On The Classification of Neutrosophic Complex Inner Product Spaces

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Abstract

The objective of this paper is to study the neutrosophic complex inner product spaces over neutrosophic complex field C(I). Also, it determines the necessary and sufficient condition of a neutrosophic complex vector space to be a complex inner product space by using semi module isomorphisms.

Key words: Neutrosophic vector space; neutrosophic complex number; neutrosophic inner product.

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1.Introduction

Neutrosophy is a generalization of fuzzy ideas presented by Smarandache [13], where it was built over the idea of extending the degree of truth (T) and falsity (F) to a third logical case (the indeterminacy I).

Neutrosophic algebraic structures have been studied by many authors such as neutrosophic groups, neutrosophic rings, and matrices [2-7].

The study of neutrosophic real inner product spaces began in [15], where we find the structure of canonical neutrosophic inner products and orthogonality. This lead to many interesting results about isometries and norms in neutrosophic Euclidean geometry [11].

In this work, we combine the classical complex inner products , with neutrosophic vector spaces defined over neutrosophic complex numbers to get the complex neutrosophic inner product spaces. Also, we use the concept of semi-module isomorphisms [16], to classify the complex neutrosophic inner product spaces.

Main Concepts and Discussion

Definition :

Let V be a vector space over C, the corresponding neutrosophic complex vector space is defined as follows:

$V(I) = \{a + bI; a, b \in V\}$.

The scalars are taken for the neutrosophic complex field C(I):

$C(I) = \{(m + in) + (t + il)I; m, n, t, l \in C\}$.

The following example clarifies operations on $V(I)$.

Example :

Let $V = C^2 = C \times C$ be the Euclidean complex vector space over C.

The corresponding neutrosophic complex vector space is:

$C^2(I) = \{(x, y) + (z, t)I = (x + zI, y + tI); x, y, z, t \in C\}$.

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Multiplication by a neutrosophic complex scalar can be showed as follows:
Take $X = \big(1 + i + (2 - i)I, i + I(1 + i)\big), \lambda = 1 - i + I(3i)$.
$\lambda X = \big([1 + i + (2 - i)I][1 - i + 3iI], \big[i + I(1 + i).[1 - i + 3iI]\big]\big)$
$\lambda X = (2 + 3iI - 3I + (1 - 3i)I + (3 + 6i)I, 1 + i - 2I + 2I + (-3 + 3i)I)$
$\lambda X = (2 + (1 + 6i)I, 1 + i + (-4 + 3i)I)$.
$V(I)$ is a module over $C(I)$, that is because the neutrosophic complex field $C(I)$ is a ring in the ordinary meaning not a field.

Definition :

Let V be a vector space over $C, f: V \times V \to C$ is called a complex inner product if:

1) $f(a, a) = 0 \Longrightarrow a = 0, f(a, a) \geq 0, f(a, a)$ is a real number.
2) $f(a, b) = \overline{f(b, a)}$
3) $f(a + b, c) = f(a, c) + f(b, c)$
4) $f(\lambda a, \mu b) = \lambda\bar{\mu} f(a, b);\ a, b, c \in V, \lambda, \mu \in C$

Example :

Consider the complex vector space C over itself.
Define: $f: C \times C \to C;\ f(a, b) = a\bar{b}$.
We remark that: $f(a, a) = |a|^2 \geq 0$.
If $f(a, a) = 0 \Longrightarrow a = 0, f(a, b) = a\bar{b} = \overline{(b.\bar{a})} = \overline{f(b, a)}$.
$f(a + b, c) = (a + b)\bar{c} = a\bar{c} + b\bar{c} = f(a, c) + f(b, c)$.
$f(\lambda a, \mu b) = \lambda a\overline{(\mu b)} = \lambda\bar{\mu} f(a, b)$ for all $a, b, c, \lambda, \mu \in C$

Definition :

Let $V(I)$ be a neutrosophic complex vector space over $C(I)$, consider the mapping $f: V(I) \times V(I) \to C(I)$ We say that $f$is a neutrosophic complex inner product if:

1) $f(a + bI, c + dI) = \overline{f(c + dI, a + bI)}$.
2) $f(a + bI, a + bI) = 0 \Longrightarrow a + bI = 0,\ f(a + bI, a + bI) \geq 0\ and\ \ f(a + bI, a + bI) \in R(I)$.
3) $f([a + bI] + [c + dI], m + nI) = f(a + bI, m + nI) + f(c + dI, m + nI)$.
4) $f\big(\lambda(a + bI), \mu(c + dI)\big) = \lambda\bar{\mu} f(a + bI, c + dI);\ a, b, c, d, m, n \in V\ and\ \lambda, \mu \in C(I)$.

Theorem 10:

Let V be a vector space over C, with$g: V \times V \to C$as a complex inner product.
Let $V(I)$be the corresponding neutrosophic vector space over $C(I)$, then $V(I)$ has a neutrosophic complex inner product generated by g.

Proof.

Define $f: V(I) \times V(I) \to C(I); f(a + bI, c + dI) = g(a, c) + I[g(a + b, c + d) - g(a, c)]$.
f is a neutrosophic complex inner product for the following reasons:

1) $f(a + bI, a + bI) = g(a, a) + I[g(a + b, a + b) - g(a, a)]$.

If $f(a + bI, a + bI) = 0$, then $g(a, a) = 0 \Longrightarrow a = 0$, $g(a + b, a + b) = 0 \Longrightarrow a + b = 0 \Longrightarrow b = 0$
Thus $a + bI = 0$.
On other hand, $g(a, a)$,$g(a + b, a + b)$ are two real positive numbers, hence:
$g(a, a) + I[g(a + b, a + b) - g(a, a)] \geq 0$ with respect to the neutrosophic partial order relation defined in [ ].

2) $f(a + bI, c + dI) = g(a, c) + I[g(a + b, c + d) - g(a, c)]$=$\overline{g(c, a)} + I\big[\overline{g(c + d, a + b)} - \overline{g(c, a)}\big] = \overline{f(c + dI, a + bI)}$.
3) Let $\lambda = p + qI, \mu = m + nI; p, q, m, n \in C$ be two neutrosophic complex scalars, hence

$f\big(\lambda(a + bI), \mu(c + dI)\big) = f(pa + [(p + q)(a + b) - pa]I, mc + [(m + n)(c + d) - mc]I)$,
$= g(pa, mc) + I\big[g\big((p + q)(a + b), (m + n)(c + d)\big) - g(pa, mc)\big] = p\bar{m} g(a, c) + I\big[(p + q)\overline{(m + n)} g(a + b, c + d) - p\bar{m} g(a, c)\big]$,
=$(p + qI)\overline{(m + nI)} f(a + bI, c + dI)$.

4) $f([a + bI] + [c + dI], m + nI) = f\big((a + c) + (b + d)I, m + nI\big)$

=$g(a + c, m) + I[g(a + c + b + d, m + n) - g(a + c, m)]$
=$g(a, m) + g(c, m) + I[g(a + b, m + n) + g(c + d, m + n) - g(a, m) - g(c, m)]$
=$f(a + bI, m + nI) + f(c + dI, m + nI)$.

Example:

Let $V = C^2$ be the Euclidean complex vector space over C.
V has an inner product defined as follows:
$g: C^2 \times C^2 \to C$ such that $g\big((x, y), (z, t)\big) = x\bar{z} + y\bar{t}; x, y, z, t \in C$
The corresponding neutrosophic complex inner product generated by g can be obtained as follows:
$f: C^2(I) \times C^2(I) \to C(I)$ such that:

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$f\big((a+bI,c+dI),(x+yI,z+tI)\big)=f\big((a,c)+(b,d)I,(x,z)+(y,t)I\big)$
$=g\big((a,c),(x,z)\big)+I\big[g\big((a+b,c+d),(x+y,z+t)\big)-g\big((a,c),(x,z)\big)\big]$
$=a\bar{x}+c\bar{z}+I\big[(a+b).\overline{(x+y)}+(c+d).\overline{(z+t)}-a\bar{x}-c\bar{z}\big]$
Let $X=(1+iI,2-i+I),Y=(1+i+(1-i)I,(3i)I)\in C(I)$, hence
$f(X,Y)=f\big((1{,}2-i)+I(i,1),(1+i,0)+I(1-i,3i)\big)=$
$g\big((1{,}2-i),(1+i,0)\big)+I\big[g\big((1+i,3-i),(2{,}3i)\big)-g\big((1{,}2-i),(1+i,0)\big)\big]=$
$1\overline{(1+\imath)}+(2-i)\bar{0}+I\big[(1+i).\bar{2}+(3-i).\overline{(3\imath)}-1\overline{(1+\imath)}-(2-i)\bar{0}\big]=$
$1-i+I[2+2i-9i-3-1+i]=1-i+I[2+2i-9i-3-1+i]=1-i+I[-2-6i]$.

Definition:

Let $V(I)$ be a neutrosophic complex vector space over $C(I)$ with a neutrosophic complex inner product f We say:

1) $a+bI\perp c+dI$ if and only if $f(a+bI,a+bI)=0$.
2) The norm of $a+bI$ is defined as follows:

$\|a+bI\|=\sqrt{f(a+bI,a+bI)}$.

Theorem :

Let $V(I)$ be a neutrosophic complex inner product space over $C(I)$, with a neutrosophic complex inner product f generated by a classical complex inner product $g\colon V\times V\to C$, we have:

1) $a+bI\perp c+dI$ if and only if $a\perp c$ and $a+b\perp c+d$.
2) $\|a+bI\|=\|a\|+I(\|a+b\|-\|a\|)$.
3) $\|(a+bI)+(c+dI)\|\leq\|a+bI\|+\|c+dI\|$.
4) $\|\lambda(a+bI)\|=|\lambda|\|a+bI\|;a,b,c,d\in V$ and $\lambda\in C(I)$.

Proof.

1) $f(a+bI,a+bI)=g(a,c)+I[g(a+b,c+d)-g(a,c)]$
$f(a+bI,a+bI)=0$ if and only if $g(a,c)=g(a+b,c+d)=0$, thus $a\perp c$ and $a+b\perp c+d$.
2) $\|a+bI\|=\sqrt{f(a+bI,a+bI)}=\sqrt{g(a,a)+I[g(a+b,a+b)-g(a,a)]}$
$=\sqrt{g(a,a)}+I\big[\sqrt{g(a+b,a+b)}-\sqrt{g(a,a)}\big]=\|a\|+I(\|a+b\|-\|a\|)$.
3) $\|(a+bI)+(c+dI)\|=\|(a+c)+(b+d)I\|=\|a+c\|+I(\|a+c+b+d\|-\|a+c\|)$
$\leq\|a\|+\|c\|+I(\|a+b\|+\|c+d\|-\|a\|-\|c\|)\leq\|a\|+I(\|a+b\|-\|a\|)+\|c\|+I(\|c+d\|-\|c\|)\leq$ $\|a+bI\|+\|c+dI\|$.

4) $\|\lambda(a+bI)\|^2=f\big(\lambda(a+bI),\lambda(a+bI)\big)=\lambda\bar{\lambda}\,(a+bI)\overline{(a+bI)}=(|\lambda|\|a+bI\|)^2$. By taking the square root of the two sides, we get the desired result.

Theorem :

Let V be a vector space over C, with $g\colon V\times V\to C$ as a complex inner product, let $V(I)$ be the corresponding neutrosophic complex inner product space over $C(I)$, with f as a neutrosophic complex inner product generated by g, hence $V(I)$ is a semi isomorphic to V.

Proof.

Define $T\colon V(I)\to V\times V$ such that $T(a+bI)=(a,a+b);\ a,b\in V$.
Consider the AH-isometry $\emptyset\colon C(I)\to C\times C;\emptyset(x+yI)=(x,x+y);\ x,y\in C$.
$\emptyset$ is a ring isomorphism.
Now, we must prove that T is a semi isomorphism.
$\forall\ a+bI,c+dI\in V(I)$, it is clear that:
a neutrosophic complex inner product for the following reasons:
$T[(a+bI)+(c+dI)]=T(a+bI)+T(c+dI)$.
Also, it is easy to check that T is a bijection.
Now, we must prove that $T[(x+yI).(a+bI)]=\emptyset(x+yI).T(a+bI)$ for all $x,y\in C,a,b\in V$.
$T[(x+yI).(a+bI)]=T[xa+I(xb+ya+yb)]=(xa,xa+xb+ya+yb)=$
$(x,x+y).(a,a+b)=\emptyset(x+yI).T(a+bI)$, thus T is a semi isomorphism.
According to the previous theorem, we get the following result.
If $V(I)$ is a complex neutrosophic inner product space with f as a neutrosophic complex inner product generated by a classical complex inner product $g\colon V\times V\to C$, hence $V(I)\cong_s V\times V$.
The converse of this result is hard to prove.
If $V(I)$ is a complex neutrosophic inner product space with $f\colon V(I)\times V(I)\to C(I)$, then can we find a classical complex inner product $g\colon V\times V\to C$ such that f generated by g and $V(I)\cong_s V\times V$?
To solve the inverse problem, it is sufficient to find a classical complex inner product $g\colon V\times V\to C$ such that f is generated by g.

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Theorem :

Let $V(I)$ be a complex neutrosophic inner product space over $C(I)$ with $f: V(I) \times V(I) \to C(I)$ as complex neutrosophic inner product space, then V is a complex inner product space over C.

Proof

Define $g: V \times V \to C;\ g(a,c) = f(a + 0I, c + 0I)$ we have:

a) $g(a,a) = 0 \Longrightarrow f(a,a) = 0 \Longrightarrow a = 0$.

$g(a,a) = f(a,a) \geq 0 \in R$.

b) $g(a+b,c) = f(a+b+0I, c+0I) = f(a+0I, c+0I) + f(b+0I, c+0I)$

$= g(a,c) + g(b,c)$.

c) $g(\lambda a, \mu c) = f(\lambda a + 0I, \mu c + 0I) = \lambda\bar{\mu} f(a+0I, c+0I) = \lambda\bar{\mu} g(a,c)$

Thus $g$is a complex inner product on the classical vector space V.

Remark.

From the previous theorems, we get the following result.

According complex inner product space $V(I)$ is semi isomorphic to the direct product of the corresponding classical inner product space$V$.

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