



On Some Properties of Dedekind Lattices

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Abstract

The objective of this paper is to study the Dedekind lattices, where we prove if L is a R -lattice and domain, then L will be a Dedekind domain if and only if every non zero maximal element is invertible.

On the other hand, we prove that if L is a domain and R -lattices, then L is a dedehind domain if and only if L is a UFD lattices and every non zero prime element is maximal.

Keywords: Lattice; UFD; Principal Ideal; Domain; PG-Lattice

Introduction.

In the theory of ideals in commutative rings, the multiplicative lattices were used and defined [3].

Many special elements in multiplicative lattices were studied in [4,7] such as principal elements, M -lattices and modules.

Dedekind lattices were defined in [1] with lattices that are UFD_s (unique factorization domain).

In this work, we will discuss some properties and characterizations of I -lattices and UFD-lattices, where we will present some novel result in terms of theorems.

Main discussion.

Definition.

A multiplicative lattice is a complete lattice with a binary operation $(.)$ which is associative, commutative and distributive.

Also, the maximal element I is the identity.

Definition.

An element $a \in L$ is called meet principal(join principal) if $((b:a) \wedge c) = b \wedge ca$, $((b \vee c:a) = (b:a) \vee c$; $\forall a, b \in L$.

The element a is called principal if a meet/join principal.

Definition.

An element a is called principal (weak meet principal/weak join principal) if $ba:a = b \vee (0:a)/(b:a) a = b \wedge a$; $\forall b \in L$.

The element a is called weak principal if a is weak meet/join principal.

For concepts such as compact element, we define $(a:b) = \bigvee \{x \in L; xb \leq a\}$.

An element m is called maximal if $a \geq m$ implies $a = 1$.

The multiplicative L is called local if it has only one maximal element.

Definition.

An element $p \in L$ is called prime if $p \neq 1$ and $xy \leq p$ with $x \not\leq p$ implies $y \leq p$ for all $x, y \in L$.

If L is a K -lattice with a compact maximal element 1 , we call $x \in L$ a non zero divisor if $(0 : x) = 0$.

A multiplicative lattice L is called domain if it has no zero divisor.

Definition.

Let $x \in L$, we say that x is invertible if x principal and not a zero divisor.

x is called regular if there exists a principal $y \in L$ such that $(0 : y) = 0$ and $y \leq x$.

Definition.

A multiplicative lattice L is called π -lattice if there exists a set $S \subseteq L$ such that S generates L with respect to the join operation and for every $x \in S$, we get x as a finite product of prime elements.

For the definition of UFD-lattice see [1].

Lemma.

Let L a multiplicative lattice with a compact maximal element 1 , consider $x \in L$ with $(0 : x) = 0$, then every factor of x will be principal.

Proof.

Let $x = md$; $m, d \in L$, m is a weak join principal.

Let $a, b \in L$, then we have:

$(am \vee b : m)m \leq am \vee b$, hence:

$(am \vee b : m)x = (am \vee b : m)md \leq (am \vee b)d = amd \vee bd = ax \vee bd$, this means that:

$(am \vee b : m) = (am \vee b : m)x : x \leq (ax \vee bd) : x = a \vee (bd : x) = a \vee (bd : md) = a \vee b : m = a \vee 0 : m \vee b =$

$am : m \vee b : m \leq (am \vee b)m$, so that $(am \vee b) : m = a \vee b : m$.

Now, we will prove that m is a meet principal:

For $a, b \in L$, we have:

$(a \wedge mb)d \leq ad \wedge xb = ((ad : x) \wedge b)x = ((ad : x) \wedge b)md$, thus:

$$a \wedge mb \leq ((ad : x) \wedge b)m = ((ad : md) \wedge b)m = ((a : m) \wedge b)m \leq a \wedge mb$$

Theorem .

Let L a PG -multiplicative lattice with a compact maximal element 1 , then L is a UFD-lattice if and only if L is a domain and every principal element is a product of primes.

Proof.

The proof is clear.

Theorem.

Let L be a multiplicative lattice with a maximal compact element 1 , if L is a UFD-lattice and each prime element which is not equal to zero is maximal, then every element in L will be principal.

Proof.

Let p a prime with $p \neq 0$, and let $a \leq p$ is principal with $a \neq 0$. Since L is a UFD then $a = p_1 \dots p_n$; p_i are prime and principal, we have $(0 : p_i) = 0$ for all $1 \leq i \leq n$.

Also, p_i are maximal and $a = p_1 \dots p_n \leq p$, then there exists i such that $p_i \leq p$, thus $p_i = p$ and p is principal.

Definition.'

A multiplicative lattice L is called Dedekind lattice if every element $x \neq 1$ is a finite product of primes.

If L has no zero divisors, then L is called a Dedekind domain.

Lemma.

Let L a multiplicative lattice with maximal compact element 1 . Assume that L is domain and every prime of L which is not zero divisor and weak joint principal.

Let $x \in L$ be written as a finite product of primes in L , then this expression is unique.

Proof.

Assume that $x = p_1 \dots p_m = p'_1 \dots p'_n$ such that p_i, p'_i are primes and not zeros.

Suppose, that p_1 is a minimal between $p_1 \dots p_m$.

Since $p'_1 \dots p'_n \leq p_1$, then there exists $p'_i \in \{p'_1 \dots p'_n\}$ such that $p'_i \leq p_1$, say that $i = 1$, so you will get $p_i \leq p'_1$, hence $p_i = p'_1$ and then $p_1 p_2 \dots p_m = p'_1 p'_2 \dots p'_n$.

Now, since p_1 is a weak principal joint, we get $p_2 \dots p_m = p'_2 \dots p'_n$, we continue the same to get that $n = m$ with $p_i = p_j$.

Theorem.

Let L a PG -multiplicative lattice with maximal compact element 1 . If L is a dedekind domain, then every non zero prime and principal element $x \in L$ is maximal.

Proof.

Let L a Dedekind domain, assume that p is non zero/prime and principal element of L .

Suppose that p is not maximal, then there exists a maximal element m such that $p < m$.

Under our assumption, there exists a principal element a such that $a \not\leq p$ and $a \leq m$.

By the assumption that L is Dedekind domain, we can see $p \vee a = p_1 \dots p_k, p \vee a^2 = q_1 \dots q_n$, where p_i, q_i are non zero prime elements.

$\bar{a} = a \vee p \in [1, p]$, $\overline{a^2} \in [1, p]$, also $\bar{a} = \overline{p_1} \circ \dots \circ \overline{p_k}$, $\overline{a^2} = \overline{q_1} \circ \dots \circ \overline{q_n}$, where $\overline{p_i} = p_i \vee p = p_i$, $\overline{q_j} = q_j \vee p = q_j$; $1 \leq i \leq k, 1 \leq j \leq n$.

This implies that $\overline{p_1^2} \circ \dots \circ \overline{p_k^2} = \overline{q_1} \circ \dots \circ \overline{q_n}$, so that $n = 2k$.

On the other hand, we can renumber q_j with $j = 1, \dots, k$ to get $q_{2i-1} = q_{2i} = p_j$.

$(p \vee a)^2 = p \vee a^2$, $p \leq (p \vee a)^2$ with $a \not\leq p$, hence $1 = (p^2 \vee a(a \vee p)) : p$, since p is principal, we get $1 = (p^2 \vee a(a \vee p)) : p = p \vee (a(a \vee p) : p)$, thus $1 = p \vee (a(a \vee p) : p) = p \vee a \vee (a^2 : p)$, but $a^2 : p \leq a$, then $p \vee a = 1$, thus $m = 1$ which is a contradiction.

Theorem.

Let L be a PG-multiplicative lattice with a maximal compact element 1. If L is a domain in which every non zero principal element is a product of finite number of maximal elements, then every $x \in L$ will be principal.

Proof.

Let p be a prime/non zero element of L , then there exists a principal non zero element $a \leq p$.

$a = m_1 \dots m_n$ where m_i are maximal.

$m_1 \dots m_n \leq p$, so that there exists i such that $m_i \leq p$ hence $m_i = p$ and $a = \prod_{i \neq j} m_j$.

This implies that p is principal and the proof is complete.

Theorem.

Let L be a PG-multiplicative lattice with a maximal compact element 1. If L is a Dedekind domain, then every non zero prime element of L is maximal.

Proof.

Assume that L is Dedekind domain, with p as non zero/prime element, there exists a non zero principal element $a \leq p$.

We have $a = p_1 \dots p_n$, p_i are non zero primes, so that a has a principal factor, this means that each p_i ; $1 \leq i \leq n$ is principal and then maximal.

$a = p_1 \dots p_n \leq p$, so that exists i such that $p_i \leq p$ and then $p_i = p$, thus p is maximal.

Theorem.

Let L be a PG-lattice with compact maximal element 1, and L is Dedekind domain, then it is principal element lattice.

a principal element lattice if and only if L has the property of and every maximal element is a weak meet principal.

The proof is easy.

Theorem.

If L be a PG-lattice with compact maximal compact element 1. And L is Dedekind domain, then it is principal element lattice.

Remark.

If L is R-modular domain, then L is Dedekind domain if and only if every finitely generated element and every non zero maximal element is invertible.

References

- [1] Anderson, D. D., Abstract commutative ideal theory without chain condition. Algebra Universalis 6, (1976), 131-145.
- [2] Burton, R. G., Fractional elements in multiplicative Lattices. Pacific Journal of Mathematics, vol. 56, No. 1, (1975), 35-49.
- [3] Dilworth, R. P., Abstract commutative ideal theory. Pacific Journal of Mathematics, 12, (1962), 487 - 498.
- [4] Janowitz, M. F., Principal Multiplicative Lattices. Pacific Journal of Mathematics, vol. 33, (1970), 653 - 656.
- [5] Johnson, E. W. and Johnson, J. A., P-Lattices as ideal Lattices and submodule Lattices., commen. Math. Univ. San. Pauli., 38 (1989), 21-27.
- [6] McCarthy, P. J. Principal elements of Lattice of ideals, American Mathematical Society, Vol. 30, No. 1, (1971), 43 - 45.
- [7] Nakkar, H.M. Localization in multiplicative Lattice modules, (In Russia), Mat. Issledovanija, Kishinev, 2(33), (1974), 88-108.
- [8] Nakkar, H. M. AL-Khouja, E. A., A Locally Noetherian Lattice Modules. Tamkang Journal of Mathematics. Volume 22, Number 3, Autumn (1991), 253- 257.
- [9]. Merkepçi, M.; Sarkis, M. An Application of Pythagorean Circles in Cryptography and Some Ideas for Future Non Classical Systems. Galoitica Journal of Mathematical Structures and Applications 2022, 2, 28-30, doi: 10.54216/GJMSA.020205.

- [10] Merkepci, M., and Ali, R., " A Study of Some Neutrosophic Algebraic Games and Their Winning Strategies", *Galoitica Journal Of Mathematical Structure And Applications*, Vol.2, 2022.