Some Algebraic structures of Neutrosophic fuzzy sets

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Abstract

The mathematical operations of convergence, association, supplement, arithmetical total, logarithmic item, scalar increase, and exponentiation are the main topics of this article. We show certain important logarithmic features of idempotency, commutativity, associativity, retention, distributivity, and De Morgan's laws over the addition of Neutrosophic fuzzy sets. We also outline new fixations and NFS widening and show some concepts in action. Last but not least, we define a further operation (@)on Neutrosophic fuzzy sets and investigate distributive laws for the case where the responsibilities of $\oplus$, $\otimes$, $\cup$, and $\cap$ are combined.

Keywords: Neutrosophic fuzzy set; Algebraic sum; Algebraic product; Scalar multiplication and Exponentiation operations; Intuitionistic fuzzy set.

1 introduction

The fuzzy set (FS) hypothesis was first put forth by Zadeh[18] in 1965. In this idea, Zadeh only considered the capacity's positive engagement degree. Several current reality disciplines, including grouping research, dynamic problems, clinical discovery, and design recognition, have used the FS hypothesis. Unfortunately, the lack of essential information regarding the capacity's negative enrollment level has led to the failure of the FS hypothesis. By incorporating the negative enrollment level of the capacity, Atanassov filled up the holes in the FS theory. Atanassov cite1 first put forth the intuitionistic fuzzy set (IFS) theory in 1986. The FS hypothesis is intended to be strengthened by the IFS hypothesis. He examined both the capacity's negative enrollment level and its positive participation level in order to evaluate this theory. As a result, the negative participation work and positive enrollment capacity are ≤ 1. In his[2] work, Atanassov characterised certain essential AIFS activities and relationships, such as crossing point, association, supplement, logarithmic aggregate, mathematical item, and so forth, and showed how closely IFSs are related. Numerous experts worked hard to create the IFS hypothesis after it was first presented.[4,5,13,15,16]

F.Smarandache cite8 introduced the neutrosophic set in 1999. Each component had three related characterising capacities, particularly the participation work (T), non-enrollment (F) work, and indeterminacy (I) work, which are all characterised on the universe of talk X and are wholly autonomous from one another. Sometimes it will

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be impossible to make a decision while comparing the characteristic issues. One may surely make decisions when the Neutrosophic set hypothesis has been improved, and the situation's nondeterministic component is the set's capacity for indeterminacy. One may cite[6,7] for examples of how the hypothesis has been used in various fields to manage ambiguous and contradictory data. The origins, nature, and breadth of neutralities, as well as how they interact with idealisation spectra, were all topics of interest to the neutrosophic set, a subset of neutrosophy. The neutrosophic set contains all of the concepts of the old-style fuzzy set, span esteemed fuzzy set, intuitionistic fuzzy set, and so forth.[8,10–12,14,17] for examples of recent work that successfully applied the concept of neutrosophic set.

Because of this, the aim of this study is to create some fundamental algebraic procedures for NFSs and investigate their appealing characteristics.

The document's portion is shown below. In the *Preliminaries* section, we give some basic definitions of IFS and PFS. We demonstrated various algebraic characteristics of NFSs in the section *Some Results on NFSs.* In the section titled *New operation(@) on NFSs,* we defined the term New operation(@) and looked into its algebraic properties. We write the paper's conclusion in the last section.

2 Preliminaries

Some basic concepts linked to the intuitionistic fuzzy set (IFS) and the Neutrosophic fuzzy set (PFS) have been presented here.

Definition 2.1. [1] "A intuitionistic fuzzy set U on a universe X is an object of the form

$$U = \{(\iota, \zeta_U(\hat{\iota}), \varrho_U(\hat{\iota}))\,|\iota \in X\},$$

where $\zeta_U(\hat{\iota}) \in [0,1]$ is called the degree of membership of ι in U, $\varrho_U(\hat{\iota}) \in [0,1]$ is called the degree of non-membership of ι in U, and where $\zeta_U(\hat{\iota})$ and $\varrho_U(\hat{\iota})$ satisfy the following condition:

$$0 \leq \zeta_U(\hat{\iota}) + \varrho_U(\hat{\iota}) \leq 1 \text{ for all } \iota \in X.$$

Let $\pi_U(\iota) = 1 - \zeta_U(\hat{\iota}) - \varrho_U(\hat{\iota})$, then it is usually called the intuitionistic fuzzy index of $\iota \in U$, representing the degree of inderminancy or hesitation of ι to U. It is obvious that $0 \leq \pi_U(\iota) \leq 1$ for every $\iota \in X$."

Definition 2.2. [6] "A Neutrosophic fuzzy set U on a universe ι is an object of the form

$$U = \{(\iota, \zeta_U(\hat{\iota}), \psi_U(\check{\iota}), \varrho_U(\hat{\iota}))\,|\iota \in X\},$$

where $\zeta_U(\hat{\iota}) \in [0,1]$ is called the degree of positive membership of ι in U, $\psi_U(\check{\iota}) \in [0,1]$ is called the degree of neutral membership of ι in U and $\varrho_U(\hat{\iota}) \in [0,1]$ is called the degree of negative membership of ι in U, and where $\zeta_U(\hat{\iota})$, $\psi_U(\check{\iota})$ and $\varrho_U(\hat{\iota})$ satisfy the following condition:

$$0 \leq \zeta_U(\hat{\iota}) + \psi_U(\check{\iota}) + \varrho_U(\hat{\iota}) \leq 3 \text{ for all } \iota \in X."$$

Let $NFS(\iota)$ denotes the set of all the Neutrosophic fuzzy set on a universe ι.

Definition 2.3. [6] "Let ι be a nonempty set and I be the unit interval [0,1]. A Neutrosophic fuzzy set U and V of the form, $U = \{(\iota, \zeta_U(\hat{\iota}), \psi_U(\check{\iota}), \varrho_U(\hat{\iota}))\,|\iota \in X\}$ and $V = \{(\iota, \zeta_V(\hat{\iota}), \psi_V(\check{\iota}), \varrho_V(\hat{\iota}))\,|\iota \in X\}$. Then
•$U \subset V$ iff $\forall \iota \in X$,
$\zeta_U(\hat{\iota}) \leq \zeta_V(\hat{\iota}), \psi_U(\check{\iota}) \leq \psi_V(\check{\iota})$ or $\psi_U(\check{\iota}) \geq \psi_V(\check{\iota}), \varrho_U(\hat{\iota}) \geq \varrho_V(\hat{\iota})$
•$U^C = \{(\iota, \varrho_U(\hat{\iota}), \psi_U(\check{\iota}), \zeta_U(\hat{\iota}))\,|\iota \in X\}$
•$U \cup V = \{(\iota, \max(\zeta_U(\hat{\iota}), \zeta_V(\hat{\iota})), \min(\psi_U(\check{\iota}), \psi_V(\check{\iota})), \min(\varrho_U(\hat{\iota}), \varrho_V(\hat{\iota})))\,|\iota \in X\}$
•$U \cap V = \{(\iota, \min(\zeta_U(\hat{\iota}), \zeta_V(\hat{\iota})), \max(\psi_U(\check{\iota}), \psi_V(\check{\iota})), \max(\varrho_U(\hat{\iota}), \varrho_V(\hat{\iota})))\,|\iota \in X\}$
•$U \oplus V = \{(\iota, \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota}), \psi_U(\check{\iota})\psi_V(\check{\iota}), \varrho_U(\hat{\iota})\varrho_V(\hat{\iota}))\,|\iota \in X\}$
•$U \otimes V = \Big[(\iota, \zeta_U(\hat{\iota})\zeta_V(\hat{\iota}), \psi_U(\check{\iota}) + \psi_V(\check{\iota}) - \psi_U(\check{\iota})\psi_V(\check{\iota}),$

$\varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota}))|\iota \in X\Big]$".

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Definition 2.4. [6] "The scalar multiplication operation over NFSs U of the universe ι is denoted by nU and is defined by

$$nU = \{(\iota, 1 - [1 - \zeta_U(\hat{\iota})]^n, [\psi_U(\check{\iota})]^n, [\varrho_U(\hat{\iota})]^n) \,|\iota \in X\}\,."$$

Definition 2.5. [6] "The exponentiation operation over NFSs U of the universe ι is denoted by nU and is defined by

$$U^n = \{(\iota, [\zeta_U(\hat{\iota})]^n, 1 - [1 - \psi_U(\check{\iota})]^n, 1 - [1 - \varrho_U(\hat{\iota})]^n) \,|\iota \in X\}."$$

Lemma 2.6. [3] *"Let x, y, z be real numbers. Then the following equalities hold:*
$(i)\iota - \min(y, z) = \max(\iota - y, \iota - z)$
$(ii)\iota - \max(y, z) = \min(\iota - y, \iota - z)$
$(iii)\min(\iota, y) - z = \min(\iota - z, y - z)$
$(iv)\max(\iota, y) - z = \max(\iota - z, y - z)$."

Lemma 2.7. [3] *"(i) If U, V, c are real numbers with $U \geq 0$ then the following holds:*

$$\begin{aligned} a.\max(V, c) &= \max(aV, ac), \\ a.\min(V, c) &= \min(aV, ac). \end{aligned}$$

(ii) *For real numbers U, V, c with $U \geq 0$ then addition distributes over the maximum operation and also over the minimum operation:*

$$\begin{aligned} a + \max(V, c) &= \max(a + V, a + c), \\ a + \min(V, c) &= \min(a + V, a + c). \end{aligned}$$

(iii) *For real numbers, the maximum operation is distributive over the minimum operation and vice versa:*

$$\begin{aligned} \max(a, \min(V, c)) &= \min(\max(a, V), \max(a, c)), \\ \min(a, \max(V, c)) &= \max(\min(a, V), \min(a, c))." \end{aligned}$$

3 Some results on Neutrosophic fuzzy sets

In this part, we prove idempotency, commutativity, associativity, absorption, distributivity, and De Morgan's laws over complement as basic algebraic properties of Neutrosophic fuzzy sets.

The theorem below describes the relationship between algebraic sum and algebraic product.

Theorem 3.1. *For every $U, V \in NFS(\iota)$,then $U \otimes V \subseteq U \oplus V$.*

Proof. Let $U \oplus V = \{\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota}), \psi_U(\check{\iota})\psi_V(\check{\iota}), \varrho_U(\hat{\iota})\varrho_V(\hat{\iota})\}$
$U \otimes V = \{\zeta_U(\hat{\iota})\zeta_V(\hat{\iota}), \psi_U(\check{\iota}) + \psi_V(\check{\iota}) - \psi_U(\check{\iota})\psi_V(\check{\iota}), \varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota})\}$.
Assume that,
$\zeta_U(\hat{\iota})\zeta_V(\hat{\iota}) \leq \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})$,
$(i.e)\zeta_U(\hat{\iota})\zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota}) \geq 0$,
$(i.e)\zeta_U(\hat{\iota})(1 - \zeta_V(\hat{\iota})) + \zeta_V(\hat{\iota})(1 - \zeta_U(\hat{\iota})) \geq 0$,
which is true as $0 \leq \zeta_U(\hat{\iota}) \leq 1$ and $0 \leq \zeta_V(\hat{\iota}) \leq 1$.
And
$\psi_U(\check{\iota})\psi_V(\check{\iota}) \leq \psi_U(\check{\iota}) + \psi_V(\check{\iota}) - \psi_U(\check{\iota})\psi_V(\check{\iota})$,
$(i.e)\psi_U(\check{\iota})\psi_V(\check{\iota}) - \psi_U(\check{\iota}) + \psi_V(\check{\iota}) - \psi_U(\check{\iota})\psi_V(\check{\iota}) \geq 0$,
$(i.e)\psi_U(\check{\iota})(1 - \psi_V(\check{\iota})) + \psi_V(\check{\iota})(1 - \psi_U(\check{\iota})) \geq 0$,
which is true as $0 \leq \psi_U(\check{\iota}) \leq 1$ and $0 \leq \psi_V(\check{\iota}) \leq 1$.
And
$\varrho_U(\hat{\iota})\varrho_V(\hat{\iota}) \leq \varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota})$,
$(i.e)\varrho_U(\hat{\iota})\varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota}) \geq 0$,
$(i.e)\varrho_U(\hat{\iota})(1 - \varrho_V(\hat{\iota})) + \varrho_V(\hat{\iota})(1 - \varrho_U(\hat{\iota})) \geq 0$,
which is true as $0 \leq \varrho_U(\hat{\iota}) \leq 1$ and $0 \leq \varrho_V(\hat{\iota}) \leq 1$.
Hence $U \otimes V \subseteq U \oplus V$. □

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Theorem 3.2. *For every $U \in NFS(\iota)$,then*
$(i) U \oplus U \supseteq U$,
$(ii) U \otimes U \subseteq U$.

Proof. (i) $U \oplus U = \left\{\left(\iota, 2\zeta_U(\hat{\imath}) - (\zeta_U(\hat{\imath}))^2, (\psi_U(\breve{\imath}))^2, (\varrho_U(\hat{\imath}))^2\right) | \iota \in X\right\}$.
$2\zeta_U(\hat{\imath}) - (\zeta_U(\hat{\imath}))^2 = \zeta_U(\hat{\imath}) + \zeta_U(\hat{\imath})(1 - \zeta_U(\hat{\imath})) \geq \zeta_U(\hat{\imath})$ for all $\iota \in X$,
and $(\psi_U(\breve{\imath}))^2 \leq \psi_U(\breve{\imath})$ for all $\iota \in X$,
and $(\varrho_U(\hat{\imath}))^2 \leq \varrho_U(\hat{\imath})$ for all $\iota \in X$.
Hence $U \oplus U \supseteq U$.
Similarly, we get (ii) $U \otimes U \subseteq U$. □

Theorem 3.3. *For each U, V and $C \in NFS(\iota)$,then*
$(i) U \oplus V = V \oplus U$,
$(ii) U \otimes V = V \otimes V$,
$(iii)(U \oplus V) \oplus C = U \oplus (V \oplus C)$,
$(iv)(U \otimes V) \otimes C = U \otimes (V \otimes C)$.

Proof. $(i) U \oplus V = \{\zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_V(\hat{\imath}), \psi_U(\breve{\imath})\psi_V(\breve{\imath}), \varrho_U(\hat{\imath})\varrho_V(\hat{\imath})\}$

$= \{\zeta_V(\hat{\imath}) + \zeta_U(\hat{\imath}) - \zeta_V(\hat{\imath})\zeta_U(\hat{\imath}), \psi_V(\breve{\imath})\psi_U(\breve{\imath}), \varrho_V(\hat{\imath})\varrho_U(\hat{\imath})\}$

$= V \oplus U$.
$(ii) U \otimes V$

$= \{\zeta_U(\hat{\imath})\zeta_V(\hat{\imath}), \psi_U(\breve{\imath}) + \psi_V(\breve{\imath}) - \psi_U(\breve{\imath})\psi_V(\breve{\imath}), \varrho_U(\hat{\imath}) + \varrho_V(\hat{\imath}) - \varrho_U(\hat{\imath})\varrho_V(\hat{\imath})\}$

$= \{\zeta_V(\hat{\imath})\zeta_U(\hat{\imath}), \psi_V(\breve{\imath}) + \psi_U(\breve{\imath}) - \psi_V(\breve{\imath})\psi_U(\breve{\imath}), \varrho_V(\hat{\imath}) + \varrho_U(\hat{\imath}) - \varrho_V(\hat{\imath})\varrho_U(\hat{\imath})\}$

$= V \otimes U$.
(iii) Let $(U \oplus V) \oplus C$

$= \Big[x, (\zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_V(\hat{\imath})) + \zeta_C(\hat{\imath}) - (\zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath})$

$- \zeta_U(\hat{\imath})\zeta_V(\hat{\imath}))\zeta_C(\hat{\imath}), \psi_U(\breve{\imath})\psi_V(\breve{\imath})\psi_C(\breve{\imath}), \varrho_U(\hat{\imath})\varrho_V(\hat{\imath})\varrho_C(\hat{\imath}) | \iota \in X\Big]$

$= \Big[x, \zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath}) + \zeta_C(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_V(\hat{\imath})\zeta_C(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_C(\hat{\imath}) - \zeta_V(\hat{\imath})\zeta_C(\hat{\imath})$

$+ \zeta_U(\hat{\imath})\zeta_V(\hat{\imath})\zeta_C(\hat{\imath}), \psi_U(\breve{\imath})\psi_V(\breve{\imath})\psi_C(\breve{\imath}), \varrho_U(\hat{\imath})\varrho_V(\hat{\imath})\varrho_C(\hat{\imath}) | \iota \in X\Big]$

$= \Big[x, \zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath}) + \zeta_C(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_V(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_C(\hat{\imath}) - \zeta_V(\hat{\imath})\zeta_C(\hat{\imath}) +$
$\zeta_U(\hat{\imath})\zeta_V(\hat{\imath})\zeta_C(\hat{\imath}), \psi_U(\breve{\imath})\psi_V(\breve{\imath})\psi_C(\breve{\imath}), \varrho_U(\hat{\imath})\varrho_V(\hat{\imath})\varrho_C(\hat{\imath}) | \iota \in X\Big]$

$U \oplus (V \oplus C)$

$= \Big[x, \zeta_U(\hat{\imath}) + (\zeta_V(\hat{\imath}) + \zeta_C(\hat{\imath}) - \zeta_V(\hat{\imath})\zeta_C(\hat{\imath})) - \zeta_U(\hat{\imath})(\zeta_V(\hat{\imath}) + \zeta_C(\hat{\imath})$

$- \zeta_V(\hat{\imath})\zeta_C(\hat{\imath})), \psi_U(\breve{\imath})\psi_V(\breve{\imath})\psi_C(\breve{\imath}), \varrho_U(\hat{\imath})\varrho_V(\hat{\imath})\varrho_C(\hat{\imath}) | \iota \in X\Big]$

$= \Big[x, \zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath}) + \zeta_C(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_V(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_C(\hat{\imath}) - \zeta_V(\hat{\imath})\zeta_C(\hat{\imath}) +$
$\zeta_U(\hat{\imath})\zeta_V(\hat{\imath})\zeta_C(\hat{\imath}), \psi_U(\breve{\imath})\psi_V(\breve{\imath})\psi_C(\breve{\imath}), \varrho_U(\hat{\imath})\varrho_V(\hat{\imath})\varrho_C(\hat{\imath}) | \iota \in X\Big]$.
Hence $(U \oplus V) \oplus C = U \oplus (V \oplus C)$.
Similarly, we get (iv)$(U \otimes V) \otimes C = U \otimes (V \otimes C)$. □

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Theorem 3.4. *For each $U, V \in NFS(\iota)$,then*
$(i) U \oplus (U \otimes V) \supseteq U$,
$(ii) U \otimes (U \oplus V) \subseteq U$.

Proof. (i) Let $U \oplus (U \otimes V)$

$= \Big[x, \zeta_U(\hat{\imath}) + \zeta_U(\hat{\imath})\zeta_V(\hat{\imath}) - \zeta_U(\hat{\imath})[\zeta_U(\hat{\imath})\zeta_V(\hat{\imath})], \psi_U(\breve{\imath})[\psi_U(\breve{\imath}) + \psi_V(\breve{\imath})$

$- \psi_U(\breve{\imath})\psi_V(\breve{\imath})], \varrho_U(\hat{\imath})[\varrho_U(\hat{\imath}) + \varrho_V(\hat{\imath}) - \varrho_U(\hat{\imath})\varrho_V(\hat{\imath})]|\iota \in X\Big]$

$= \Big[x, \zeta_U(\hat{\imath}) + \zeta_U(\hat{\imath}) + \zeta_U(\hat{\imath})\zeta_V(\hat{\imath})[1 - \zeta_U(\hat{\imath})], \psi_U(\breve{\imath})(1 - [1 - \psi_U(\breve{\imath})]$

$[1 - \psi_V(\breve{\imath})]), \varrho_U(\hat{\imath})\left(1 - [1 - \varrho_U(\hat{\imath})][1 - \varrho_V(\hat{\imath})]\right)|\iota \in X\Big]$

$\geq U$.
Hence $U \oplus (U \otimes V) \supseteq U$.
Similarly, we have $(ii) U \otimes (U \oplus V) \subseteq A$. □

Theorem 3.5. *For each $U, V \in NFS(\iota)$,then*
$(i) U \cup V = V \cup U$,
$(ii) U \cap V = V \cap U$,

Theorem 3.6. *For each $U, V, C \in NFS(\iota)$,then*
$(i) U \oplus (V \cup C) = (U \oplus V) \cup (U \oplus C)$,
$(ii) U \otimes (V \cup C) = (U \otimes V) \cup (U \otimes C)$,
$(iii) U \oplus (V \cap C) = (U \oplus V) \cap (U \oplus C)$,
$(iv) U \otimes (V \cap C) = (U \otimes V) \cap (U \otimes C)$.

Proof. Now we prove (i), and $(ii) - (iv)$ can be proved similarly.
(i) Let $U \oplus (V \cup C)$
$= \Big[x, \zeta_U(\hat{\imath}) + \max(\zeta_V(\hat{\imath}), \zeta_C(\hat{\imath})) - \zeta_U(\hat{\imath}). \max(\zeta_V(\hat{\imath}), \zeta_C(\hat{\imath}))$,
$\psi_U(\breve{\imath}). \max(\psi_V(\breve{\imath}), \psi_C(\breve{\imath})), \varrho_U(\hat{\imath}). \max(\varrho_V(\hat{\imath}), \varrho_C(\hat{\imath}))|\iota \in X\Big]$
$= \Big[x, \max(\zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath}), \zeta_U(\hat{\imath}) + \zeta_C(\hat{\imath})) - \max(\zeta_U(\hat{\imath})\zeta_V(\hat{\imath}), \zeta_U(\hat{\imath})\zeta_C(\hat{\imath}))$,
$\min(\psi_U(\breve{\imath})\psi_V(\breve{\imath}), \psi_U(\breve{\imath})\psi_C(\breve{\imath})), \min(\varrho_U(\hat{\imath})\varrho_V(\hat{\imath}), \varrho_U(\hat{\imath})\varrho_C(\hat{\imath}))|\iota \in X\Big]$,
by Lemma 2.2.
$= \Big[x, \max(\zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_V(\hat{\imath}), \zeta_U(\hat{\imath}) + \zeta_C(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_C(\hat{\imath}))$,
$\min(\psi_U(\breve{\imath})\psi_V(\breve{\imath}), \psi_U(\breve{\imath})\psi_C(\breve{\imath})), \min(\varrho_U(\hat{\imath})\varrho_V(\hat{\imath}), \varrho_U(\hat{\imath})\varrho_C(\hat{\imath}))|\iota \in X\Big]$
$= (U \oplus V) \cup (U \oplus C)$. □

Theorem 3.7. *For each $U, V \in NFS(\iota)$,then*
$(i)(U \cap V) \oplus (U \cup V) = U \oplus V$,
$(ii)(U \cap V) \otimes (U \cup V) = U \otimes V$,
$(iii)(U \oplus V) \cap (U \otimes V) = U \otimes V$,
$(iv)(U \oplus V) \cup (U \otimes V) = U \oplus V$.

Proof. Now we prove (i), and $(ii) - (iv)$ can be proved similarly.
(i) Let $(U \cap V) \oplus (U \cup V)$
$= \Big[x, \min(\zeta_U(\hat{\imath}), \zeta_V(\hat{\imath})) + \max(\zeta_U(\hat{\imath}), \zeta_V(\hat{\imath})) - \min(\zeta_U(\hat{\imath}), \zeta_V(\hat{\imath}))$.
$\max(\zeta_U(\hat{\imath}), \zeta_V(\hat{\imath})), \max(\psi_U(\breve{\imath}), \psi_V(\breve{\imath})) . \min(\psi_U(\breve{\imath}), \psi_V(\breve{\imath}))$,
$\max(\varrho_U(\hat{\imath}), \varrho_V(\hat{\imath})) . \min(\varrho_U(\hat{\imath}), \varrho_V(\hat{\imath}))|\iota \in X\Big]$
$= \{(\iota, \zeta_U(\hat{\imath}) + \zeta_V(\hat{\imath}) - \zeta_U(\hat{\imath})\zeta_V(\hat{\imath}), \psi_U(\breve{\imath})\psi_V(\breve{\imath}), \varrho_U(\hat{\imath})\varrho_V(\hat{\imath}))|\iota \in X\}$
$= U \oplus V$. □

In the following, the operator complement obey the De Morgan's laws for the operation $\oplus, \otimes, \cup, \cap$.

Theorem 3.8. *For each $U, V \in NFS(\iota)$, then*
$(i)(U \oplus V)^C = U^C \otimes V^C$,
$(ii)(U \otimes V)^C = U^C \oplus V^C$,
$(iii)(U \oplus V)^C \subseteq U^C \oplus V^C$,
$(iv)(U \otimes V)^C \supseteq U^C \otimes V^C$.

Proof. We prove $(iii), (iv)$, and $(i), (ii)$ are clear.
$(iii)(U \oplus V)^C = \{\varrho_U(\hat{\iota})\varrho_V(\hat{\iota}), \psi_U(\breve{\iota}) + \psi_V(\breve{\iota}) - \psi_U(\breve{\iota})\psi_V(\breve{\iota}), \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})\}$.
$U^C \oplus V^C = \{(\iota, \varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota}), \psi_U(\breve{\iota})\psi_V(\breve{\iota}), \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})) \,|\iota \in X\}$.
Since $\varrho_U(\hat{\iota})\varrho_V(\hat{\iota}) \leq \varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota})$,
$\psi_U(\breve{\iota}) + \psi_V(\breve{\iota}) - \psi_U(\breve{\iota})\psi_V(\breve{\iota}) \geq \psi_U(\breve{\iota})\psi_V(\breve{\iota})$,
$\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota}) \geq \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})$.
Hence $(U \oplus V)^C \subseteq U^C \oplus V^C$.
$(iv)(U \otimes V)^C$

$= \{(\iota, \varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota}), \psi_U(\breve{\iota})\psi_V(\breve{\iota}), \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})) \,|\iota \in X\}$.
$U^C \otimes V^C$

$= \{\varrho_U(\hat{\iota})\varrho_V(\hat{\iota}), \psi_U(\breve{\iota}) + \psi_V(\breve{\iota}) - \psi_U(\breve{\iota})\psi_V(\breve{\iota}), \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})\}$.
Since $\varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota}) \geq \varrho_U(\hat{\iota})\varrho_V(\hat{\iota})$,
$\psi_U(\breve{\iota})\psi_V(\breve{\iota}) \leq \psi_U(\breve{\iota}) + \psi_V(\breve{\iota}) - \psi_U(\breve{\iota})\psi_V(\breve{\iota})$,
$\zeta_U(\hat{\iota})\zeta_V(\hat{\iota}) \leq \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})$.
Hence $(U \otimes V)^C \supseteq U^C \otimes V^C$. □

Theorem 3.9. *For each $U, V, C \in NFS(\iota)$,then*
$(i)(U^C)^C = U$,
$(ii)(U \cup V)^C = U^C \cap V^C$,
$(iii)(U \cap V)^C = U^C \cup V^C$.

Proof. We prove (ii) only, (i) is clear.
$U \cup V = \{(\iota, \max(\zeta_U(\hat{\iota}), \zeta_V(\hat{\iota})), \min(\psi_U(\breve{\iota}), \psi_V(\breve{\iota})), \min(\varrho_U(\hat{\iota}), \varrho_V(\hat{\iota}))) \,|\iota \in X\}$
$(U \cup V)^C$

$= \{(\iota, \min(\varrho_U(\hat{\iota}), \varrho_V(\hat{\iota})), \min(\psi_U(\breve{\iota}), \psi_V(\breve{\iota})), \max(\zeta_U(\hat{\iota}), \zeta_V(\hat{\iota}))) \,|\iota \in X\}$
$\Rightarrow U^C = \{(\iota, \varrho_U(\hat{\iota}), \psi_U(\breve{\iota}), \zeta_U(\hat{\iota})) \,|\iota \in X\}$.
$V^C = \{(\iota, \varrho_V(\hat{\iota}), \psi_V(\breve{\iota}), \zeta_V(\hat{\iota})) \,|\iota \in X\}$.
$\Rightarrow U^C \cap V^C$

$= \{(\iota, \min(\varrho_U(\hat{\iota}), \varrho_V(\hat{\iota})), \min(\psi_U(\breve{\iota}), \psi_V(\breve{\iota})), \max(\zeta_U(\hat{\iota}), \zeta_V(\hat{\iota}))) \,|\iota \in X\}$.
Hence $(U \cup V)^C = U^C \cap V^C$,
Similarly, we have $(iii)(U \cap V)^C = U^C \cup V^C$. □

Based on the Definition 4 & 5., next we prove the algebraic properties of Neutrosophic fuzzy sets under the operations of scalar multiplication and exponentiation.

Theorem 3.10. *For each $U, V \in NFS(\iota)$,then for any positive number n,*
$(i)n(U + V) = nU \oplus nV, n > 0$,
$(ii)n_1U \oplus n_2U = (n_1 + n_2)U, n_1, n_2 > 0$,
$(iii)(U \otimes V)^n = U^n \otimes V^n, n > 0$,
$(iv)U_1^n \otimes U_2^n = U^{(n_1+n_2)}, n_1, n_2 > 0$,

Proof. For the three NFSs U and V, and $n, n_1, n_2 > 0$, according to definition , we get
$(i)n(U \oplus V)$

$= n\{\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota}), \psi_U(\breve{\iota})\psi_V(\breve{\iota}), \varrho_U(\hat{\iota})\varrho_V(\hat{\iota})\}$

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$= \{1 - [1 - \zeta_U(\hat{\iota})]^n[1 - \zeta_U(\hat{\iota})]^n, [\psi_U(\breve{\iota})\psi_V(\breve{\iota})]^n, [\varrho_U(\hat{\iota})\varrho_V(\hat{\iota})]^n\}$

$= \{1 - [1 - \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})]^n, [\psi_U(\breve{\iota})\psi_V(\breve{\iota})]^n, [\varrho_U(\hat{\iota})\varrho_V(\hat{\iota})]^n\}$
$nU \oplus nV$
$= \{(1 - [1 - \zeta_U(\hat{\iota})]^n, [\psi_U(\breve{\iota})]^n, [\varrho_U(\hat{\iota})]^n) \oplus (1 - [1 - \zeta_V(\hat{\iota})]^n, [\psi_V(\breve{\iota})]^n, [\varrho_V(\hat{\iota})]^n)\}$
$= \Big[(1 - [1 - \zeta_U(\hat{\iota})]^n + 1 - [1 - \zeta_U(\hat{\iota})]^n) - (1 - [1 - \zeta_U(\hat{\iota})]^n)$

$(1 - [1 - \zeta_V(\hat{\iota})]^n), [\psi_U(\breve{\iota})\psi_V(\breve{\iota})]^n, [\varrho_U(\hat{\iota})\varrho_V(\hat{\iota})]^n\Big]$
$= \{1 - [1 - \zeta_U(\hat{\iota})]^n[1 - \zeta_V(\hat{\iota})]^n, [\psi_U(\breve{\iota})\psi_V(\breve{\iota})]^n, [\varrho_U(\hat{\iota})\varrho_V(\hat{\iota})]^n\}$
$= \{1 - [1 - \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})]^n, [\psi_U(\breve{\iota})\psi_V(\breve{\iota})]^n, [\varrho_U(\hat{\iota})\varrho_V(\hat{\iota})]^n\}$
$= n(U \oplus V).$
$(ii) n_1U \oplus n_2V$
$= \Big[1 - [1 - \zeta_U(\hat{\iota})]^{n_1} + 1 - [1 - \zeta_U(\hat{\iota})]^{n_2} - (1 - [1 - \zeta_U(\hat{\iota})]^{n_1})(1 - [1 - \zeta_U(\hat{\iota})]^{n_2}),$
$[\psi_U(\breve{\iota})]^{n_1}[\psi_U(\breve{\iota})]^{n_2}, [\varrho_U(\hat{\iota})]^{n_1}[\varrho_U(\hat{\iota})]^{n_2}\Big]$
$= \{1 - [1 - \zeta_U(\hat{\iota})]^{n_1+n_2}, [\psi_U(\breve{\iota})]^{n_1+n_2}, [\varrho_U(\hat{\iota})]^{n_1+n_2}\}$
$= (n_1 + n_2)U.$
$(iii)(U \otimes V)^n$
$= \Big[(\zeta_U(\hat{\iota})\zeta_V(\hat{\iota}))^n, 1 - [1 - \psi_U(\breve{\iota}) + \psi_V(\breve{\iota}) - \psi_U(\breve{\iota})\psi_V(\breve{\iota})]^n,$
$1 - [1 - \varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota}) - \varrho_U(\hat{\iota})\varrho_V(\hat{\iota})]^n\Big]$
$= \Big[(\zeta_U(\hat{\iota})\zeta_V(\hat{\iota}))^n, 1 - [1 - \psi_U(\breve{\iota})]^n[1 - \psi_V(\breve{\iota})]^n,$
$1 - [1 - \varrho_U(\hat{\iota})]^n[1 - \varrho_V(\hat{\iota})]^n\Big]$
$U^n \otimes V^n$
$= \Big[(\zeta_U(\hat{\iota})\zeta_V(\hat{\iota}))^n, 1 - [1 - \psi_U(\breve{\iota})]^n + 1 - [1 - \psi_V(\breve{\iota})]^n - (1 - [1 - \psi_U(\breve{\iota})]^n) 1 - [1 - \psi_V(\breve{\iota})]^n,$
$1 - [1 - \varrho_U(\hat{\iota})]^n + 1 - [1 - \varrho_V(\hat{\iota})]^n - (1 - [1 - \varrho_U(\hat{\iota})]^n)(1 - [1 - \varrho_V(\hat{\iota})]^n)\Big]$
$= \{(\zeta_U(\hat{\iota})\zeta_V(\hat{\iota}))^n, 1 - [1 - \psi_U(\breve{\iota})]^n[1 - \psi_V(\breve{\iota})]^n, 1 - [1 - \varrho_U(\hat{\iota})]^n[1 - \varrho_V(\hat{\iota})]^n\}$
$= (U \otimes V)^n.$
$(iv) U^{n_1} \otimes U^{n_2}$
$= \Big[(\zeta_U(\hat{\iota}))^{n_1+n_2}, 1 - [1 - \psi_U(\breve{\iota})]^{n_1} + 1 - [1 - \psi_U(\breve{\iota})]^{n_2} - (1 - [1 - \psi_U(\breve{\iota})]^{n_1})$
$\big(1 - [1 - \psi_U(\breve{\iota})]^{n-2}\big), 1 - [1 - \varrho_U(\hat{\iota})]^{n_1} + 1 - [1 - \varrho_U(\hat{\iota})]^{n-2} - (1 - [1 - \varrho_U(\hat{\iota})]^{n_1})$
$(1 - [1 - \varrho_U(\hat{\iota})]^{n_2})\Big]$
$= \Big\{(\zeta_U(\hat{\iota}))^{n_1+n_2}, 1 - [1 - \psi_U(\breve{\iota})]^{n_1+n_2}, 1 - [1 - \varrho_U(\hat{\iota})]^{n_1+n_2}\Big\}$
$= U^{(n_1+n_2)}.$

□

Theorem 3.11. *For each $U, V \in NFS(\iota)$,then for any positive number n,*
$(i) nU \subseteq nV,$
$(ii) U^n \subseteq V^n.$

Proof. (i) Let $U \subseteq V$
$\Rightarrow \zeta_U(\hat{\iota}) \leq \zeta_V(\hat{\iota})$ and $\psi_U(\breve{\iota}) \geq \psi_V(\breve{\iota})$ and $\varrho_U(\hat{\iota}) \geq \varrho_V(\hat{\iota})$ for all $\iota \in X$.
$\Rightarrow 1 - [1 - \zeta_U(\hat{\iota})]^n \leq 1 - [1 - \zeta_V(\hat{\iota})]^n$,
$[\psi_U(\breve{\iota})]^n \geq [\psi_V(\breve{\iota})]^n$ and
$[\varrho_U(\hat{\iota})]^n \geq [\varrho_V(\hat{\iota})]^n$. for all $\iota \in X$.
(ii) Also, $[\zeta_U(\hat{\iota})]^n \geq [\zeta_V(\hat{\iota})]^n$,
$1 - [1 - \psi_U(\breve{\iota})]^n \leq 1 - [1 - \psi_V(\breve{\iota})]^n$,
$1 - [1 - \varrho_U(\hat{\iota})]^n \leq 1 - [1 - \varrho_V(\hat{\iota})]^n$, for all $\iota \in X$. □

Theorem 3.12. *For each $U, V \in NFS(\iota)$,then for any positive number n,*
$(i) n(U \cap V) = nU \cap nV,$
$(ii) n(U \cup V) = nU \cup nV.$

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Proof. (i) Let $n(U \cap V)$
$= \Big[x, 1-[1-\min(\zeta_U(\hat{\iota}), \zeta_V(\hat{\iota}))]^n, \max([\psi_U(\breve{\iota})]^n, [\psi_V(\breve{\iota})]^n),$
$\max([\varrho_U(\hat{\iota})]^n, [\varrho_V(\hat{\iota})]^n) \,|\iota \in X\Big]$
$= \Big[x, 1-[\max(1-\zeta_U(\hat{\iota}), 1-\zeta_V(\hat{\iota}))]^n, \max([\psi_U(\breve{\iota})]^n, [\psi_V(\breve{\iota})]^n),$
$\max([\varrho_U(\hat{\iota})]^n, [\varrho_V(\hat{\iota})]^n) \,|\iota \in X\Big]$ by Lemma 1.
$= \Big[x, 1-(\max([1-\zeta_U(\hat{\iota})]^n, [1-\zeta_V(\hat{\iota})]^n)), \max([\psi_U(\breve{\iota})]^n, [\psi_V(\breve{\iota})]^n),$
$\max([\varrho_U(\hat{\iota})]^n, [\varrho_V(\hat{\iota})]^n) \,|\iota \in X\Big]$
$= \Big[x, \max(1-[1-\zeta_U(\hat{\iota})]^n, 1-[1-\zeta_V(\hat{\iota})]^n), \max([\psi_U(\breve{\iota})]^n, [\psi_V(\breve{\iota})]^n),$
$\max([\varrho_U(\hat{\iota})]^n, [\varrho_V(\hat{\iota})]^n) \,|\iota \in X\Big]$
$= nU \cap nV$. Hence $n(U \cap V) = nU \cap nV$.
Similarly, we prove that $(ii) n(U \cup V) = nU \cup nV$. □

Theorem 3.13. *For each $U, V \in NFS(\iota)$, then for any positive number n,*
$(i)(U \cap V)^n = U^n \cap V^n$,
$(ii)(U \cup V)^n = U^n \cup V^n$.

Proof. (i) Let

$(U \cap V)^n = \Big[x, \min([\zeta_U(\hat{\iota})]^n, [\zeta_V(\hat{\iota})]^n), 1-[\max(1-\psi_U(\breve{\iota}), 1-\psi_V(\breve{\iota}))]^n,$
$1-[\max(1-\varrho_U(\hat{\iota}), 1-\varrho_V(\hat{\iota}))]^n |\iota \in X\Big]$
$= \Big[x, \min([\zeta_U(\hat{\iota})]^n, [\zeta_V(\hat{\iota})]^n), 1-(\min([1-\psi_U(\breve{\iota})]^n, [1-\psi_V(\breve{\iota})]^n)),$
$1-(\min([1-\varrho_U(\hat{\iota})]^n, [1-\varrho_V(\hat{\iota})]^n)) \,|\iota \in X\Big]$
$= \Big[x, \min([\zeta_U(\hat{\iota})]^n, [\zeta_V(\hat{\iota})]^n), \max(1-[1-\psi_U(\breve{\iota})]^n, 1-[1-\psi_V(\breve{\iota})]^n),$
$\max(1-[1-\varrho_U(\hat{\iota})]^n, 1-[1-\varrho_V(\hat{\iota})]^n) \,|\iota \in X\Big]$.
$U^n \cap V^n = \Big[x, ([\zeta_U(\hat{\iota})]^n, 1-[1-\psi_U(\breve{\iota})]^n, 1-[1-\varrho_U(\hat{\iota})]^n) \bigcap$
$([\zeta_V(\hat{\iota})]^n, 1-[1-\psi_V(\breve{\iota})]^n, 1-[1-\varrho_V(\hat{\iota})]^n) \,|\iota \in X\Big]$
$= \Big[x, \min([\zeta_U(\hat{\iota})]^n, [\zeta_V(\hat{\iota})]^n), \max(1-[1-\psi_U(\breve{\iota})]^n, 1-[1-\psi_V(\breve{\iota})]^n),$
$\max(1-[1-\varrho_U(\hat{\iota})]^n, 1-[1-\varrho_V(\hat{\iota})]^n) \,|\iota \in X\Big]$.
$= (U \cap V)^n$.
Hence $(U \cap V)^n = U^n \cap V^n$,
Similarly, we get $(ii)(U \cup V)^n = U^n \cup V^n$. □

Theorem 3.14. *For each $U, V \in NFS(\iota)$, then for any positive number n,*
$(U \oplus V)^n \neq U^n \oplus V^n$.

Proof. Let $(U \oplus V)^n$

$= \Big[x, [\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota})]^n, 1-[1-\psi_U(\breve{\iota})\psi_V(\breve{\iota})]^n,$
$1-[1-\varrho_U(\hat{\iota})\varrho_V(\hat{\iota})]^n |\iota \in X\Big]$.
$U^n = \{(\iota, [\zeta_U(\hat{\iota})]^n, 1-[1-\psi_U(\breve{\iota})]^n, 1-[1-\varrho_U(\hat{\iota})]^n) \,|\iota \in X\}$.
$V^n = \{(\iota, [\zeta_V(\hat{\iota})]^n, 1-[1-\psi_V(\breve{\iota})]^n, 1-[1-\varrho_V(\hat{\iota})]^n) \,|\iota \in X\}$.
$U^n \oplus V^n = \Big[x, [\zeta_U(\hat{\iota})]^n + [\zeta_V(\hat{\iota})]^n - [\zeta_U(\hat{\iota})]^n[\zeta_V(\hat{\iota})]^n, [1-[1-\psi_U(\breve{\iota})]^n]^n.$
$[1-[1-\psi_V(\breve{\iota})]^n]^n, [1-[1-\varrho_U(\hat{\iota})]^n]^n.[1-[1-\varrho_U(\hat{\iota})]^n]^n |\iota \in X\Big]$.
Hence $(U \oplus V)^n \neq U^n \oplus V^n$. □

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Based on the Definition 4 & 5, scalar multiplication and exponentiation operations of NFSs, we can define new concentration and dilation of the PFS as follows.

Definition 3.15. The concentration of a PFS U in the universe ι is noted by $CON(U)$ and is determined by
$CON(U) = \left\{ \left(\iota, \zeta_{CON(U)}(\hat{\iota}), \psi_{CON(U)}(\breve{\iota}), \varrho_{CON(U)}(\hat{\iota}) \right) | \iota \in X \right\}$,
where,
$\zeta_{CON(U)}(\hat{\iota}) = [\zeta_U(\hat{\iota})]^2$,
$\psi_{CON(U)}(\breve{\iota}) = 1 - [1 - \psi_U(\breve{\iota})]^2$,
$\varrho_{CON(U)}(\hat{\iota}) = 1 - [1 - \varrho_U(\hat{\iota})]^2$.
In other way, concentration of a PFS is defined by $CON(U) = U^2$.

Definition 3.16. The dilation of a PFS U in the universe ι is noted by $DIL(U)$ and is determined by
$DIL(U) = \left\{ \left(\iota, \zeta_{DIL(U)}(\iota), \psi_{DIL(U)}(\breve{\iota}), \varrho_{DIL(U)}(\hat{\iota}) \right) | \iota \in X \right\}$,
where,
$\zeta_{DIL(U)}(\hat{\iota}) = [\zeta_U(\hat{\iota})]^{\frac{1}{2}}$,
$\psi_{DIL(U)}(\breve{\iota}) = 1 - [1 - \psi_U(\breve{\iota})]^{\frac{1}{2}}$,
$\varrho_{DIL(U)}(\hat{\iota}) = 1 - [1 - \varrho_U(\hat{\iota})]^{\frac{1}{2}}$

In other words, dilation of a PFS is determined by $DIL(U) = U^{\frac{1}{2}}$.

The following theorem are clear.

Theorem 3.17. *For each $U \in NFS(\iota)$, then*
(i) $CON(U) \subseteq U \subseteq DIL(U)$,
(ii) If $\pi_U(\iota) = 0$, then $\pi_{CON(U)}(\iota) = 0$,
(iii) If $\pi_U(\iota) = 0$, then $\pi_{DIL(U)}(\iota) = 0$.

4 New operation on Neutrosophic fuzzy sets

In this section, we define and verify the algebraic characteristics of a new operation (@) on Neutrosophic fuzzy sets. We also go through the Distributivity rules in the scenario when the $\oplus, \otimes, \cup$, and $\cap$ operations are coupled.

Definition 4.1. Let ι be a nonempty set and I be the unit interval [0,1]. A Neutrosophic fuzzy set U and V of the form, $U = \{(\iota, \zeta_U(\hat{\iota}), \psi_U(\breve{\iota}), \varrho_U(\hat{\iota})) | \iota \in X\}$ and $V = \{(\iota, \zeta_V(\hat{\iota}), \psi_V(\breve{\iota}), \varrho_V(\hat{\iota})) | \iota \in X\}$. Then
$U@V = \left\{ \left(\iota, \frac{\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota})}{2}, \frac{\psi_U(\breve{\iota}) + \psi_V(\breve{\iota})}{2}, \frac{\varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota})}{2} \right) | \iota \in X \right\}$.

Remark 4.2. Clearly, for all two NFSs U and V, then U@V is a PFS.
Simple illustration given: For U@V,
$0 \le \frac{\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota})}{2} + \frac{\psi_U(\breve{\iota}) + \psi_V(\breve{\iota})}{2} + \frac{\varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota})}{2}$
$\le \frac{\zeta_U(\hat{\iota}) + \psi_U(\breve{\iota}) + \varrho_U(\hat{\iota})}{2} + \frac{\zeta_V(\hat{\iota}) + \psi_V(\breve{\iota}) + \varrho_V(\hat{\iota})}{2} \le \frac{1}{2} + \frac{1}{2} = 1.$

Theorem 4.3. *For every $U \in NFS(\iota)$, then $U@U = U$.*

Proof. Let $U@U$

$$= \left\{ \left(\iota, \frac{\zeta_U(\hat{\iota}) + \zeta_U(\hat{\iota})}{2}, \frac{\psi_U(\breve{\iota}) + \psi_U(\breve{\iota})}{2}, \frac{\varrho_U(\hat{\iota}) + \varrho_U(\hat{\iota})}{2} \right) | \iota \in X \right\}$$

$$= \left\{ \left(\iota, \frac{2\zeta_U(\hat{\iota})}{2}, \frac{2\psi_U(\breve{\iota})}{2}, \frac{2\varrho_U(\hat{\iota})}{2} \right) | \iota \in X \right\}$$

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$= \{(\iota, \zeta_U(\hat{\iota}), \psi_U(\check{\iota}), \varrho_U(\hat{\iota})) \,|\iota \in X\}$

$= U.$ □

Theorem 4.4. *For every $U, V, C \in NFS(\iota)$, then*
$(i)U@(V \cup C) = (U@V) \cup (U@C),$
$(ii)U@(V \cup C) = (U@V) \cup (U@C).$

Proof. (i) Let $(U@V) \cup (U@C)$

$$= \left[\max\left(\frac{\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota})}{2}, \frac{\zeta_U(\hat{\iota}) + \zeta_C(\hat{\iota})}{2}\right),\right.$$
$$\left.\min\left(\frac{\psi_U(\check{\iota}) + \psi_V(\check{\iota})}{2}, \frac{\psi_U(\check{\iota}) + \psi_C(\check{\iota})}{2}\right), \min\left(\frac{\varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota})}{2}, \frac{\varrho_U(\hat{\iota}) + \varrho_C(\hat{\iota})}{2}\right)\right].$$

$U@(V \cup C)$

$$= \left[x, \frac{\zeta_U(\hat{\iota}) + \max(\zeta_V(\hat{\iota}), \zeta_C(\hat{\iota}))}{2}, \frac{\psi_U(\check{\iota}) + \min(\psi_V(\check{\iota}), \psi_C(\check{\iota}))}{2},\right.$$
$$\left.\frac{\varrho_U(\hat{\iota}) + \min(\varrho_V(\hat{\iota}), \varrho_C(\hat{\iota}))}{2} |\iota \in X\right]$$
$$= \left[\max\left(\frac{\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota})}{2}, \frac{\zeta_U(\hat{\iota}) + \zeta_C(\hat{\iota})}{2}\right),\right.$$
$$\left.\min\left(\frac{\psi_U(\check{\iota}) + \psi_V(\check{\iota})}{2}, \frac{\psi_U(\check{\iota}) + \psi_C(\check{\iota})}{2}\right), \min\left(\frac{\varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota})}{2}, \frac{\varrho_U(\hat{\iota}) + \varrho_C(\hat{\iota})}{2}\right)\right],$$

by Lemma 2.
Hence, $U@(V \cup C) = (U@V) \cup (U@C)$.
(ii) Similarly, $U@(V \cup C) = (U@V) \cup (U@C)$. □

Remark 4.5. For $U, V \in [0, 1]$,then $Ub \le \frac{a+b}{2}, \frac{a+b}{2} \le a + b - ab$.

Theorem 4.6. *For every $U, V \in NFS(\iota)$, then*
$(i)(U \oplus V) \cup (U@V) = U \oplus V,$
$(ii)(U \otimes V) \cap (U@V) = U \otimes V,$
$(iii)(U \oplus V) \cap (U@V) = U@V,$
$(iv)(U \otimes V) \cup (U@V) = U@V.$

Proof. We prove only (i),and (ii),(iii) and (iv) can be proved similarly.
(i) Let $(U \oplus V) \cup (U@V)$

$$= \left[\max\left(\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota}), \frac{\zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota})}{2}\right),\right.$$
$$\left.\min\left(\psi_U(\check{\iota})\psi_V(\check{\iota}), \frac{\psi_U(\check{\iota}) + \psi_V(\check{\iota})}{2}\right), \min\left(\varrho_U(\hat{\iota})\varrho_V(\hat{\iota}), \frac{\varrho_U(\hat{\iota}) + \varrho_V(\hat{\iota})}{2}\right)\right]$$

$= \{(\iota, \zeta_U(\hat{\iota}) + \zeta_V(\hat{\iota}) - \zeta_U(\hat{\iota})\zeta_V(\hat{\iota}), \psi_U(\check{\iota})\psi_V(\check{\iota}), \varrho_U(\hat{\iota})\varrho_V(\hat{\iota})) \,|\iota \in X\}$
$= U \oplus V.$ □

Remark 4.7. Under the Neutrosophic fuzzy set operation of algebraic sum and algebraic product, the Neutrosophic fuzzy set forms a semilattice, associativity, commutativity, and idempotency. When $\oplus, \otimes$ and $\wedge, \vee, @$ are combined, the distributive law also holds.

5 Applications

The results are applicable to the development of Neutrosophic fuzzy semilattice structure, Neutrosophic fuzzy set, and algebraic structure on this set.

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6 Conclusion

Idempotency, commutativity, associativity, absorption, distributivity, and De Morgan's laws over complement are some algebraic features of Neutrosophic fuzzy sets that we have established in this study. We also established several theorems and defined new concentration and dilation of NFSs. Finally, we established a new operation (@) on Neutrosophic fuzzy sets and analysed distributive laws in the situation of combining the operations $\oplus, \otimes, \cup$, and $\cap$. This solution can be used in many Neutrosophic fuzzy set theory applications. The results of this research will be useful in the creation of Neutrosophic fuzzy semi-lattice and its algebraic property. The applicability of the suggested aggregating operators of NFSs in decision making, risk analysis, and many other uncertain and fuzzy environments will need to be studied in the future.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

author contributions

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