|u(t)|}{t^2}]^p \le A \quad (13) \;\rightarrow\; \frac{|u(t)|}{t^2} \le [A]^{\frac{1}{p}}$$

$t \ge 1$, we get: for

$$\frac{|u(t)|}{t^2} < +\infty$$

$$\int_1^t \left|f(s,u(s)) - e(s)\right| ds \le \int_1^t f_{uu}(s,0).|u(s)|^r\, ds + \int_1^t e(s)ds$$

$$\le k + \int_1^t s^{2r}.f_{uu}(s,0).[\frac{|u(s)|}{s^2}]^r ds$$

$$\le k + (A)^{\frac{r}{p}}\int_1^t s^{2r}.f_{uu}(s,0)ds$$

$$\le k + (A)^{\frac{r}{p}}\int_1^t s^{2r}.f_{uu}(s,0)ds < +\infty$$

This means that

$\lim_{t\to\infty}\int_1^t [f(s,u(s)) - e(s)]ds$, and there exists $a \in R$ such that $\lim_{t\to\infty} u''(t) = a$.

Now, we can write:

$\lim_{t\to\infty}\frac{u(t)}{t^2} = \frac{u_1+\int_1^t u'(s)ds}{t^2} = \lim_{t\to\infty} u''(t) = a$, for any constants a,b

$$\lim_{t\to\infty}\left\{\frac{u(t) - (at^2 + bt + c)}{t^2}\right\} = 0$$

Example:

$$[|u''(t)|^{p-1}u''(t)]' + \frac{u}{t^{2+2r}}.\sin u = \frac{1}{1+t^2} \qquad (*)$$

$$f(t,u) = \frac{u}{t^{2+2r}}.\sin u \quad , \quad e(t) = \frac{1}{1+t^2}$$

Doi : https://doi.org/10.54216/GJMSA.030202
Received: October 25, 2022 Accepted: January 24, 2023

$$f_u(t,u) = \frac{.\sin u}{t^{2+2r}} + \frac{u}{t^{2+2r}} .\cos u$$

$$f_{uu}(t,0) = \frac{2}{t^{2+2r}}$$

$$|f(t,u)| \le \frac{|u|}{t^{2+2r}} < 2.\frac{|u|^2}{t^{2+2r}} = f_{uu}(t,0).|u|^2$$

$$\int_1^{\infty} t^{2r} f_{uu}(t,0)dt = \int_1^{\infty} t^{2r}.\frac{2}{t^{2+2r}}dt = 2 < \infty$$

$\int_1^{\infty} e(t)dt = \int_1^{\infty} \frac{1}{1+t^2}dt = \frac{\pi}{4} < \infty$, so it has the property (L_1).

Theorem:

If the following conditions are true:

[1] the function $f(t,u)$ is continuous on $D = \{(t,u) : \ge 1, u \in R\}$.

[2] there exists a continuous non-negative functions $g_1, g_2, g_3 : R^+ \to R^+$such that
$\int_1^{\infty} g_1(t).g_2(t).g_3(t) < +\infty$, $\int_1^{\infty} e(s)ds < +\infty$.

[3] There exists a function g(u) which is continuous for $u \ge 0$ and positive and non-descending for $u > 0$. If we put $G(x) = \int_1^{x} \frac{du}{g(u^{\frac{r}{p}})} = \infty$,then $G(+\infty) = \int_1^{+\infty} \frac{du}{g(u^{\frac{r}{p}})} = \frac{p}{r}\int_1^{+\infty} \frac{s^{\frac{p}{r}-1}}{g(s)}ds = +\infty$,

$|f(t,u) - e(s)| \le g_1(t).g_2(t).g_3(t).g\left(\left[\frac{|u(t)|}{t^2}\right]^r\right) + e(s)$ (iv),

Then every global solution to (1) has the property (L_1).

Proof:

$$u(1) = |u_1|;\ u'(1) = |u_2|\ ;\ u''(1) = |u_3|^p$$

$$|u''(t)|^{p-1}u''(t) = c_3 - \int_1^t [f(s,u(s)) - e(s)]ds\ ;\ c_3 = |u_3|^p$$

$$(u''(t))^p \le |u''(t)|^p \le c_3 + \int_1^t |f(s,u(s)) - e(s)|ds \quad (2)$$

)3 $(Q(t) = c_3 + \int_1^t |f(s,u(s)) - e(s)|ds$

$$(u''(t))^p \le Q(t) \quad (4)$$

$$u''(t) \le [Q(t)]^{\frac{1}{p}} \quad (5)$$

$$u'(t) \le c_2 + \int_1^t [Q(t)]^{\frac{1}{p}}ds \le c_2 + (t-1)[Q(t)]^{\frac{1}{p}}$$

$$\le t\left[c_2 + (Q(t))^{\frac{1}{p}}\right]\ ; t \in [1,\infty)\ ;\ c_2 = |u_2|$$

$$u'(t) \le t\left[c_2 + (Q(t))^{\frac{1}{p}}\right]\ ;\ c_2 = |u_2| \quad (6)$$

$$u(t) \le t^2\left[c_1 + (Q(t))^{\frac{1}{p}}\right]\ ;\ c_1 = |u_1| \quad (7)$$

Doi : https://doi.org/10.54216/GJMSA.030202
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$$[\frac{|u'(t)|}{t}]^P \le \left[c_2 + (Q(t))^{\frac{1}{P}}\right]^P \qquad (8)$$

$$[\frac{|u(t)|}{t^2}]^P \le \left[c_1 + (Q(t))^{\frac{1}{P}}\right]^P \qquad (9)$$

$(a+b)^p \le 2^{p-1}(a^p + b^p)$; $a, b \ge 0$

$$[\frac{|u(t)|}{t^2}]^P \le 2^{p-1}[c_1^p + Q(t)] = 2^{p-1}c_1^p + 2^{p-1}Q(t)$$

$$\le 2^{p-1}c_1^p + 2^{p-1}[c_3 + \int_1^t |f(s,u(s)) - e(s)|ds]$$

$$[\frac{|u(t)|}{t^2}]^P \le e_1 + 2^{p-1}\int_1^t g_1(s).g_2(s).g_3(s).g\left(\left[\frac{|u(s)|}{s^2}\right]^r\right) ds \qquad (14)$$

$$e_1 = 2^{p-1}(c_1^p + c_3 + k)$$

$$A(t) = e_1 + 2^{p-1}\int_1^t g_1(s).g_2(s).g_3(s).g\left(\left[\frac{|u(s)|}{s^2}\right]^r\right) ds$$

$$[\frac{|u(t)|}{t^2}]^P \le A(t) \qquad (16)$$

$)17 ([\frac{|u(t)|}{t^2}]^r \le [A(t)]^{\frac{r}{p}}$

$$g([\frac{|u(t)|}{t^2}]^r) \le g\left([A(t)]^{\frac{r}{p}}\right) \qquad (18)$$

$$A(t) = e_1 + 2^{p-1}\int_1^t g_1(s).g_2(s).g_3(s).g\left([A(t)]^{\frac{r}{p}}\right) ds \qquad (19)$$

$$A(t) \le G^{-1}[G(e_1) + 2^{p-1}\int_1^t g_1(s).g_2(s).g_3(s)\, ds \qquad (20)$$

$$G(w) = \int_1^w \frac{ds}{g\left(s^{\frac{r}{p}}\right)}$$

$$G^{-1}(w[\frac{|u(t)|}{t^2}]^P \le G^{-1}(k) \quad \rightarrow \quad \frac{|u(t)|}{t^2} \le [G^{-1}(k)]^{\frac{1}{p}}$$

$$\int_1^t |f(s,u(s)) - e(s)|ds \le \int_1^t g_1(s).g_2(s).g_3(s).g\left(\left[\frac{|u(s)|}{s^2}\right]^r\right) ds + \int_1^t e(s)ds$$

$$\le e_1 + 2^{p-1}\int_1^t g_1(s).g_2(s).g_3(s).g\left([A(s)]^{\frac{r}{p}}\right) ds$$

$$= A(t) \le G^{-1}(k) < +\infty \;; \quad t \ge 1$$

$a \in R$ $thus, \lim_{t\to\infty} u''(t) = a$

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$\lim_{t\to\infty}\frac{u(t)}{t^2}=\frac{u_1+\int_1^t u'(s)ds}{t^2}=\lim_{t\to\infty}u''(t)=a$, for any constants a,b

$$\lim_{t\to\infty}\left\{\frac{u(t)-(at^2+bt+c)}{t^2}\right\}=0$$

Thus the proof is complete.

Example:

$$[|u''(t)|^{p-1}u''(t)]'-\frac{2}{t^2}.(\frac{u}{t^2})^{p-r}.\ln\left(1+\left(\frac{u}{t^2}\right)^r\right)=e^{-t}\ ;\ r>0\,;p\geq 1\ ;t\geq 1\,(**)\,\Big)$$

$$f(t,u)=-\frac{2}{t^2}.\left(\frac{u}{t^2}\right)^{p-r}.\ln(1+\left(\frac{u}{t^2}\right)^r)\qquad ,\ e(t)=e^{-t}$$

$$|f(t,u)|\leq -\frac{2}{t^2}.|u|^{p-r}.\ln(1+|u|^r)$$

$$g_1(t)=\frac{2}{t^2}\quad ,\quad g_2(t)=g_3(t)=1\quad ,\ g(u)=u^{\frac{p}{r}-1}.\ln(1+u)$$

$$\int_1^\infty g_1(t).g_2(t).g_3(t)dt=\int_1^\infty\frac{2}{t^2}dt=2<+\infty$$

$$\int_1^\infty e(t)dt=\int_1^\infty e^{-1}dt=\frac{1}{e}<+\infty$$

$$G(+\infty)=\int_1^{+\infty}\frac{du}{g(u^{\frac{r}{p}})}=\frac{p}{r}\int_1^{+\infty}\frac{s^{\frac{p}{r}-1}}{g(s)}ds$$

$$=\frac{p}{r}\int_1^{+\infty}\frac{s^{\frac{p}{r}-1}}{s^{\frac{p}{r}-1}.\ln(1+s)}ds>\frac{p}{r}\int_1^{+\infty}\frac{1}{(1+s)}ds=+\infty$$

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Received: October 25, 2022 Accepted: January 24, 2023

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Doi : https://doi.org/10.54216/GJMSA.030202
Received: October 25, 2022 Accepted: January 24, 2023