On Some Results about Schrodinger-Hermite Equation

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Abstract

This work is dedicated to study the equation of Schrodinger-Hermite on some well-known spaces as $L_2(R^n)$ by using Hermite operator $H = -\Delta + |x|^2$.

Keywords: Hermite operator; Schrodinger equation; Hermite function.

1.Introduction

Schrodinger equation was defined for physical purposes as follows:

$$\begin{cases} i\,\dfrac{\partial u}{\partial t} = -\Delta u \\ u(x,0) = f(x) \end{cases}$$

This equation was studied by many researchers [1,4,8]. Many works and results have been proved.

The function $u = u(x,t)$ is defined an $R^n \times [0,\infty[$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$ denotes to Laplace operator.

In [5,11,12], we find many results about Schrodinger-Hermite equation $\begin{cases} i\,\frac{\partial u}{\partial t} = Hu \\ u(x,0) = f(x) \end{cases}$.

The problem is to find $u = u(x,t)$ which is defined on $R^n \times R$, where $H = -\Delta + |x|^2$.

In this work, we try to give some novel results in this direction .

Main Discussion

Definition 1:

Hermite polynomial is defined as follow:

$$H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dx^k}\left(e^{-t^2}\right); k = 0,1, \dots \ , t \in R$$

Definition 2:

We define Hermite orthogonal functions on R as follows:

$$h_n(t) = \left(2^k.k!.\sqrt{\pi}\right)^{\frac{-1}{2}} e^{-\frac{t^2}{2}.H_k(t)} \quad ; k = 0,1, \dots \ , t \in R$$

Definition 3:

For $t \in R$ we put:

$e^{-itH} f = \sum_\alpha e^{-it(2|\alpha|+n)} \langle f, h_\alpha \rangle h_\alpha \,; f \in L_2(R^n)$ where $\lambda_\alpha = 2|\alpha| + n$.

Theorem 4:

The operator e^{-itH} is unitary in $L_2(R^n)$.

proof.

For any $f \in L_2(R^n)$, we have:

$$\left\|e^{-itH}\right\|_{L_2(R^n)} = \sum_\alpha \left|e^{-it(2|\alpha|+n)}\right|^2 . |\langle f, h_\alpha \rangle|^2 = \sum_\alpha |\langle f, h_\alpha \rangle|^2 = \|f\|^2{}_{L_2(R^n)}$$

So that $D(e^{-itH}) = L_2(R^n)$.

On the other hand, the inverse operator has the form:

$$(e^{-itH})^{-1} = e^{itH} : D(e^{-itH}) \to D(e^{itH}) = L_2(R^n)$$

; but $e^{-itH} e^{itH} = I\ (the\ identity\ map)$, thus $R(e^{-itH}) = L_2(R^n)$.

Result 5:

The equation $e^{-itH} f = g$ has one solution $f = e^{itH} g = \sum_\alpha e^{i(2|\alpha|+n)t} \langle g, h_\alpha \rangle h_\alpha \in L_2(R^n)$.

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On the other hand, for any complex number $\lambda \neq e^{-si(2|\alpha|+n)t}$, then $e^{-itH}f - \lambda f = g$has one solution in $L_2(R^n)$:
$f = \sum_\alpha \left(e^{-it\lambda\alpha} - \lambda\right)^{-1} \langle g, h_\alpha \rangle h_\alpha$.
Theorem 6:

$$\frac{\partial}{\partial t}(e^{-itH}) = -iHe^{-itH}$$

Proof.

$$\frac{\partial}{\partial t}(e^{-itH}) = \lim_{y\to 0} \frac{e^{-i(t+y)H} - e^{-itH}}{y}$$

For $f \in D(e^{-itH}) = L_2(R^n)$, we have:

$$\left\| \frac{1}{y}\left[e^{-i(t+y)\lambda_\alpha} - e^{-itH} f\right] + iHe^{-itH} f \right\|^2_{L_2(R^n)}$$

$$= \left\| \sum_\alpha \frac{1}{y}\left[e^{-i(t+y)\lambda_\alpha} - e^{-it\lambda_\alpha}\right]\langle f, h_\alpha\rangle h_\alpha + i \sum_\alpha \lambda_\alpha e^{-it\lambda_\alpha} \langle f, h_\alpha \rangle h_\alpha \right\|^2$$

$$= \left\| \sum_\alpha \left(\frac{1}{y}\left[e^{-i(t+y)\lambda_\alpha} - e^{-it\lambda_\alpha}\right] + i\lambda_\alpha e^{-it\lambda_\alpha} \right) \langle f, h_\alpha \rangle h_\alpha \right\|^2$$

$$= \sum_\alpha \left| \frac{1}{y}\left[e^{-i(t+y)\lambda_\alpha} - e^{-it\lambda_\alpha}\right] + i\lambda_\alpha e^{-it\lambda_\alpha} \right|^2 |\langle f, h_\alpha \rangle h_\alpha|^2$$

Now, we write:

$$sup_\alpha \left| \frac{1}{y}\left[e^{-i(t+y)\lambda_\alpha} - e^{-it\lambda_\alpha}\right] + i\lambda_\alpha e^{-it\lambda_\alpha} \right|^2 \|f\|^2_{L_2(R^n)} \underset{y\to 0}{\to} sup_\alpha \left|\left(e^{-it\lambda_\alpha}\right) + i\lambda_\alpha e^{-it\lambda_\alpha}\right|^2 \|f\|_{L_2(R^n)} = 0$$

Theorem 7:

For $f \in L_2(R^n)$, the equation $\begin{cases} i\,\frac{\partial u}{\partial t} = Hu \\ u(x,0) = f(x) \end{cases}$ has one solution.

On the other hand, for $t \in R$:

$$\|u\|_{L_2(R^n)} = \|f\|_{L_2(R^n)}$$

Proof.
Consider $u(x,t) = e^{-itH} f(x)$, then:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(e^{-itH} f) = -iHe^{-itH} f$$

Hence $i\frac{\partial u}{\partial t} = He^{-itH} f = Hu$.
Now, for $f_1, f_2 \in L_2(R^n)$ with $u_1(x,t), u_2(x,t)$ as the corresponding solutions, then we get:

$$\|u_1(x,t) - u_1(x,t)\|_{L_2(R^n)} = \left\|e^{-itH}(f_1 - f_2)\right\|_{L_2(R^n)} = \|f_1 - f_2\|_{L_2(R^n)}$$

Definition 8:
For a complex number z, with $Re(z) \geq 0$:

$$e^{-ztH} = \sum_\alpha e^{-zt\lambda_\alpha} \langle f, h_\alpha \rangle h_\alpha \ ; f \in L_2(R^n)$$

Remark:e^{-ztH} is a linear-bounded operator.

Remark: The equation $\begin{cases} \frac{\partial u}{\partial t} = -zHu \\ u(x,0) = f(x) \end{cases}$ has only one solution for any $f \in L_2(R^n)$ with the formula $u(x,t) = e^{-ztH} f(x)$.

Remark: The operator e^{-ztH} is integral with $k(x,y)$ as a kernel.

Theorem:
For $z = r + it\,; r > 0$ and $0 < |t| < \pi$.

$$|k_z(x,y)| \leq \frac{e^{-nr}}{|sin2t|^{\frac{n}{2}}}$$

Proof.
Consider $w = e^{-2(r+it)}$, then:

$$1 - w^2 = 2e^{-2(r+it)}\, sh(2r)$$

So that.

$$sh(2z) = cos(2t)sh(2r) + i\, sin(2t)ch(2r)$$

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And.

$$|sh(2z)| \geq |sin(2t)ch(2r)|$$

Thus.

$$|sh(2z)|^{\frac{-n}{2}} \leq \frac{e^{-nr}}{|sin2t|^{\frac{n}{2}}}$$

For $w = e^{-2(r+it)}$, the kernel can be written as follows:

$$k_z(x,y) = \pi^{\frac{-n}{2}} e^{-2(r+it)} (1-w^2)^{\frac{-n}{2}} .exp\left[\frac{-1}{2} . \frac{1+w^2}{1-w^2}(|x|^2+|y|^2) + \frac{2w}{1-w^2} x.y\right]$$

On the other hand, we have:

$$x.y\ cos2t.|x.y| \leq \frac{1}{2}(|x|^2+|y|^2)$$

$$Re\left(\frac{1+w^2}{1-w^2}\right) = \frac{1-e^{-\delta r}}{1+e^{-\delta r}-2e^{-4r}\,cos4t}$$

Hence:

$$Re\left[\frac{-1}{2} . \frac{1+w^2}{1-w^2}(|x|^2+|y|^2) + \frac{2w}{1-w^2} x.y\right] \leq \frac{\frac{-1}{2}\left[\left(1-e^{-\delta r}-2e^{-2r}\,(1-e^{-4r})\right)(|x|^2+|y|^2)\right]}{1+e^{-\delta r}-2e^{-4r}\,cos4t}$$

$$\leq \frac{\frac{-1}{2}(|x|^2+|y|^2)(1-e^{-4r})}{1+e^{-\delta r}-2e^{-4r}\,cos4t}[1-e^{-2r}]^2$$

This implies that:

$$Re\left[\frac{-1}{2} . \frac{1+w^2}{1-w^2}(|x|^2+|y|^2) + \frac{2w}{1-w^2} x.y\right] \leq 0$$

Thus.

$$exp\left[\frac{-1}{2} . \frac{1+w^2}{1-w^2}(|x|^2+|y|^2) + \frac{2w}{1-w^2} x.y\right] \leq 1$$

Remark:

Now, we can solve Schrodinger-Hermite equation as follows:

$$i\,\frac{\partial u}{\partial t} = Hu\,; u = u(x,t)\,, x \in R^n\,, 0 < t \leq \pi$$

$$u(x,0) = e^{-rH} f(x)\,; f \in L_2(R^n)$$

The solution is:

$$u_r(x,t) = e^{-itH}\left(e^{-rH}f(x)\right) = \sum_{\alpha} e^{-(r+it)(2|\alpha|+n)}\,\langle f, h_\alpha\rangle h_\alpha$$

If we put $z = r + it$, then we get:

$$u_r(x,t) = \int_{R^n} k_z(x,y) f(y)\, dy$$

Conclusion

In this paper we have studied the Schrodinger – Hermite equation over the space $L_2(R^n)$, where we have presented some novel results about this equation by using Hermite operators.

As a future research direction, we aim to study more differential equations that have applications in theoretical physics.

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