New Approach in Logarithmic Summability of Sequences in Neutrosophic Normed Spaces

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Abstract

We introduce logarithmic summability in Neutrosophic Normed Spaces [NNS] and give some Taubarian conditions for which logarithmic summability yields convergence in NNS. Besides we define the concept of slow oscillation with respect to logarithmic summability in NNS, Investigate its relation with the concept of q-boundedness and give Taubarian theorems by means of q-boundedness and slow oscillation with respect to logarithmic summability. A comparison theorem between CesaroSummability method and logarithmic summability method in NNS is also proved in the paper.

Keywords: Neutrosophic Normed Spaces; Logarithm Summability; Slow Oscillation; Taubarian Theorem.

1. Introduction

Fuzzy Sets (FSs) put forward by Zadeh [19] has influenced deeply all the scientific fields since the publication of the paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [1] initiated Intuitionistic fuzzy sets (IFSs) for such cases.

Recently Talo and Yavuz [17] introduced CesaroSummability of sequences in Intuitionistic fuzzy normed set and gave Taubarian theorems for CesaroSummability theory and Taubarian theory Intuitionistic fuzzy normed spaces. In their study they also defined the concept of slow oscillation in intuitionistic fuzzy normed spaces and gave related theorems. Fuzzy metric spaces only deal with membership functions. An intuitionistic fuzzy metric space was established by Park [10] that is used to deal with both membership and nonmembership functions. Neutrosophic Set (NS) is a new version of the idea of the classical set which is defined by Smarandache [14], Kirisci and Simsek [15] introduced the notion of neutrosophic metric spaces that is used to deal with membership, non-membership, and naturalness. Sowndrarajan et al. [16] proved some fixed-point results in the setting of neutrosophic metric spaces. In 2022, Jeyaraman, Ramachandran and Shakila [3] proved Approximate Fixed Point Theorems for Weak Contractions on NNS.

We define the notion of slow oscillation with respect to logarithmic summability in NNS and give slowly oscillating type Tauberian conditions for which logarithmic summability yields convergence in NNS. Besides we compare Cesarosummabiltiy and logarithmic summability in NNS. Before continuing with main results, we now give some preliminaries.

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2. Preliminaries

Definition 2.1:
The (N, μ, ν, ω) is said to be an NNS if N is a real vector space, and μ, ν, ω are fuzzy sets on $N \times \mathbb{R}$ satisfying the following conditions for every u. w and $t, s \in \mathbb{R}$.

a) $0 \leq \mu(u,t) \leq 1;\quad 0 \leq \nu(u,t) \leq 1;\quad 0 \leq \nu(u,t) \leq 1;$
b) $\mu(u,t) + \nu(u,t) + \omega(u,t) \leq 3,$
c) $\mu(u,t) = 0 \ for \ t \leq 0,$
d) $\mu(u,t) = 1 \ for \ all \ t \in \mathbb{R}^+ \ if \ and \ only \ if \ u = 0,$
e) $\mu(cu,t) = \mu\left(u, \frac{t}{|c|}\right) \ for \ all \ t \in \mathbb{R}^+ \ and \ c \neq 0,$
f) $\mu(u+w, t+s) \geq min\{\mu(u,t), \mu(w,s)\},$
g) $\lim_{t\to\infty} \mu(u,t) = 1 \ and \ \lim_{t\to 0} \mu(u,t) = 0,$
h) $\nu(u,t) = 0 \ for \ t \leq 0,$
i) $\nu(u,t) = 1 \ for \ all \ t \in \mathbb{R}^+ \ if \ and \ only \ if \ u = 0,$
j) $\nu(cu,t) = \nu\left(u, \frac{t}{|c|}\right) \ for \ all \ t \in \mathbb{R}^+ \ and \ c \neq 0,$
k) $\nu(u+w, t+s) \leq max\{\nu(u,t), \nu(w,s)\},$
l) $\lim_{t\to\infty} \nu(u,t) = 1 \ and \ \lim_{t\to 0} \nu(u,t) = 0,$
m) $\omega(u,t) = 0 \ for \ t \leq 0,$
n) $\omega(u,t) = 1 \ for \ all \ t \in \mathbb{R}^+ \ if \ and \ only \ if \ u = 0,$
o) $\omega(cu,t) = \omega\left(u, \frac{t}{|c|}\right) \ for \ all \ t \in \mathbb{R}^+ \ and \ c \neq 0,$
p) $\omega(u+w, t+s) \leq max\{\omega(u,t), \omega(w,s)\},$
q) $\lim_{t\to\infty} \omega(u,t) = 1 \ and \ \lim_{t\to 0} \omega(u,t) = 0,$

We call (μ, ν, ω)an Neutrosophic Norm on N.

Example 2.2:
Let $(N, \|\cdot\|)$be a normed space μ_0, ν_0, ω_0 and $N \times \mathbb{R}$ be F-sets on defined by
$\mu_0(u,t) = \begin{Bmatrix} 0 & t \leq 0 \\ \frac{t}{t+\|u\|} & t > 0 \end{Bmatrix}$, $\nu_0(u,t) = \begin{Bmatrix} 0 & t \leq 0 \\ \frac{\|u\|}{t+\|u\|} & t > 0 \end{Bmatrix}$and
$\omega_0(u,t) = \begin{Bmatrix} 0 & t \leq 0 \\ \frac{\|u\|}{t} & t > 0 \end{Bmatrix}$. Then (μ_0, ν_0, ω_0) is NN on N.

Definition 2.3:
A sequence (u_n) in (N, μ, ν, ω) is said to be convergent to $a \in N$ and denoted by $u_n \to a$ if for every $\varepsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(u_n - a, t) > 1 - \varepsilon$, $\nu(u_n - a, t) < \varepsilon$ and $\omega(u_n - a, t) < \varepsilon$, for all $n \geq n_0$.

Definition 2.4:
A sequence (u_n)in $(N, \mu, \nu, \omega)$is said to be Cauchy if for every $\varepsilon > 0$ and t >0, there exists $n_0 \in \mathbb{N}$ such that $\mu(u_k - u_n, t) > 1 - \varepsilon, \nu(u_k - u_n, t) < \varepsilon$and
$\omega(u_k - u_n, t) < \varepsilon$,for all $n, k \geq n_0$.
Every convergent sequence is Cauchy in NNS.

Definition 2.5:
A sequence (u_n) in $(N, \mu, \nu, \omega)$is called q-bounded if $\lim_{t\to\infty} \inf_{n\in\mathbb{N}} \mu(u_n, 1) = 1$, $\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \nu(u_n, 1) = 0$ and
$\lim\limits_{t\to\infty} \sup\limits_{n\in\mathbb{N}} \omega(u_n, 1) = 0$.

3. Main results:

Now, we introduce logarithmic summability in NNS and prove corresponding Tauberian theorems.

Definition 3.1:
Let sequence (u_n) be in (N, μ, ν, ω). Logarithmic mean τ_n of (u_n) is defined by $\tau_n = \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{u_k}{k}$ and $\ell_n = \sum_{k=1}^{n} \frac{1}{k}$. (u_n) is said to be logarithmic summable to $a \in N$ if $\lim_{n\to\infty} \tau_n = a$.
Following theorem shows that convergence yields logarithmic summability in NNS.

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Theorem 3.2:
Let sequence (u_n) be in (N, μ, ν, ω). If (u_n) is convergent to $a \in N$, then (u_n) is logarithmic summable to a.
Proof:
Let sequence (u_n) converge to $a \in N$.Fix $t > 0$. For $\varepsilon > 0$

- There exists $n_0 \in \mathbb{N}$ such that $\mu(u_n - a, t) > 1 - \varepsilon$, $\nu(u_n - a, t) < \varepsilon$ and $\omega(u_n - a, t) < \varepsilon$, for $n \geq n_0$.
- There exists $n_1 \in \mathbb{N}$ such that $\mu\left(\sum_{k=1}^{n_0} \frac{u_{k-a}}{k}, \frac{\ell_n t}{2}\right) > 1 - \varepsilon$, $\nu\left(\sum_{k=1}^{n_0} \frac{u_{k-a}}{k}, \frac{\ell_n t}{2}\right) < \varepsilon$and $\omega\left(\sum_{k=1}^{n_0} \frac{u_{k-a}}{k}, \frac{\ell_n t}{2}\right) < \varepsilon$, for $n > n_1$. Since, we have

$$\lim_{n\to\infty} \mu\left(\sum_{k=1}^{n_0} \frac{u_{k-a}}{k}, \frac{\ell_n t}{2}\right) = 1, \lim_{n\to\infty} \nu\left(\sum_{k=1}^{n_0} \frac{u_{k-a}}{k}, \frac{\ell_n t}{2}\right) = 0 \ and$$

$$\lim_{n\to\infty} \omega\left(\sum_{k=1}^{n_0} \frac{u_{k-a}}{k}, \frac{\ell_n t}{2}\right) = 0.$$

Hence, we get

$$\begin{aligned}
\mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k}{k} - a, t\right) &= \mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k - a}{k}, t\right) = \mu\left(\sum_{k=1}^{n}\frac{u_k - a}{k}, \ell_n t\right) \\
&\geq min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right), \mu\left(\sum_{k=n_0+1}^{n}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right)\right\} \\
&\geq min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right), \mu\left(\sum_{k=n_0+1}^{n}\frac{u_k - a}{k}, \frac{(\ell_n - \ell_{n_0})t}{2}\right)\right\} \\
&\geq min\left\{\begin{matrix}\mu\left(\sum_{k=1}^{n_0}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right), \mu\left(\frac{u_{n_0+1} - a}{n_0 + 1}, \frac{t}{2(n_0 + 1)}\right), \\ , \dots \mu\left(\frac{u_n - a}{n}, \frac{t}{2n}\right)\end{matrix}\right\} \\
&= min\left\{\begin{matrix}\mu\left(\sum_{k=1}^{n_0}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right), \mu\left(u_{n_0+1} - a, \frac{t}{2}\right), \\ , \dots \mu\left(u_n - a, \frac{t}{2n}\right)\end{matrix}\right\} > 1 - \varepsilon,
\end{aligned}$$

$$\begin{aligned}
\nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k}{k} - a, t\right) &= \nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k - a}{k}, t\right) = \nu\left(\sum_{k=1}^{n}\frac{u_k - a}{k}, \ell_n t\right) \\
&\leq max\left\{\nu\left(\sum_{k=1}^{n_0}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right), \nu\left(\sum_{k=n_0+1}^{n}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right)\right\} \\
&\leq max\left\{\nu\left(\sum_{k=1}^{n_0}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right), \nu\left(\sum_{k=n_0+1}^{n}\frac{u_k - a}{k}, \frac{(\ell_n - \ell_{n_0})t}{2}\right)\right\} \\
&\leq max\left\{\begin{matrix}\nu\left(\sum_{k=1}^{n_0}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right), \nu\left(\frac{u_{n_0+1} - a}{n_0 + 1}, \frac{t}{2(n_0 + 1)}\right), \\ , \dots \nu\left(\frac{u_n - a}{n}, \frac{t}{2n}\right)\end{matrix}\right\} \\
&= max\left\{\begin{matrix}\nu\left(\sum_{k=1}^{n_0}\frac{u_k - a}{k}, \frac{\ell_n t}{2}\right), \nu\left(u_{n_0+1} - a, \frac{t}{2}\right), \\ , \dots \nu\left(u_n - a, \frac{t}{2n}\right)\end{matrix}\right\} < \varepsilon \ and
\end{aligned}$$

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$$\omega\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k}{k}-a,t\right)=\omega\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k-a}{k},t\right)=\omega\left(\sum_{k=1}^{n}\frac{u_k-a}{k},\ell_n t\right)$$
$$\leq max\left\{\omega\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\omega\left(\sum_{k=n_0+1}^{n}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right)\right\}$$
$$\leq max\left\{\omega\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\omega\left(\sum_{k=n_0+1}^{n}\frac{u_k-a}{k},\frac{(\ell_n-\ell_{n_0})t}{2}\right)\right\}$$
$$\leq max\left\{\begin{array}{c}\omega\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\omega\left(\frac{u_{n_0+1}-a}{n_0+1},\frac{t}{2(n_0+1)}\right),\\ ,\ldots\omega\left(\frac{u_n-a}{n},\frac{t}{2n}\right)\end{array}\right\}$$
$$=max\left\{\begin{array}{c}\omega\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\omega\left(u_{n_0+1}-a,\frac{t}{2}\right),\\ ,\ldots\omega\left(u_n-a,\frac{t}{2n}\right)\end{array}\right\}<\varepsilon.\ .$$

Whenever $n > max\{n_0, n_1\}$, which completes the proof.
Logarithmic summability does not imply convergence in NNS by the next example.

Example 3.3:
Take $(u_n) = ((-1)^n)$ in NNS $(\mathbb{R}, \mu_0, \nu_0, \omega_0)$ where μ_0, ν_0 and ω_0 are as in Example (2.2). Sequence $(u_n)$is logarithmic summable to 0 in view of theorem (3.13), but it is not convergent.
We now give some Tauberian condition for which logarithmic summability yields convergence in NNS.

Theorem 3.4:
Let sequence (u_n) be in (N, μ ,ν, ω). If (u_n) is logarithmic summable to a $\in N$, then it converges to 'a' if and only if for each $t > 0$

$$\sup_{\lambda>1}\lim_{n\to\infty}\mu\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),t\right)=1, \qquad (3.4.1)$$

$$\inf_{\lambda>1}\lim_{n\to\infty}\nu\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),t\right)=0, \qquad (3.4.2)$$

$$\inf_{\lambda>1}\lim_{n\to\infty}\omega\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),t\right)=0. \qquad (3.4.3)$$

Proof:
Necessity. Let (u_n) converge to a. For all $\lambda > 1$ and large enough n, that is when $[n^\lambda] > n$,we can write

$$u_n-\tau_n=\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right)-\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{(u_k-u_n)}{k}. \qquad (3.4.4)$$

Since (τ_n) is Cauchy, for each $t > 0$ we have
$\lim_{n\to\infty}\mu\left(\tau_{[n^\lambda]}-\tau_n,t\right)=1, \lim_{n\to\infty}\nu\left(\tau_{[n^\lambda]}-\tau_n,t\right)=0$and
$\lim_{n\to\infty}\omega\left(\tau_{[n^\lambda]}-\tau_n,t\right)=0.$

Hence, for sufficiently large n such that $\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\leq\frac{2\lambda}{\lambda-1}$ is satisfied, we have

$$\mu\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),t\right)=\mu\left(\tau_{[n^\lambda]}-\tau_n,\frac{t}{\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}}\right)\geq\mu\left(\tau_{[n^\lambda]}-\tau_n,\frac{t}{\frac{2\lambda}{\lambda-1}}\right)\to 1,$$

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$$\nu\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),t\right)=\nu\left(\tau_{[n^\lambda]}-\tau_n,\frac{t}{\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}}\right)$$

$$\leq \nu\left(\tau_{[n^\lambda]}-\tau_n,\frac{t}{\frac{2\lambda}{\lambda-1}}\right)\to 0 \quad \text{and}$$

$$\omega\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),t\right)=\omega\left(\tau_{[n^\lambda]}-\tau_n,\frac{t}{\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}}\right)$$

$$\leq \omega\left(\tau_{[n^\lambda]}-\tau_n,\frac{t}{\frac{2\lambda}{\lambda-1}}\right)\to 0.\,(n\to\infty)$$

Revealing that $\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right)\to 0$.So, by equation (3.4.3). We conclude

$$\lim_{n\to\infty}\mu\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),t\right)=1,$$

$$\lim_{n\to\infty}\nu\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),t\right)=0 \text{ and}$$

$$\lim_{n\to\infty}\omega\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),t\right)=0.$$

Which means that (3.4.1), (3.4.2) and (3.4.3) are satisfied.
Sufficiency. Let condition (3.4.1), (3.4.2) and (3.4.3) be satisfied.
Let $t>0$ be fixed. For $\varepsilon>0$ we have:

- There exist $\lambda>1$ and $n_0\in\mathbb{N}$ such that $\mu\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),\frac{t}{3}\right)>1-\varepsilon$

$\nu\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),\frac{t}{3}\right)<\varepsilon$ and $\omega\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),\frac{t}{3}\right)<\varepsilon$,
for$n>n_0$.

- There exist $n_1\in\mathbb{N}$ such that $\mu\left(\tau_n-a,\frac{t}{3}\right)>1-\varepsilon$, $\nu\left(\tau_n-a,\frac{t}{3}\right)<\varepsilon$ and

$\omega\left(\tau_n-a,\frac{t}{3}\right)<\varepsilon$, for $n>n_1$.

- There exist $n_2\in\mathbb{N}$ such that $\mu\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),\frac{t}{3}\right)>1-\varepsilon,$

$\nu\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),\frac{t}{3}\right)<\varepsilon$ and$\omega\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),\frac{t}{3}\right)<\varepsilon$.

For $n>n_2$, since $\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right)\to 0$.

Hence, by equation (3.4.4), we get

$$\mu(\tau_n-a,t)=\mu(u_n-\tau_n+\tau_n-a,t)$$

$$=\mu\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right)-\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right)+\tau_n-a,t\right)$$

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$$\geq min\begin{Bmatrix} \mu\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),\frac{t}{3}\right), \\ \mu\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),\frac{t}{3}\right), \\ \mu\left(\tau_n-a,\frac{t}{3}\right) \end{Bmatrix} > 1-\varepsilon,$$

$$\nu(\tau_n-a,t)=\nu(u_n-\tau_n+\tau_n-a,t)$$

$$=\nu\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right)-\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right)+\tau_n-a,t\right)$$

$$\leq max\begin{Bmatrix} \nu\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),\frac{t}{3}\right), \\ \nu\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),\frac{t}{3}\right), \\ \nu\left(\tau_n-a,\frac{t}{3}\right) \end{Bmatrix} < \varepsilon \quad \text{and}$$

$$\omega(\tau_n-a,t)=\omega(u_n-\tau_n+\tau_n-a,t)$$

$$=\omega\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right)-\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right)+\tau_n-a,t\right)$$

$$\leq max\begin{Bmatrix} \omega\left(\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(\tau_{[n^\lambda]}-\tau_n\right),\frac{t}{3}\right), \\ \omega\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),\frac{t}{3}\right), \\ \omega\left(\tau_n-a,\frac{t}{3}\right) \end{Bmatrix} < \varepsilon$$

For $n > max\{n_0, n_1, n_2\}$, which completes the proof.

Theorem 3.5:
Let sequence (u_n) be in (N, μ, ν, ω). If (u_n) is logarithmic summable to $a \in N$, then it converges to a if and only if for each $t > 0$

$$\sup_{0<\lambda<1} \lim_{n\to\infty} \mu\left(\frac{1}{\ell_n-\ell_{[n^\lambda]}}\sum_{k=[n^\lambda]+1}^{n}\left(\frac{u_k-u_n}{k}\right),t\right) = 1,$$

$$\inf_{0<\lambda<1} \lim_{n\to\infty} \nu\left(\frac{1}{\ell_n-\ell_{[n^\lambda]}}\sum_{k=[n^\lambda]+1}^{n}\left(\frac{u_k-u_n}{k}\right),t\right) = 0 \quad \text{and}$$

$$\inf_{0<\lambda<1} \lim_{n\to\infty} \omega\left(\frac{1}{\ell_n-\ell_{[n^\lambda]}}\sum_{k=[n^\lambda]+1}^{n}\left(\frac{u_k-u_n}{k}\right),t\right) = 0.$$

Proof:
The proof is done similarly to that of theorem (3.4) by using equation

$$u_n-\tau_n=\frac{\ell_{[n^\lambda]}}{\ell_n-\ell_{[n^\lambda]}}\left(\tau_n-\tau_{[n^\lambda]}\right)+\frac{1}{\ell_n-\ell_{[n^\lambda]}}\sum_{k=[n^\lambda]+1}^{n}\left(\frac{u_k-u_n}{k}\right)(0<\lambda<1)\text{instead of } (3.4.3)$$

Now, we introduce the concept of slow oscillation with respect to logarithmic summability in NNS.

Definition 3.6:
Let (u_n) in (N, μ, ν, ω) is said to be slowly oscillating with respect to logarithmic summability if

$$\sup_{0<\lambda<1} \liminf_{n\to\infty} \min_{n<k\leq[n^\lambda]} \mu(u_k-u_n,t)=1, \quad (3.6.1)$$

$$\inf_{0<\lambda<1} \liminf_{n\to\infty} \min_{n<k\leq[n^\lambda]} \nu(u_k-u_n,t)=0, \quad (3.6.2)$$

$$\inf_{0<\lambda<1} \liminf_{n\to\infty} \min_{n<k\leq[n^\lambda]} \omega(u_k-u_n,t)=0, \quad (3.6.3)$$

for each $t > 0$. " $\sup_{\lambda>1}$ "in (3.6.1), " $\inf_{\lambda>1}$ " in (3.6.2) and " $\inf_{\lambda>1}$ " in (3.6.3) can be replaced by " $\lim_{\lambda\to 1^+}$ ".

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A sequence (u_n) in (N, μ, ν, ω) is slowly oscillating with respect to logarithmic summability if for each $\varepsilon > 0$and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(u_k - u_n, t) > 1 - \varepsilon$, $\nu(u_k - u_n, t) > \varepsilon$ and $\omega(u_k - u_n, t) < \varepsilon$, whenever $n_0 \le n < k \le [n^\lambda]$.

Theorem 3.7:
Let sequence (u_n) be in (N, μ, ν, ω). For $t > 0$, condition (3.6.1), (3.6.2) and (3.6.3) are equivalent to

$$\sup_{0<\lambda<1} \liminf_{n\to\infty} \min_{[n^\lambda]<k\le n} \mu(u_k - u_n, t) = 1, \quad (3.7.1)$$

$$\inf_{0<\lambda<1} \liminf_{n\to\infty} \min_{[n^\lambda]<k\le n} \nu(u_k - u_n, t) = 0, \quad (3.7.2)$$

$$\inf_{0<\lambda<1} \liminf_{n\to\infty} \min_{[n^\lambda]<k\le n} \omega(u_k - u_n, t) = 0, \quad (3.7.3)$$

respectively" $\sup_{0<\lambda<1}$ " in (3.7.1), " $\inf_{0<\lambda<1}$ " in (3.7.2) and " $\inf_{0<\lambda<1}$ " in (3.7.3) can be replaced by " $\lim_{\lambda\to 1^-}$ ".

Example 3.8:
Consider NNS $(\mathbb{R}, \mu_0, \nu_0, \omega_0)$, where μ_0, ν_0 and ω_0 are as in Example (2.2).
$u_n = \sum_{j=1}^{n} \frac{1}{j\, In\, j}$is slowly oscillating with respect to logarithmic summability by the calculations below:
Fix $t > 0$. For $\varepsilon > 0$,take $\lambda = e^{\frac{t\varepsilon}{1-\varepsilon}}$. Then for $n_0 < n < k \le [n^\lambda]$, we have

$$\mu_0(u_k - u_n, t) = \frac{t}{t + |u_k - u_n|} > \frac{t}{t + \frac{t\varepsilon}{1-\varepsilon}} = 1 - \varepsilon,$$

$$\nu_0(u_k - u_n, t) = \frac{|u_k - u_n|}{t + |u_k - u_n|} < \frac{\frac{t\varepsilon}{1-\varepsilon}}{t + \frac{t\varepsilon}{1-\varepsilon}} = \varepsilon \quad and$$

$$\omega_0(u_k - u_n, t) = \frac{|u_k - u_n|}{t} < \frac{\frac{t\varepsilon}{1-\varepsilon}}{t} = \varepsilon.$$

Since $|u_k - u_n| = \sum_{j=n+1}^{n} \frac{1}{j\, In\, j} < \int_n^k \frac{du}{u\, In\, u} \le In\left(\frac{Ink}{Inn}\right) \le In\, \lambda = \frac{t\varepsilon}{1-\varepsilon}$.

Theorem 3.9:
Let sequence (u_n) be in (N, μ, ν, ω). If (u_n) is slowly oscillating with respect to logarithmic summability then (3.4.1), (3.4.2) and (3.4.3) are satisfied.

Proof:
A sequence (u_n) in (N,μ, ν, ω) is slowly oscillating with respect to logarithmic summability. Fix $t > 0$.For $\varepsilon > 0$ there exists $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that
$\mu(u_k - u_n, t) > 1 - \varepsilon$, $\nu(u_k - u_n, t) > \varepsilon$, $\omega(u_k - u_n, t) < \varepsilon$, when ever

$$n_0 \le n < k \le [n^\lambda].$$

$$\begin{aligned}
\mu\left(\frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \left(\frac{u_k - u_n}{k}\right), t\right) &= \mu\left(\sum_{k=n+1}^{[n^\lambda]} \left(\frac{u_k - u_n}{k}\right), (\ell_{[n^\lambda]} - \ell_n)t\right) \\
&\ge min\left\{\mu\left(\frac{u_{n+1} - u_n}{n+1}, \frac{t}{n+1}\right), \dots, \mu\left(\frac{u_{[n^\lambda]} - u_n}{[n^\lambda]}, \frac{t}{[n^\lambda]}\right)\right\} \\
&= min\left\{\mu(u_{n+1} - u_n, t), \dots, \mu\left(u_{[n^\lambda]} - u_n, t\right)\right\} \\
&> 1 - \varepsilon,
\end{aligned}$$

$$\begin{aligned}
\nu\left(\frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \left(\frac{u_k - u_n}{k}\right), t\right) &= \nu\left(\sum_{k=n+1}^{[n^\lambda]} \left(\frac{u_k - u_n}{k}\right), (\ell_{[n^\lambda]} - \ell_n)t\right) \\
&\le max\left\{\nu\left(\frac{u_{n+1} - u_n}{n+1}, \frac{t}{n+1}\right), \dots, \nu\left(\frac{u_{[n^\lambda]} - u_n}{[n^\lambda]}, \frac{t}{[n^\lambda]}\right)\right\} \\
&= min\left\{\nu(u_{n+1} - u_n, t), \dots, \nu\left(u_{[n^\lambda]} - u_n, t\right)\right\} \\
&< \varepsilon \text{and}
\end{aligned}$$

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$$\omega\left(\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),t\right)=\omega\left(\sum_{k=n+1}^{[n^\lambda]}\left(\frac{u_k-u_n}{k}\right),(\ell_{[n^\lambda]}-\ell_n)t\right)$$
$$\le max\left\{\omega\left(\frac{u_{n+1}-u_n}{n+1},\frac{t}{n+1}\right),\dots,\omega\left(\frac{u_{[n^\lambda]}-u_n}{[n^\lambda]},\frac{t}{[n^\lambda]}\right)\right\}$$
$$=min\left\{\omega(u_{n+1}-u_n,t),\dots,\omega\left(u_{[n^\lambda]}-u_n,t\right)\right\}$$
$$<\ \varepsilon.\text{For}\quad n\ge n_0\ .$$

In view of Theorem (3.4) and Theorem (3.9) we give the following Tauberian theorem.

Theorem 3.10:
Let sequence (u_n) be in (N, μ, ν, ω). If (u_n) is logarithmic summable to $a\in N$ and slowly oscillating with respect to logarithmic summability, then (u_n) converges to a.

Theorem 3.11:
Let sequence (u_n) be in (N, μ, ν, ω). If $\{n\ \text{In}\ n(u_n-u_{n-1})\}$ is q-bounded then (u_n) is slowly oscillating with respect to logarithmic summability.

Proof:
Let $\{n\ \text{In}\ n(u_n-u_{n-1})\}$ is q-bounded. In view of Definition (2.5), for given $\varepsilon>0$ there exist $M_\varepsilon>0$ such that $t>M_\varepsilon\Rightarrow \inf_{n\in\mathbb{N}}\mu(n\ \text{In}\ n(u_n-u_{n-1}),t)>1-\varepsilon$,

$$\sup_{n\in\mathbb{N}}\nu(n\ \text{In}\ n(u_n-u_{n-1}),t)<\varepsilon\quad and\quad \sup_{n\in\mathbb{N}}w(n\ \text{In}\ n(u_n-u_{n-1}),t)<\varepsilon.$$

For every $t>0$, choose $\lambda<1+\frac{t}{M_\varepsilon}$, then $n_0<n<k\le[n^\lambda]$ we have

$$\mu(u_k-u_n,t)=\mu\left(\sum_{j=n+1}^{k}(u_j-u_{j-1}),t\right)\ge\min_{n+1\le j\le k}\mu\left((u_j-u_{j-1}),\frac{t}{j(\ell_k-\ell_n)}\right)$$
$$=\min_{n+1\le j\le k}\mu\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t\ \text{In}\ j}{(\ell_k-\ell_n)}\right)\ge\min_{n+1\le j\le k}\mu\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t\ \text{In}\ n}{(\ell_k-\ell_n)}\right)$$
$$\ge\min_{n+1\le j\le k}\mu\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t}{\frac{\text{Ink}}{\text{Inn}}-1}\right)\ge\min_{n+1\le j\le k}\mu\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t}{\lambda-1}\right)$$
$$\ge\inf_{n\in\mathbb{N}}\mu\left(n\ \text{In}\ n(u_n-u_{n-1}),\frac{t}{\lambda-1}\right)>1-\varepsilon,$$

$$\nu(u_k-u_n,t)=\nu\left(\sum_{j=n+1}^{k}(u_j-u_{j-1}),t\right)\le\max_{n+1\le j\le k}\nu\left((u_j-u_{j-1}),\frac{t}{j(\ell_k-\ell_n)}\right)$$
$$=\max_{n+1\le j\le k}\nu\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t\ \text{In}\ j}{(\ell_k-\ell_n)}\right)\le\max_{n+1\le j\le k}\nu\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t\ \text{In}\ n}{(\ell_k-\ell_n)}\right)$$
$$\le\max_{n+1\le j\le k}\nu\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t}{\frac{\text{Ink}}{\text{Inn}}-1}\right)\le\max_{n+1\le j\le k}\nu\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t}{\lambda-1}\right)$$
$$\le\sup_{n\in\mathbb{N}}\nu\left(n\ \text{In}\ n(u_n-u_{n-1}),\frac{t}{\lambda-1}\right)<\varepsilon\text{and}$$

$$\omega(u_k-u_n,t)=\omega\left(\sum_{j=n+1}^{k}(u_j-u_{j-1}),t\right)\le\max_{n+1\le j\le k}\omega\left((u_j-u_{j-1}),\frac{t}{j(\ell_k-\ell_n)}\right)$$
$$=\max_{n+1\le j\le k}\omega\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t\ \text{In}\ j}{(\ell_k-\ell_n)}\right)\le\max_{n+1\le j\le k}\omega\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t\ \text{In}\ n}{(\ell_k-\ell_n)}\right)$$
$$\le\max_{n+1\le j\le k}\omega\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t}{\frac{\text{Ink}}{\text{Inn}}-1}\right)\le\max_{n+1\le j\le k}\omega\left(j\ \text{In}\ j(u_j-u_{j-1}),\frac{t}{\lambda-1}\right)$$
$$\le\sup_{n\in\mathbb{N}}\omega\left(n\ \text{In}\ n(u_n-u_{n-1}),\frac{t}{\lambda-1}\right)<\varepsilon.$$

Hence, (u_n) is slowly oscillating with respect to logarithmic summability.
By theorem (3.10) and theorem (3.11), we conclude following Tauberian theorem.

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Theorem 3.12:
Let sequence (u_n) be in (N, μ, ν, ω). If (u_n) is logarithmic summable to $a \in N$ and $\{n \ln n(u_n - u_{n-1})\}$ is q-bounded then (u_n) converges to a.
Now, we prove a comparison theorem.

Theorem 3.13:
Let sequence (u_n) be in (N, μ, ν, ω). If (u_n) is Cesarosummable to $a \in N$, then (u_n) is logarithmic summable to a.

Proof:
Let (u_n) be Cesaro summable to $a \in N$.
Then, Cesaro means $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} u_k$ converges to a and $\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a \to a$ by Theorem (3.2) with the agreement $\sigma_0 = 0$. Fix $t > 0$. For $\varepsilon > 0$

- There exist $n_0 \in \mathbb{N}$ such that $\mu\left(\sigma_n - a, \frac{t}{2}\right) > 1 - \varepsilon$, $\nu\left(\sigma_n - a, \frac{t}{2}\right) < \varepsilon$ and $\omega\left(\sigma_n - a, \frac{t}{2}\right) < \varepsilon$, for $n > n_0$
- There exist $n_1 \in \mathbb{N}$ such that $\mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right) > 1 - \varepsilon$, $\nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right) < \varepsilon$and$\omega\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right) < \varepsilon$. Whenever $n > n_1$.
- There exist $n_2 \in \mathbb{N}$ such that $\mu\left(a\frac{(\ell_n - 1)t}{2}\right) > 1 - \varepsilon$, $\nu\left(a\frac{(\ell_n - 1)t}{2}\right) < \varepsilon$ and $\nu\left(a\frac{(\ell_n - 1)t}{2}\right) < \varepsilon$. Whenever $n > n_1$, since $\lim_{n\to\infty}\mu\left(a\frac{(\ell_n - 1)t}{2}\right) = 1$, $\lim_{n\to\infty}\nu\left(a\frac{(\ell_n - 1)t}{2}\right) = 0$and $\lim_{n\to\infty}\omega\left(a\frac{(\ell_n - 1)t}{2}\right) = 0$.

Then, we have

$$\mu(\tau_n - a, t) = \mu\left(\frac{\sigma_n}{\ell_n} + \frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, t\right)$$
$$\geq \min\left\{\mu\left(\frac{\sigma_n}{\ell_n}, \frac{t}{2}\right),\ \mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$\geq \min\left\{\mu\left(\sigma_n, \frac{\ell_n t}{2}\right),\ \mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$\geq \min\left\{\mu\left(\sigma_n - a, \frac{t}{2}\right), \mu\left(a\frac{(\ell_n - 1)t}{2}\right), \mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$> 1 - \varepsilon,$$
$$\nu(\tau_n - a, t) = \nu\left(\frac{\sigma_n}{\ell_n} + \frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, t\right)$$
$$\leq \max\left\{\nu\left(\frac{\sigma_n}{\ell_n}, \frac{t}{2}\right),\ \nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$\leq \max\left\{\nu\left(\sigma_n, \frac{\ell_n t}{2}\right),\ \nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$\leq \max\left\{\nu\left(\sigma_n - a, \frac{t}{2}\right), \nu\left(a\frac{(\ell_n - 1)t}{2}\right), \nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$< \varepsilon \quad \text{and}$$
$$\omega(\tau_n - a, t) = \omega\left(\frac{\sigma_n}{\ell_n} + \frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, t\right)$$
$$\leq \max\left\{\omega\left(\frac{\sigma_n}{\ell_n}, \frac{t}{2}\right),\ \omega\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$\leq \max\left\{\omega\left(\sigma_n, \frac{\ell_n t}{2}\right),\ \omega\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$\leq \max\left\{\omega\left(\sigma_n - a, \frac{t}{2}\right), \omega\left(a\frac{(\ell_n - 1)t}{2}\right), \omega\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\}$$
$$< \varepsilon.$$

Whenever $n > \max\{n_0, n_1, n_2\}$, which completes the proof.

Example 3.14:
Consider sequence $(u_n) = ((-1)^n)$ in NNS $(\mathbb{R}, \mu_0, \nu_0, \omega_0)$, where μ_0, ν_0 and ω_0 are as in Example (2.2). Then, since

$$\lim_{n\to\infty}\mu_0(\tau_{2n+1}, t) = \lim_{n\to\infty}\mu_0\left(-\frac{1}{\ell_{2n+1}}, t\right) = \lim_{n\to\infty}\frac{t}{t + \left|-\frac{1}{\ell_{2n+1}}\right|} = 1,$$

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$$\lim_{n\to\infty} \nu_0(\tau_{2n+1},t) = \lim_{n\to\infty} \nu_0\left(-\frac{1}{\ell_{2n+1}},t\right) = \lim_{n\to\infty} \frac{\left|-\frac{1}{\ell_{2n+1}}\right|}{t+\left|-\frac{1}{\ell_{2n+1}}\right|} = 0 \quad and$$

$$\lim_{n\to\infty} \omega_0(\tau_{2n+1}t) = \lim_{n\to\infty} \omega_0\left(-\frac{1}{\ell_{2n+1}},t\right) = \lim_{n\to\infty} \frac{\left|-\frac{1}{\ell_{2n+1}}\right|}{t} = 0.$$

We have $\tau_{2n+1} \to 0$ and since,

$$\lim_{n\to\infty} \mu_0(\sigma_{2n},t) = \lim_{n\to\infty} \mu_0(0,t) = \lim_{n\to\infty} \frac{t}{t+0} = 1,$$

$$\lim_{n\to\infty} \nu_0(\sigma_{2n},t) = \lim_{n\to\infty} \nu_0(0,t) = \lim_{n\to\infty} \frac{0}{t+0} = 0 \quad \text{and}$$

$$\lim_{n\to\infty} \omega_0(\sigma_{2n},t) = \lim_{n\to\infty} \omega_0(0,t) = \lim_{n\to\infty} \frac{0}{t} = 0.$$

We have $\sigma_{2n} \to 0$, which implies that $\lim_{n\to\infty}\tau_n = 0$.

So, $(u_n)$is logarithmic summable to 0. But sequence $(u_n)$is not Cesaro summable.

We note that converse of Theorem (3.13) is true under the condition $\text{Inn}(\tau_n - a) \to 0$, which can be seen by the following:

$$\mu(\tau_n - a,t) = \mu\left(\ell_n(\tau_n - a) - \tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}\ell_k(\tau_k - a),t\right)$$
$$\geq min\left\{\mu\left(\ell_n(\tau_n - a),\tfrac{t}{2}\right),\mu\left(\tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}\ell_k(\tau_k - a),\tfrac{t}{2}\right)\right\} \to 1 \text{as } n \to \infty,$$

$$\nu(\tau_n - a,t) = \nu\left(\ell_n(\tau_n - a) - \tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}\ell_k(\tau_k - a),t\right)$$
$$\leq max\left\{\nu\left(\ell_n(\tau_n - a),\tfrac{t}{2}\right),\nu\left(\tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}\ell_k(\tau_k - a),\tfrac{t}{2}\right)\right\} \to 1 \text{as } n \to \infty,$$

$$\omega(\tau_n - a,t) = \omega\left(\ell_n(\tau_n - a) - \tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}\ell_k(\tau_k - a),t\right)$$
$$\leq max\left\{\omega\left(\ell_n(\tau_n - a),\tfrac{t}{2}\right),\omega\left(\tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}\ell_k(\tau_k - a),\tfrac{t}{2}\right)\right\} \to 1 \text{as } n \to \infty,$$

By Theorem (3.13) and Example (3.14), we see that logarithmic summability method is stronger than Cesarosummability method in summing up sequences in NNS.

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