



An Introduction to Weak Fuzzy Complex Numbers

Ahmed Hatip

Gaziantep University, Department of Mathematics, Gaziantep, Turkey

Email: Kollnar5@gmail.com

Abstract

The objective of this paper is to define for the first time the concept of weak fuzzy complex numbers as a novel generalization of real numbers. Also, it presents some of their elementary properties and algorithms for solving some related algebraic equations.

Keywords: weak fuzzy complex number; weak fuzzy algebraic equation; split-complex numbers.

Introduction

In the history of mathematics, many generalizations of real number were introduced with deep insight in geometry.

The main concept was the field of complex number $\mathcal{C} = \{x + iy; i^2 = -1, x, y \in \mathbf{R}\}$. In addition, we find concepts such as:

Dual numbers $\mathcal{D} = \{x + yj; j^2 = 0, x, y \in \mathbf{R}\}$, [1]

Split- complex numbers $\mathcal{S} = \{x + yJ; J^2 = 1, x, y \in \mathbf{R}\}$, [2]

Neutrosophic numbers $\mathcal{N} = \{x + yI; I^2 = I, x, y \in \mathbf{R}\}$, [3,6, 8-10]

In the literature, we find the concept of fuzzy set [4-5], which concerns the probability of truth and falsity.

If \mathbf{M} is a non empty subset and $f: \mathbf{m} \rightarrow [0, 1]$, the fuzzy subset is defined as a duplet (\mathbf{M}, f) .

This approach motivated us to use a fuzzy operator (J) to extend the real field (\mathbf{R}) to a similar ring of the split-complex ring \mathcal{S} .

A fuzzy weak complex operator (J) will be defined to be compatible with the soul of fuzzy logic, i.e. $J^2 \in]0, 1[$, where $J \notin \mathbf{R}$.

From this point of view, we use the previous approach to build the ring of weak fuzzy complex numbers.

Main discussion.

Definition.

Let J be a weak fuzzy complex operator, we define the set of weak fuzzy complex numbers as follows:

$$F_j = \{a + bj; a, b \in \mathbf{R}, J^2 \in]0, 1[\}$$

For example $F_{\frac{1}{3}} = \{a + bJ ; a, b \in R, J^2 = \frac{1}{3}\}$

Remark.

F_J contains the real field R .

Definition.

Let F_J be a weak fuzzy complex numbers set, with $J^2 = t \in]0, 1[$, the operations on F_J are defined as follows:

Addition: $(a + bJ) + (c + dJ) = (a + c) + (b + d)J$.

Multiplication $(a + bJ) \cdot (c + dJ) = ac + adJ + bcJ + bdJ^2 = (ac + bdt) + (ad + bc)J$

Remark.

$(F_J, +, \cdot)$ Is a commutative ring.

Definition.

Let $x = a + bJ \in F_J$, we define the conjugate of x as follows:

$$\bar{x} = a - bJ$$

The norm of x is define as follows:

$$\|x\| = \sqrt{|x \cdot \bar{x}|} = \sqrt{|a^2 - b^2t|}$$

Example.

Take the ring $(F_{\frac{1}{2}}, +, \cdot)$, $x = 1 + 2J$, we have:

$$\bar{x} = 1 - 2J, \|x\| = \sqrt{|1^2 - 2^2(\frac{1}{2})|} = 1$$

Theorem.

Let $(F_J, +, \cdot)$ be the ring of weak fuzzy complex numbers with $J^2 = t \in]0, 1[$. Let $x = a + bJ, y = c + dJ$ be two arbitrary elements of F_J , then:

1. $\overline{(x + y)} = \bar{x} + \bar{y}, \overline{(x - y)} = \bar{x} - \bar{y}$
2. $\overline{\bar{x} \cdot \bar{y}} = x \cdot y$
3. $\|x \cdot y\| = \|x\| \cdot \|y\|$
4. x is invertible if and only if $t \neq \frac{a^2}{b^2}$, and $\frac{1}{x} = x^{-1} + J \frac{-b}{a^2 - tb^2}$.

Proof:

1. $\overline{(x + y)} = [(a + c) - (b + d)J] = (a - bJ) + (c - dJ) = \bar{x} + \bar{y}$
 $\overline{(x - y)} = [(a - c) - (b - d)J] = (a - bJ) - (c - dJ) = \bar{x} - \bar{y}$
2. $\overline{\bar{x} \cdot \bar{y}} = (ac + bdt) - (ad + bc)J$
 $\overline{\bar{x} \cdot \bar{y}} = (a - bJ) \cdot (c - dJ) = (ac + bdt) + J(-bc - ad) = \overline{\bar{x} \cdot \bar{y}}$
3. $\|x \cdot y\| = \sqrt{|(ac + bdt)^2 - (ad + bc)^2t|}$
 $\|x\| \cdot \|y\| = \sqrt{|a^2 - b^2t|} \cdot \sqrt{|c^2 - d^2t|} = \sqrt{|a^2c^2 - a^2d^2t - b^2c^2t + b^2d^2t^2|}$

On other hand, we have:

$$\|x \cdot y\| = \sqrt{|a^2c^2 + b^2d^2t^2 + 2acbd t - a^2d^2t - b^2c^2t - 2abcdt|} = \sqrt{|a^2c^2 - a^2d^2t - b^2c^2t + b^2d^2t^2|}$$

$$= \|x\| \cdot \|y\|$$

4. $\frac{1}{x} = \frac{1}{a + bJ} = \frac{a - bJ}{(a + bJ)(a - bJ)} = \frac{a}{a^2 - tb^2} + J \frac{-b}{a^2 - tb^2}$

So that x is invertible if and only if $a^2 - tb^2 \neq 0$, hence $t \neq \frac{a^2}{b^2}$.

example.

For F_J with $J^2 = t = \frac{1}{3}$, and $x = 3 + 3J$, we have:

$$\frac{1}{x} = \frac{1}{3 + 3J} = \frac{3}{3^2 - (\frac{1}{3})3^2} + J \frac{-3}{3^2 - (\frac{1}{3})3^2} = \frac{3}{6} + J \frac{-3}{6} = \frac{1}{2} - \frac{1}{2}J$$

Definition.

Let F_J be the weak fuzzy complex numbers with $J^2 = t$.

Let $x = a + bJ, y = c + dJ \in F_J$, then y is called a square root of x if $y^2 = x$.

Which implies that:

$$\begin{cases} c^2 + d^2t = a \dots (1) \\ 2cd = b \dots (2) \\ |c^2 - d^2t| = \|x\| = \sqrt{|a^2 - b^2t|} \dots (3) \end{cases}$$

Example.

For $J^2 = \frac{1}{2}$, we consider $x = 2 + 2J, y = c + dJ$ be a square root of x , hence:

$$\begin{cases} c^2 + \frac{1}{2}t = 2 \dots (1) \\ 2cd = 2 \dots (2) \\ |c^2 - d^2t| = \sqrt{2} \dots (3) \end{cases}$$

If $c^2 - d^2t = \sqrt{2}$, we get $2c^2 = 2 + \sqrt{2}$, thus $c = \mp \sqrt{\frac{2+\sqrt{2}}{2}}$.

From equation (2), we get $d = \sqrt{\frac{2}{2+\sqrt{2}}}$, thus:

$$y = \sqrt{\frac{2+\sqrt{2}}{2}} + J\sqrt{\frac{2}{2+\sqrt{2}}} \text{ or } y = -\sqrt{\frac{2+\sqrt{2}}{2}} - J\sqrt{\frac{2}{2+\sqrt{2}}}$$

On the other hand, if $c^2 - \frac{1}{2}d^2 = -\sqrt{2}$, then:

$$c = \mp \sqrt{\frac{2-\sqrt{2}}{2}}, d = \sqrt{\frac{2}{2-\sqrt{2}}}$$

which implies $y = \sqrt{\frac{2-\sqrt{2}}{2}} + J\sqrt{\frac{2}{2-\sqrt{2}}}$ or $y = -\sqrt{\frac{2-\sqrt{2}}{2}} - J\sqrt{\frac{2}{2-\sqrt{2}}}$.

Definition.

Let $A = a_1 + a_2J, X = x_1 + x_2J, B = b_1 + b_2J \in F_J$, the linear weak fuzzy equation is defined as follows:

$A.X + B = 0$, with X as the variable.

Theorem.

Let $AX + B = 0$ be a linear weak fuzzy equation, then:

1. If $a_1^2 - a_2^2t \neq 0$, then it is solvable unqudy.
2. If $a_1^2 - a_2^2t = b_1 = b_2 = 0$, then it has infinite solutions.
3. If $a_1^2 - a_2^2t = 0$ or $b_1 \neq 0$ or $b_2 \neq 0$, then it has no solutions.

Proof.

The equation $AX + B = 0$ is equivalent to:

$$(a_1 + a_2J)(x_1 + x_2J) + (b_1 + b_2J) = 0, \text{ hence:}$$

$$\begin{cases} a_1x_1 + a_2x_2t + b_1 = 0 \\ a_1x_2 + a_2x_1 + b_2 = 0 \end{cases}$$

Which is equivalent to:

$$\begin{cases} a_1x_1 + a_2tx_2 = -b_1 \dots (1) \\ a_1x_2 + a_2x_1 = -b_2 \dots (2) \end{cases}$$

by using Cramer's method, we get:

$$\begin{vmatrix} a_1 & a_2t \\ a_2 & a_1 \end{vmatrix} = a_1^2 - a_2^2t$$

According to cramer's method, we get the proof of 1,2 and 3.

Example.

Let $J^2 = \frac{1}{2}$, $A = 1 + J$, $B = 2 + 0J = 2$, the linear equation $AX + B = 0 \Leftrightarrow \begin{cases} x_1 + \frac{1}{2}x_2 = 2 \dots (1) \\ x_2 + x_1 = 0 \dots (2) \end{cases}$

This implies that, $x_1 = -4, x_2 = 4$, thus $X = -4 + 4J$.

The weak fuzzy quadratic equations.

Definition.

Let $AX^2 + BX + C = 0$, with $A = a_1 + a_2J, B = b_1 + b_2J, C = c_1 + c_2J, X = x_1 + x_2J \in F_J$, it is called a weak fuzzy quadratic equations.

Remark.

By easy computing of $AX^2 + BX + C = 0$, we get:

$$\begin{cases} a_1x_1^2 + a_2x_2^2t + 2a_2x_1x_2t + b_1x_1 + b_2x_2t + c_1 = 0 \\ a_2x_1^2 + a_2x_2^2t + 2a_1x_1x_2 + b_1x_2 + b_2x_1 + c_2 = 0 \end{cases}$$

Example.

Consider the following quadratic equation:

$$(1 - J)X^2 + JX + 2 + 4J = 0, \text{ with } t = J^2 = \frac{1}{3}.$$

It is equivalent to:

$$\begin{cases} x_1^2 + \frac{1}{3}x_2^2 - \frac{2}{3}x_1x_2 + \frac{1}{3}x_2 + 2 = 0 \\ -x_1^2 - \frac{1}{3}x_2^2 + 2x_1x_2 - x_1 + 4 = 0 \end{cases}$$

The solution may be hard in this way.

We will try another method for searching the solutions.

Theorem.

Let $AX^2 + BX + C = 0$ be a quadratic weak fuzzy complex equation, where $A = a_1 + a_2J, B = b_1 + b_2J, C = c_1 + c_2J, X = x_1 + x_2J \in F_J$, then it is equivalent to:

$$\begin{cases} (a_1 + \sqrt{t}a_2)(x_1 + \sqrt{t}x_2)^2 + (b_1 + \sqrt{t}b_2)(x_1 + \sqrt{t}x_2) + c_1 + \sqrt{t}c_2 = 0 \dots (1) \\ (a_1 - \sqrt{t}a_2)(x_1 - \sqrt{t}x_2)^2 + (b_1 - \sqrt{t}b_2)(x_1 - \sqrt{t}x_2) + c_1 - \sqrt{t}c_2 = 0 \dots (2) \end{cases}$$

Proof.

By computing equation (1), we get:

$$(a_1 + \sqrt{t}a_2)(x_1^2 + tx_2^2 + 2x_1x_2\sqrt{t}) + b_1x_1 + b_1x_2\sqrt{t} + b_2x_1\sqrt{t} + b_2x_2t + c_1 + c_2\sqrt{t} = 0$$

Hence:

$$a_1x_1^2 + a_1tx_2^2 + 2a_1x_1x_2\sqrt{t} + a_2\sqrt{t}x_1^2 + a_2t\sqrt{t}x_2^2 + 2a_2x_1x_2t + b_1x_1 + b_1x_2\sqrt{t} + b_2x_1\sqrt{t} + b_2x_2t + c_1 + c_2\sqrt{t} = 0 \dots (I)$$

By computing equation (2), we get:

$$(a_1 - \sqrt{t}a_2)(x_1^2 + tx_2^2 - 2x_1x_2\sqrt{t}) + b_1x_1 - b_1x_2\sqrt{t} - b_2x_1\sqrt{t} + b_2x_2t + c_1 - c_2\sqrt{t} = 0$$

Hence:

$$a_1x_1^2 + a_1tx_2^2 - 2a_1x_1x_2\sqrt{t} - a_2\sqrt{t}x_1^2 - a_2t\sqrt{t}x_2^2 + 2a_2x_1x_2t + b_1x_1 - b_1x_2\sqrt{t} - b_2x_1\sqrt{t} + b_2x_2t + c_1 - c_2\sqrt{t} = 0 \dots (II)$$

We add (I)to(II):

$$2a_1x_1^2 + 2a_1tx_2^2 + 4a_2x_1x_2t + 2b_1x_1 + 2b_2x_2t + 2c_1 = 0, \text{ thus:}$$

$$a_1x_1^2 + a_1tx_2^2 + 2a_2x_1x_2t + b_1x_1 + b_2x_2t + c_1 = 0 \dots (*)$$

We subtract (II)from(I):

$$4a_1x_1x_2\sqrt{t} + 2a_2\sqrt{t}x_1^2 + 2a_2t\sqrt{t}x_2^2 + 2b_1x_2\sqrt{t} + 2b_2x_1\sqrt{t} + 2c_2\sqrt{t} = 0, \text{ thus:}$$

$$+a_2x_1^2 + a_2tx_2^2 + 2a_1x_1x_2 + b_1x_2 + b_2x_1 + c_2 = 0 \dots (**)$$

Equations (*) and (**)are equivalent to $AX^2 + BX + C = 0$ according to remark.

Example.

Consider the equation $(3 + 2J)X^2 + JX - 15 - \frac{45}{2}J = 0; a_1 = 3, a_2 = 2, b_1 = 0, b_2 = 1, c_1 = -15, c_2 = -\frac{45}{2}, J^2 = \frac{1}{4} = t.$

It is equivalent to:

$$\begin{cases} (3 + 1)\left(x_1 + \frac{1}{2}x_2\right)^2 + \left(0 + \frac{1}{2}\right)\left(x_1 + \frac{1}{2}x_2\right) - 15 - \frac{45}{4} = 0 \\ (3 - 1)\left(x_1 - \frac{1}{2}x_2\right)^2 + \left(0 - \frac{1}{2}\right)\left(x_1 - \frac{1}{2}x_2\right) - 15 + \frac{45}{4} = 0 \end{cases}$$

Which are equivalent to:

$$\begin{cases} 4\left(x_1 + \frac{1}{2}x_2\right)^2 + \frac{1}{2}\left(x_1 + \frac{1}{2}x_2\right) - \frac{105}{4} = 0 \dots (1) \\ 2\left(x_1 - \frac{1}{2}x_2\right)^2 - \frac{1}{2}\left(x_1 - \frac{1}{2}x_2\right) - \frac{15}{4} = 0 \dots (2) \end{cases}$$

Firstly, we solve equation(1).

$$\Delta = \frac{1}{4} - 4(4)\left(-\frac{105}{4}\right) = \frac{1}{4} + 420 = \frac{1681}{4} > 0, \sqrt{\Delta} = \frac{41}{2}$$

So that: $x_1 + \frac{1}{2}x_2 = \frac{-\frac{1}{2} + \frac{41}{2}}{4} = \frac{20}{8} = \frac{5}{2}$ or $x_1 + \frac{1}{2}x_2 = \frac{-\frac{1}{2} - \frac{41}{2}}{4} = \frac{-22}{8} = \frac{-11}{4}.$

We solve equation (2).

$$\Delta = \frac{1}{4} - 4(2)\left(-\frac{15}{4}\right) = \frac{121}{4} > 0, \sqrt{\Delta} = \frac{11}{2}$$

So that: $x_1 - \frac{1}{2}x_2 = \frac{\frac{1}{2} + \frac{11}{2}}{4} = \frac{6}{4} = \frac{3}{2}$ or $x_1 - \frac{1}{2}x_2 = \frac{\frac{1}{2} - \frac{11}{2}}{4} = \frac{-5}{4}$.

$$\text{if } \begin{cases} x_1 + \frac{1}{2}x_2 = \frac{5}{2} \\ x_1 - \frac{1}{2}x_2 = \frac{3}{2} \end{cases}, \text{ then } x_1 = 2, x_2 = 1.$$

$$\text{if } \begin{cases} x_1 + \frac{1}{2}x_2 = \frac{5}{2} \\ x_1 - \frac{1}{2}x_2 = \frac{-5}{4} \end{cases}, \text{ then } x_1 = \frac{5}{8}, x_2 = \frac{5}{4}.$$

$$\text{if } \begin{cases} x_1 + \frac{1}{2}x_2 = \frac{-11}{4} \\ x_1 - \frac{1}{2}x_2 = \frac{3}{2} \end{cases}, \text{ then } x_1 = \frac{-5}{8}, x_2 = \frac{-17}{4}.$$

$$\text{if } \begin{cases} x_1 + \frac{1}{2}x_2 = \frac{-11}{4} \\ x_1 - \frac{1}{2}x_2 = \frac{-5}{4} \end{cases}, \text{ then } x_1 = -2, x_2 = \frac{3}{2}.$$

This means that the weak fuzzy complex solutions are:

$$\left\{ 2 + J, \frac{5}{8} + \frac{5}{4}J, \frac{-5}{8} - \frac{17}{4}J, -2 + \frac{3}{2}J \right\}.$$

Conclusion

In this paper, we have defined the concept of weak fuzzy complex numbers for the first time and presented their elementary properties such as normal operations, conjugates, and algebraic equations.

In the future, we aim to study weak fuzzy complex matrices and their algebraic properties.

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