Neutrosophic Submodule of Direct Sum $M \oplus N$

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Abstract

The paper focuses on neutrosophic algebraic structures and operations applicability to the study of classical algebraic structures, particularly the R-module. The definition of neutrosophic submodules P and Q was further developed upon in this work in order to create neutrosophic submodules of $P + Q$. In this study, the neutrosophic submodule of the direct sum $M \oplus N$ is constructed, analyzed, and its associated results are examined. Additionally, several algebraic results of the neutrosophic submodule's direct sum of a non-empty arbitrary family of submodules are examined.

Keywords: R-module, Neutrosophic Set, Neutrosophic Submodule, Support, Neutrosophic Point

1 Introduction

Numerous applications in the complex environment frequently involve uncertainty, fuzziness, and indeterminacy. Florentin Smarandache[1] devised analytical neutrosophic sets to describe the information's fuzziness, uncertainty, and imprecision. Abstract algebra was one of the first few areas of research using the idea of neutrosophic set among the diverse branches of applied and pure mathematics. Some authors have studied the algebraic structure associated in pure mathematics with uncertainty. In 1971, Azriel Rosenfeld[2] presented a seminal paper on fuzzy subgroup and W.J. Liu[3] developed the idea of fuzzy normal subgroup and fuzzy subring. It was a significant milestone in the area of mathematics research and fuzzy algebra. Mordeson's and Malik's book[4] gives an account of all these concepts upto 1998. Negoita and Ralescu[5] launched the notion of a fuzzy module. Then fuzzy module was further developed by Mashinchi & Zahedi.[6] The idea of a direct sum of fuzzy modules was investigated by P. Isaac.[7] In 2011 P. Isaac, P.P.John[8] studied about the algebraic nature of intuitionistic fuzzy submodule of a classical module. The hybridization of the neutrosophic set with several conceivable algebraic structures, such as the bipolar set, soft set, and hesitant fuzzy set, is one of the major breakthroughs in the field of neutrosophic set theory.[9–11] W. B. Vasantha Kandasamy and Florentin Smarandache[12] initially presented basic neutrosophic algebraic structures and their application. Vidan Cetkin[13,14] consolidated the neutrosophic set theory and algebraic structures, creating neutrosophic subgroups and neutrosophic submodules. The basic features of single valued neutrosophic submodules of an R-module (classical module) are studied by Cetkin and Olgun N.[14,15] Neutrosophic submodule is one of the generalizations of the algebraic structure "module" that supplements the classic structure by assigning three diverse level graded features of each module component

A direct sum of modules is defined in the same way as a direct sum of vector spaces. The direct sum is an action carried out in algebraic structures which is commutative and associative up to isomorphism. The prototype of the direct sum of neutrosophic submodules of an R-modules is present in this chapter and examine some

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related properties with the assistance of *support* of a neutrosophic submodule. The remaining of the paper is organised in the following manner. Brief summaries of neutrosophic set operations, neutrosophic submodules and direct sum of a R-module $M,\ N$ are provided in section 2. Section 3 extended the definition of neutrosophic submodules $P\ and\ Q$ to construct neutrosophic submodules of $P \oplus Q$. Besides, the direct sum of a non-empty arbitrary family of neutrosophic submodules is examine and derive some algebraic results of neutrosophic submodule. Finally section 4 presents a valid summary and future work of the proposed study.

2 Preliminaries

This section presents some of the preliminary definitions and results which are basic for a better and clear cognizance of next chapters.[16] Let $A\ and\ B$ be submodules of an R-module M. The sum of A and B, denoted as a set

$$A + B = \{x + y : x \in A, y \in B\}$$

which is also a submodule and smallest submodule which contains both $A\ and\ B$.

[16] The intersection of any non empty collection of submodules of an R-module is a sub module . [17,18] "If $t_P, i_P, f_P : X \to [0, 1]$, then P is known as Single Valued Neutrosophic Set (SVNS).

This paper considers only SVNS. For simplicity SVNS will be called neutrosophic set. U^X denotes the set of all neutrosophic subsets of X or neutrosophic power set of X. [19] Let P and Q be neutrosophic sets of an R-Module M . Then their sum $P + Q$ is a neutrosophic set of M, defined as follows

$$P + Q(x) = \{x, t_{P+Q}(x), i_{P+Q}(x), f_{P+Q}(x) : x \in M\} \text{ where}$$
$$t_{P+Q}(x) = \vee\{t_P(y) \wedge t_Q(z) | x = y + z, y, z \in M\}$$
$$i_{P+Q}(x) = \vee\{i_P(y) \wedge i_Q(z) | x = y + z, y, z \in M\}$$
$$f_{P+Q}(x) = \wedge\ f_P(y) \vee f_Q(z) | x = y + z, y, z \in M\}.$$

[14] Let P be a neutrosophic set of an R-module M and $r \in R$. Define neutrosophic set $rP = \{x, t_{rP}(x), i_{rP}(x), f_{rP}(x) : x \in M\}$ of M as follows $t_{rP}(x) = \begin{cases} \vee\{t_P(y)\} & if\ y \in M, x = ry \\ 0 & otherwise \end{cases}$; $i_{rP}(x) = \begin{cases} \vee\{i_P(y)\} & if\ y \in M, x = ry \\ 0 & otherwise \end{cases}$ and $f_{rP}(x) = \begin{cases} \wedge\{f_P(y)\} & if\ y \in M, x = ry \\ 0 & otherwise \end{cases}$ [14] Let M be an R module. Let $P \in U^M$ where U^M denotes the neutrosophic power set of R-module M. Then a neutrosophic subset $P = \{x, t_P(x), i_P(x), f_P(x) : x \in M\}$ in M is called neutrosophic submodule of M if it satisfies the following;

1. $t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$
2. $t_P(x + y) \geq t_P(x) \wedge t_P(y)$
   $i_P(x + y) \geq i_P(x) \wedge i_P(y)$
   $f_P(x + y) \leq f_P(x) \vee f_P(y),\ \forall x, y \in M$
3. $t_P(rx) \geq t_P(x),\ i_P(rx) \geq i_P(x),\ f_P(rx) \leq f_P(x), \forall\, x \in M, \forall\, r \in R$.

The set of all neutrosophic submodules of R-module M represented by $U(M)$. [20,21]

For any neutrosophic subset $P = \{(x, t_P(x), i_P(x), f_P(x)) : x \in X\}$ of X, the support P^* of the neutrosophic set P can be defined as

$$P^* = \{x \in X, t_P(x) > 0, i_P(x) > 0, f_P(x) < 1\}.$$

[20] Let $P,\ Q \in U^X$. If $P \subseteq Q$, then $P^* \subseteq Q^*$. If $P, Q \in U^M$, then $\forall x, y \in M,\ r, s \in R$

1. $t_{(rP+sQ)}(rx + sy) \geq t_P(x) \wedge t_Q(y)$

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2. $i_{(rP+sQ)}(rx+sy) \geq i_P(x) \wedge i_Q(y)$
3. $f_{(rP+sQ)}(rx+sy) \leq f_P(x) \wedge f_Q(y)$

[20] Let $P_i,\ i \in J$ be an arbitrary non empty family of U^M where $P_i = \{x, t_{P_i}(x), i_{P_i}(x), f_{P_i}(x) : x \in M\}$ for each $i \in J$. Then

$$\begin{aligned}
&\textstyle\sum_{i\in J} P_i = \{x, t_{\sum\limits_{i\in J} P_i}(x), i_{\sum\limits_{i\in J} P_i}(x), f_{\sum\limits_{i\in J} P_i}(x) : x \in M\} \text{ where}\\
&t_{\sum\limits_{i\in J} P_i}(x) = \vee\{\underset{i\in J}{\wedge} t_{P_i}(x_i) : x_i \in M, \textstyle\sum_{i\in J} x_i = x\}\ \forall x \in M\\
&i_{\sum\limits_{i\in J} P_i}(x) = \vee\{\underset{i\in J}{\wedge} i_{P_i}(x_i) : x_i \in M, \textstyle\sum_{i\in J} x_i = x\}\ \forall x \in M\\
&f_{\sum\limits_{i\in J} P_i}(x) = \wedge\{\underset{i\in J}{\vee} f_{P_i}(x_i) : x_i \in M, \textstyle\sum_{i\in J} x_i = x\}\ \forall x \in M
\end{aligned}$$

in $\sum\limits_{i\in J} x_i$, at most finitely $x_i's$ are not equal to zero. [20] Let $P_i,\ i \in J$ be an arbitrary non empty family of U^M, then $r(\bigcup_{i\in J} P_i) = \bigcup_{i\in J}(rP_i)\ for\ r \in R$. [20] For any $x \in X$, the neutrosophic point $\hat{N}_{\{x\}}$ is defined as $\hat{N}_{\{x\}}(s) = \{s, t_{\hat{N}_{\{x\}}}(s), i_{\hat{N}_{\{x\}}}(s), f_{\hat{N}_{\{x\}}}(s) : s \in X\}$ where

$$\hat{N}_{\{x\}}(s) = \begin{cases} (1,1,0) & x = s\\ (0,0,1) & x \neq s \end{cases}$$

[22] Let P, Q and $S \in U(M)$, then P is said to be the direct sum of neutrosophic submodules Q and S, we write $P = Q \oplus S$, if

1. $P = Q + S$
2. $Q \cap S = \hat{N}_{\{0\}}$

[22] Let $P_i \in U(M)\ \forall i \in J$, then we say that P is the direct sum of $\{P_i : i \in J\}$ denoted by $\oplus_{i\in J} P_i$ if

1. $P = \sum_{i\in J} P_i$

2. $P_j \cap \sum_{i\in J-\{j\}} P_i = \hat{N}_{\{0\}}\ \forall j \in J$ [22] If P, Q and $S \in U(M)$ such that $P = Q \oplus S$. Then $P^* = Q^* \oplus S^*$. Let $P, Q\ S \in U(M)$. If $P^* = Q^* \oplus S^* \nRightarrow P = Q \oplus S$. i.e. The converse of the theorem 2 need not true.

3 Neutrosophic Submodule of Direct Sum $M \oplus N$

This section discusses the construction of the neutrosophic submodule of the direct sum $M \oplus N$ and analyzes the results associated with it. The direct sum of non-empty arbitrary family of neutrosophic submodules is also investigated in this part. Let M and N be R-modules. Let $P = \{m, t_P(m), i_P(m), f_P(m) : m \in M\} \in U(M)$ and $Q = \{n, t_Q(n), i_Q(n), f_Q(n) : n \in N\} \in U(N)$. Consider the direct sum $M \oplus N$. We extend the definition of P and Q to $M \oplus N$ to get the neutrosophic set $P'\ and\ Q'$ in $M \oplus N$ such that

$$P' = \{(m,n), t_{P'}(m,n), i_{P'}(m,n), f_{P'}(m,n) : (m,n) \in M \oplus N\}$$

$$Q' = \{(m,n), t_{Q'}(m,n), i_{Q'}(m,n), f_{Q'}(m,n) : (m,n) \in M \oplus N\}$$

where

$$t_{P'}(m,n) = \begin{cases} t_P(m) & if\ n = 0\\ 0 & if\ n \neq 0 \end{cases},$$

$$i_{P'}(m,n) = \begin{cases} i_P(m) & if\ n = 0\\ 0 & if\ n \neq 0 \end{cases},$$

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$$f_{P'}(m,n) = \begin{cases} f_P(m) & if\ n=0 \\ 1 & if\ n \neq 0 \end{cases}$$

$$t_{Q'}(m,n) = \begin{cases} t_Q(n) & if\ m=0 \\ 0 & if\ m \neq 0 \end{cases},$$

$$i_{Q'}(m,n) = \begin{cases} i_Q(n) & if\ m=0 \\ 0 & if\ m \neq 0 \end{cases},$$

$$f_{Q'}(m,n) = \begin{cases} f_Q(n) & if\ m=0 \\ 1 & if\ m \neq 0 \end{cases} \quad \forall\,(m,n) \in M \oplus N$$

The neutrosophic sets $P^{'}\ and\ Q^{'} \in U(M \oplus N)$.

Proof. Consider the neutrosophic set

$$P^{'} = \{(m,n), t_{P'}(m,n), i_{P'}(m,n), f_{P'}(m,n) : (m,n) \in M \oplus N\}$$

defined as

$$t_{P'}(m,n) = \begin{cases} t_P(m) & if\ n=0 \\ 0 & if\ n \neq 0 \end{cases},\ i_{P'}(m,n) = \begin{cases} i_P(m) & if\ n=0 \\ 0 & if\ n \neq 0 \end{cases},$$

$$f_{P'}(m,n) = \begin{cases} f_P(m) & if\ n=0 \\ 1 & if\ n \neq 0 \end{cases}$$

Then clearly $t_{P'}(0,0) = 1, i_{P'}(0,0) = 1\ and\ f_{P'}(0,0) = 0.$
For $(m_1,n_1),(m_2,n_2) \in M \oplus N$

$$\begin{aligned} t_{P'}((m_1,n_1)+(m_2,n_2)) &= t_{P'}(m_1+m_2, n_1+n_2) \\ &= \begin{cases} t_P(m_1+m_2) & if\ n_1+n_2=0 \\ 0 & if\ n_1+n_2 \neq 0 \end{cases} \end{aligned}$$

Case 1 If $n_1+n_2=0$, then either $n_1=0, n_2=0$ or $n_1 \neq 0, n_2=-n_1 \neq 0$
If $n_1=n_2=0,$, then

$$\begin{aligned} t_{P'}((m_1,n_1)+(m_2,n_2)) &= t_P(m_1+m_2) \\ &\geq t_P(m_1) \wedge t_P(m_2) \\ &= t_{P'}(m_1,n_1) \wedge t_{P'}(m_2,n_2). \end{aligned}$$

If $n_1 \neq 0, n_2=-n_1 \neq 0$, then

$$\begin{aligned} t_{P'}((m_1,n_1)+(m_2,n_2)) &= t_P(m_1+m_2,0) \\ &= t_P(m_1+m_2) \\ &\geq t_P(m_1) \wedge t_P(m_2) \\ &= 0 \wedge 0 \\ &= t_{P'}(m_1,n_1) \wedge t_{P'}(m_2,n_2). \end{aligned}$$

Case 2 If $n_1+n_2 \neq 0$, then either $n_1\ or\ n_2\ or\ both\ \neq 0$

If $n_1 \neq 0, n_2 \neq 0$ then,

$$t_{P'}((m_1,n_1)+(m_2,n_2)) = 0 = 0 \wedge 0 = t_{P'}(m_1,n_1) \wedge t_{P'}(m_2,n_2)$$

If $n_1=0, n_2 \neq 0$ then,

$$t_{P'}((m_1,n_1)+(m_2,n_2)) = 0 = t_P(m_1) \wedge 0 = t_{P'}(m_1,n_1) \wedge t_{P'}(m_2,n_2) \quad .$$

If $n_1 \neq 0, n_2=0$. Same as above
$\therefore$ from the above cases,

$$t_{P'}((m_1,n_1)+(m_2,n_2)) \geq t_{P'}(m_1,n_1) \wedge t_{P'}(m_2,n_2)$$

Similarly we can prove

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$$i_{P'}((m_1,n_1)+(m_2,n_2)) \geq i_{P'}(m_1,n_1) \wedge i_{P'}(m_2,n_2)$$
$$f_{P'}((m_1,n_1)+(m_2,n_2)) \leq f_{P'}(m_1,n_1) \vee f_{P'}(m_2,n_2)$$

Now for any $r \in R, (m,n) \in M \oplus N$,

$$t_{P'}(r(m,n)) = t_{P'}(rm,rn) = \begin{cases} t_P(rm) & if\ rn = 0 \\ 0 & if\ rn \neq 0 \end{cases}$$

Case 1: If $rn = 0,\ then,$

$$t_{P'}(r(m,n)) = t_P(rm) \geq t_P(m) = t_{P'}(m,n)$$

Case 2: If $rn \neq 0 \Rightarrow n \neq 0,\ then,$

$$t_{P'}(r(m,n)) = t_{P'}(rm,rn) = 0 = t_{P'}(m,n)$$

$\therefore$ from the above cases, $t_{P'}(r(m,n)) \geq t_{P'}(m,n)$

Similarly we can prove $i_{P'}(r(m,n)) \geq i_{P'}(m,n)\ and\ f_{P'}(r(m,n)) \leq f_{P'}(m,n).$

Hence $P^{'} \in U(M \oplus N)$. Similarly we can prove that $Q^{'} \in U(M \oplus N)$. □

If $P^{'}, Q^{'} \in U(M \oplus N)$, then their sum $P^{'}+Q^{'} \in U(M \oplus N)$. $(P^{'} \cap Q^{'})(m,n) = \{(m,n), t_{P' \cap Q'}(m,n), i_{P' \cap Q'}(m,n), f_{P' \cap Q'}$ $(m,n) \in M \oplus N\}$, then from the definition of $P^{'}\ and\ Q^{'}$

$$t_{P' \cap Q'}(m,n) = \begin{cases} 1 & if\ (m,n) = 0 \\ 0 & if\ (m,n) \neq 0 \end{cases}, \quad i_{P' \cap Q'}(m,n) = \begin{cases} 1 & if\ (m,n) = 0 \\ 0 & if\ (m,n) \neq 0 \end{cases}$$

$$f_{P' \cap Q'}(m,n) = \begin{cases} 0 & if\ (m,n) = 0 \\ 1 & if\ (m,n) \neq 0 \end{cases}$$

$\therefore\ P^{'} \cap Q^{'} = \hat{N}_{\{0\}}$

$\Rightarrow\ P^{'} + Q^{'}$ is a direct sum and is denoted by $P \oplus Q \in U(M \oplus N)$. If $P \oplus Q \in U(M \oplus N)$, then

1. $t_{P \oplus Q}(m,n) = t_P(m) \wedge t_Q(n)$
2. $i_{P \oplus Q}(m,n) = i_P(m) \wedge i_Q(n)$
3. $f_{P \oplus Q}(m,n) = f_P(m) \vee f_Q(n)$.

Proof. 1. We have

$$\begin{aligned} (P \oplus Q(m,n)) &= \{(m,n), t_{P \oplus Q}(m,n), i_{P \oplus Q}(m,n), f_{P \oplus Q}(m,n) : (m,n) \in M \oplus N\} \\ &= P^{'} + Q^{'}(m,n) \\ &= \{(m,n), t_{P'+Q'}(m,n), i_{P'+Q'}(m,n), f_{P'+Q'}(m,n) : (m,n) \in M \oplus N\} \end{aligned}$$

$$\begin{aligned} \text{Now} \quad t_{P \oplus Q}(m,n) &= t_{P'+Q'}(m,n) \\ &= \vee\{t_{P'}(m_1,n_1) \wedge t_{Q'}(m_2,n_2) : (m,n) = (m_1,n_1) + (m_2,n_2) : \\ &\quad (m_1,n_1),(m_2,n_2) \in M \oplus Q\} \\ &= t_{P'}(m,0) \wedge t_{Q'}(0,n) \\ &= t_P(m) \wedge t_Q(n) \end{aligned}$$

Similarly we get 2 and 3.

$i_{P \oplus Q}(m,n) = i_P(m) \wedge i_Q(n),\ f_{P \oplus Q}(m,n) = f_P(m) \vee f_Q(n).$ □

If $P \in U(M)\ and\ Q \in U(N)$, then $(P \oplus Q)^* = P^* \oplus Q^*$.

Proof. We have $P \oplus Q = P^{'} + Q^{'} \in U(M \oplus N)$. Let $(m,n) \in (P \oplus Q)^*$

$$\begin{aligned} &\Rightarrow\ t_{(P \oplus Q)^*}(m,n) > 0,\ i_{(P \oplus Q)^*}(m,n) > 0 \text{ and } f_{(P \oplus Q)^*}(m,n) < 1 \\ &\Rightarrow\ t_P(m) \wedge t_Q(n) > 0,\ i_P(m) \wedge i_Q(n) > 0 \text{ and } f_P(m) \vee f_Q(n) < 1 \\ &\Rightarrow\ t_P(m) > 0, t_Q(n) > 0,\ i_P(m) > 0\ i_Q(n) > 0 \text{ and } f_Q(m) < 1,\ f_Q(n) < 1 \\ &\Rightarrow\ m \in P^* \text{ and } n \in Q^* \\ &\Rightarrow\ (m,n) \in P^* + Q^* \end{aligned}$$

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Now $x \in P^* \cap Q^*$

$$\begin{aligned}
&\Rightarrow \quad x \in P^* \text{ and } x \in Q^* \\
&\Rightarrow \quad t_P(x) > 0, i_P(x) > 0, f_P(x) < 1 \text{ and } t_Q(x) > 0, i_Q(x) > 0, f_Q(x) < 1 \\
&\Rightarrow \quad t_P(x) \wedge t_Q(x) > 0,\ i_P(x) \wedge i_Q(x) > 0 \text{ and } f_P(x) \vee f_Q(x) < 1 \\
&\Rightarrow \quad t_{P\cap Q}(x) > 0,\ i_{P\cap Q}(x) > 0 \text{ and } f_{P\cap Q}(x) < 1 \\
&\Rightarrow \quad t_{P\cap Q}(x) = 1,\ i_{P\cap Q}(x) = 1\ and\ f_{P\cap Q}(x) = 0 \\
&\qquad [Since\ P \oplus Q\ is\ the\ direct\ sum, hence P \cap Q = \hat{N}_{\{0\}}] \\
&\Rightarrow \quad x = 0 \\
&\Rightarrow \quad P^* \cap Q^* = \{0\}
\end{aligned}$$

Hence $P^* + Q^*$ is the direct sum, denoted as $P^* \oplus Q^*$ which is a submodule of $M \oplus N$. Hence

$$(P \oplus Q)^* \subseteq P^* \oplus Q^*. \tag{1}$$

Let $(m, n) \in P^* \oplus Q^*$

$$\begin{aligned}
&\Rightarrow \quad m \in P^* \text{ and } n \in Q^* \\
&\Rightarrow \quad t_P(m) > 0, t_Q(n) > 0,\ i_P(m) > 0\ i_Q(n) > 0 \text{ and } f_P(m) < 1,\ f_Q(n) < 1 \\
&\Rightarrow \quad t_P(m) \wedge t_Q(n) > 0,\ i_P(m) \wedge i_Q(n) > 0 \text{ and } f_P(m) \vee f_Q(n) < 1 \\
&\Rightarrow \quad t_{P\oplus Q}(m,n) > 0,\ i_{P\oplus Q}(m,n) > 0 \text{ and } f_{P\oplus Q}(m,n)1 \\
&\Rightarrow \quad (m, n) \in (P \oplus Q)^*
\end{aligned}$$

Hence

$$P^* \oplus Q^* \subseteq (P \oplus Q)^* \tag{2}$$

$\therefore$ from the equations (5.3.1) and (5.3.2), we get $(P \oplus Q)^* = P^* \oplus Q^*$. ☐

If $P, Q \in U(M)$, then $(P \oplus Q)^* = P^* \oplus Q^*$. Let $P_i,\ i \in J$ be an arbitrary non empty family of neutrosophic submodule of an R-module M where $\sum_{i\in J} P_i$ is the direct sum of $\oplus_{i\in J} P_i$ and $Q \in U(M)$. If $Q \cap \sum_{i\in J} P_i = \hat{N}_{\{0\}}$, then $Q + \sum_{i\in J} P_i$ is the direct sum $Q \oplus (\oplus_{i\in J} P_i)$.

Proof. Given that $\sum_{i\in J} P_i$ is a direct sum, hence $P_j \cap (\sum_{i\in J-\{j\}} P_i) = \hat{N}_{\{0\}}\ \forall j \in J$. Also given that $Q \cap \sum_{i\in J} P_i = \hat{N}_{\{0\}}$

For $x \in M,\ j \in J$, the neutosophic components of $(P_j \cap (Q + \sum_{J-\{j\}} P_i))(x)$ are

$$\{x, t_{P_j \cap (Q + \sum_{i\in J-\{j\}} P_i)}(x), i_{P_j \cap (Q + \sum_{J-\{j\}} P_i)}(x), f_{P_j \cap (Q + \sum_{J-\{j\}} P_i)}(x)\} .$$

Now consider

$$\begin{aligned}
t_{P_j \cap (Q + \sum_{i\in J-\{j\}} P_i)}(x) &= t_{P_j}(x) \wedge t_{Q + \sum_{i\in J-\{j\}} P_i)}(x) \\
&= t_{P_j}(x) \wedge \vee\{t_Q(y) \wedge (\wedge_i t_{P_i}(x_i)) \\
&\qquad : x = y + \sum_i x_i, i \in J - \{j\}, y, x_i \in M\} \\
&= t_{P_j}(y + \sum x_i) \wedge \vee\{t_Q(y) \wedge (\wedge_i t_{P_i}(x_i)) \\
&\qquad : x = y + \sum_i x_i, i \in J - \{j\}, y, x_i \in M\} \\
&= (\vee\{t_{P_j}(y) \wedge (\wedge_i t_{P_j}(x_i)) : x = y + \sum_i x_i, i \in J - \{j\}, y, x_i \in M\}) \\
&\wedge (\vee\{t_Q(y) \wedge (\wedge_i t_{P_i}(x_i)) : x = y + \sum_i x_i, i \in J - \{j\}, y, x_i \in M\}) \\
&= \vee\{(t_{P_j}(y) \wedge (\wedge_i t_{P_j}(x_i)) \wedge (t_Q(y) \wedge (\wedge_i t_{P_i}(x_i)) :
\end{aligned}$$

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$$x = y + \sum_i x_i, i \in J - \{j\}, y, x_i \in M\}$$
$$= \vee\{(t_{P_j}(y) \wedge t_Q(y)) \wedge \wedge_i(t_{P_j}(x_i)) \wedge t_{P_i}(x_i)) :$$
$$x = y + \sum_i x_i, i \in J - \{j\}, y, x_i \in M\}$$
$$= \vee\{t_{P_j \cap Q}(y) \wedge \wedge_i(t_{P_j \cap P_i}(x_i)) : x = y + \sum_i x_i,$$
$$i \in J - \{j\}, y, x_i \in M\}$$
$$= \begin{cases} 1 & if\ y = 0, x_i = 0\ \forall i \in J - \{j\} \\ 0 & if\ y \neq 0\ or\ x_i \neq 0\ for\ some\ i \in J - \{j\} \end{cases}$$
$[Since\ P_j \cap Q = \hat{N}_{\{0\}}\ \forall j \in J$ and $P_j \cap P_i = \hat{N}_{\{0\}}\ \forall i \in J - \{j\}]$
$$= \begin{cases} 1 & if\ x = 0 \\ 0 & if\ x \neq 0 \end{cases} \quad [since\ x = y + \sum_i x_i, i \in J - \{j\}, y, x_i \in M]$$
$$= t_{\hat{N}_{\{0\}}}(x)$$

Similarly we can show that

$$i_{P_j \cap (Q + \sum_{i \in J - \{j\}} P_i)}(x) = \begin{cases} 1 & if\ x = 0 \\ 0 & if\ x \neq 0 \end{cases} = i_{\hat{N}_{\{0\}}}(x)$$

$$f_{P_j \cap (Q + \sum_{i \in J - \{j\}} P_i)}(x) = \begin{cases} 0 & if\ x = 0 \\ 1 & if\ x \neq 0 \end{cases} = f_{\hat{N}_{\{0\}}}(x)$$

$\Rightarrow\ P_j \cap (Q + \sum_{J-\{j\}} P_i) = \hat{N}_{\{0\}}\ \forall j \in J$
$\therefore\ Q + \sum_{i \in J - \{j\}} P_i$ is the direct sum $Q \oplus (\oplus_{i \in J} P_i)$
Thus the theorem is proved. □

4 Conclusion

The primary goals of this study is to infuse the contemporary speculations of neutrosophic sets and module structure into the conventional algebraic structures. This paper investigate and improve the construction of neutrosophic submodule from the direct sum of submodules in the arena of abstract algebra. The present study leads to explore the concept of Tensor product of neutrosophic submodule, injective and projective neutrosophic submodules of an R-module, semi-simple neutrosophic submodule of an R-module and a quasi neutrosophic submodule of an R-module.

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