Under Solvable Groups as a Novel Generalization of Solvable Groups

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Abstract

The objective of this paper is to define a new generalization of solvable groups by using the concept of power maps which generalize the classical concept of powers (exponents). Also, it presents many elementary properties of this new generalization in terms of theorems.

Key words: Solvable group; Power map; Under Solvable group; $\varphi - abelian\ group, \varphi - centre$.

1.Introduction

Groups are one of the most important concepts in algebra, where they have many applications in defining other algebraic concepts, and solving algebraic equations [1].

The concept of solvable groups was defined firstly by the great mathematician E. Galois, to solve the problem of solvability of a polynomial equation by radicals [2]. The solvability of a group was studied widely in [3-6], where many sufficient conditions were proved.

Solvable groups are considered as extensions by abelian series of subgroups. From this point of view, we introduce a new generalization of abelian groups by using power maps defined in [7], where we use these maps to study (under solvable) groups as a novel interesting generalization of solvability and to present their elementary properties.

We denote by map(G) to the set of all mappings $\varphi: G \to G$

Main Discussion

Definition 1.1 :

Let G be a group and $\varphi \in map(G)$. We define $Z_\varphi(G) = \{x \in G\ ; x\varphi(y) = \varphi(y)x\ \ \forall y \in G\}$. We call $Z_\varphi(G)$ by ($\varphi - center$) of G.

Definition 2.1 :

We call a group G ($\varphi - abelian$) if G=$Z_\varphi(G)$.

Lemma 3.1 :

Let G be a group and $\varphi \in map(G)$. then

(a) $Z_\varphi(G)$ is a subgroup of G and Z(G)≤ $Z_\varphi(G)$.

(b) G is abelian if and only if G is ($-abelian$) for each $\varphi \in map(G)$.

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(c) If φ is surjective, then Z(G) = $Z_{\varphi}(G)$.

(d) If G is ($-abelian$) and φ is surjective, then G is abelian.

Proof :

(a) $\forall\, x, y \in Z_{\varphi}(G)$ *and* $t \in G$ *then* $xy\varphi(t) = x\varphi(t)y = \varphi(t)xy$ *so* $xy \in Z_{\varphi}(G)$ *and we have that* $x\varphi(t) = \varphi(t)x$, *this implies that* $\varphi(t)x^{-1} = x^{-1}\varphi(t)$, so $x^{-1} \in Z_{\varphi}(G)$ *and* $Z_{\varphi}(G)$ *is a subgroup contains* $Z(G)$.

(b) If G is abelian, then each single element x of G permutes with every element $\varphi(y)$, such that y$\in G$.

(c) Holds directly from the definition.

(d) Holds directly from (b) and (c).

Definition 4.1 :

Let G be a group, and $\varphi \in map(G)$, let H be a subgroup of G. We say that H is ($\varphi - abelian$) subgroup if $Z_{\varphi}(H) = \{x \in H\,;\ x\varphi(y) = \varphi(y)x \ \ \forall y \in H\}$ = H.

Definition 5.1 :

Let G be a group, and $\varphi \in map(G)$, and H, K are two subgroups of G with H▶K. We define $\varphi_{K,H}: K/H \to G/H\,;\ \varphi_H(kH) = \varphi(k)H.$

Theorem 6.1 :

(a) If G is ($-abelian$) and $H \leq G$, then H is ($\varphi - abelian$) subgroup.

(b) If G is ($-abelian$) and H▶G, then G/H is ($\varphi_{G,H} - abelian$).

(c) If $f \circ \varphi = \varphi \circ f$ *for each* $f \in aut(G)$, then $Z_{\varphi}(G)$ is a characteristic subgroup of G.

(d) $Z_{\varphi}(G)$▶G if and only if $f_g(x) = f_{g\varphi(y)}(x)$ *where* $f_g: Z_{\varphi}(G) \to G\,;\ f_g(x) = g^{-1}xg$ *and* $f_{g\varphi(y)}: Z_{\varphi}(G) \to G\,;\ \ f_{g\varphi(y)}(x) = (g\varphi(y))^{-1}x\,\big(g\varphi(y)\big)$ *for each* $x \in Z_{\varphi}(G)$ *and* $g, y \in G$.

Proof :

(a) Holds directly from the definition.

(b) Let G be ($\varphi - abelian$) and H▶G, then
$x\varphi(y) = \varphi(y)x \ \forall\, y \in G$ *so* $(xH)\Big(\ \varphi_{G,H}(yH)\Big) = \varphi_{G,H}(yH)(xH)$, *so* G/H is ($\varphi_{G,H} - abelian$).

(c) Assume that $f \circ \varphi = \varphi \circ f$ *for each* $f \in aut(G)$, then
$\forall x \in Z_{\varphi}(G)$ *and* $y \in G$, *there is* $t \in G$ *such* $f(t) = y$ *and* $f(x)\varphi(y) = f(x)\varphi\big(f(t)\big) = f(x)f\big(\varphi(t)\big) = f\big(x\varphi(t)\big) = f(\varphi(t)x) = f\big(\varphi(t)\big)f(x) = \varphi(y)f(x)$ *so* $f(x) \in Z_{\varphi}(G)$ *and* $Z_{\varphi}(G)$ is a characteristic subgroup of G.

(d) Suppose that $Z_{\varphi}(G)$▶G, then
$g^{-1}xg \in Z_{\varphi}(G)$ *for each* $g \in G$ *and* $x \in Z_{\varphi}(G)$, $\forall\, y \in G$ *we have* $g^{-1}xg\varphi(y) = \varphi(y)g^{-1}xg$ *so* $\big(\varphi(y)\big)^{-1}g^{-1}xg\varphi(y) = g^{-1}xg$. For the converse we assume that $f_g(x) = f_{g\varphi(y)}(x)$, *then* $\big(\varphi(y)\big)^{-1}g^{-1}xg\varphi(y) = g^{-1}xg$ *this implies* $g^{-1}xg\varphi(y) = \varphi(y)g^{-1}xg$ *so* $g^{-1}xg \in Z_{\varphi}(G)$ *and* $Z_{\varphi}(G)$▶G .

Definition 7.1 :

Let x, y be two arbitrary elements of G we define the ($\varphi - commutator$) of x and y by $[x, y]_{\varphi} = [x, \varphi(y)] = x^{-1}(\varphi(y))^{-1}x\varphi(y)$, we denote by G'_{φ} to the subgroup generated by all ($\varphi - commutators$)

Theorem 8.1 :

Let G be a group, then

(a) $G'_{\varphi} = \{e\}$ *if and only if* G *is* $(\varphi - abelian)$.

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(b) If H►G, then G/H is $(\varphi_{G,H} - abelian)$ if and only if $G'_\varphi \leq H$.

The proof is obvious.

Definition 9.1 :

(a) Let G be a group, and H is a subgroup of G, and $\varphi \in map(G)$. We define $H'_\varphi = \{\langle [h,h']_\varphi \rangle \,; h\,, h' \in H\}$.

(b) We say that H is $(\varphi - normal\ factor)\ of\ G$ if and only if G/H is $(\varphi_{G,H} - abelian)\ and\ we\ denote\ it\ by\ H ►_\varphi G$, which means that $G'_\varphi \leq H$

It is easy to show that $H'_\varphi = \{e\}\ if\ and\ only\ if\ H\ is\ (\varphi - abelian)\ subgroup.$

Theorem 10.1 :

Let G be a group, and $\varphi \in map(G)$, and H,K be two subgroups of G, then

(a) If H , K are two ($\varphi - abelian$) subgroups, then H∩K is ($\varphi - abelian$) subgroup.

(b) If H , K $►_\varphi G$, then H∩K , HK $►_\varphi G$.

Proof :

(a) ∀ $x, y \in H \cap K, then\ x, y \in H\ and\ x, y \in K, so\ x\varphi(y) = \varphi(y)x\ and\ Z_\varphi(H \cap K) = H \cap K.$

(b) suppose that $G'_\varphi \leq H\ and\ G'_\varphi \leq K$ then $G'_\varphi \leq H \cap K\ and\ G'_\varphi \leq HK$.

Definition 11.1 :

Let G be a $(\varphi - abelian)$ group and $f: G \to K$ be an isomorphism we define $\varphi^*: K \to K$ by $\varphi^* = f \circ \varphi \circ f^{-1}$

It is easy to show that K is $\varphi^* - abelian$

Definition 12.1 :

Let G and K be two groups, and $\varphi \in map(G)$, then

(a) we say that K is $(\varphi - abelian)$ if there is a $(\varphi - abelian)$ subgroup H of G such that $H \cong K$.

(b) Let S►K, we say that K/S is a $(\varphi - abelian)$ factor if there are two subgroups T,H of G where T►H and $T'_\varphi \leq H$ and $T/H \cong K/S$.

Theorem 13.1 :

Let G be a group and $\varphi \in map(G)$, then

$(G/H)'_\varphi = G'_\varphi H/H$.

Proof :

It is easy to see that $(G/H)'_\varphi \leq G'_\varphi H/H$, for the converse we assume that $zhH \in G'_\varphi H/H$; $z \in G'_\varphi$, $h \in H$, we have $z = \prod_{i=1}^{n} x_i^{-1}(\varphi(y_i))^{-1} x_i \varphi(y_i)$; $x_i\,, y_i \in G$, so that $zhH = \prod_{i=1}^{n} x_i^{-1}(\varphi(y_i))^{-1} x_i \varphi(y_i)\ H \in (G/H)'_\varphi$, thus the proof is complete .

2. $(\boldsymbol{\varphi - solvable})$ $\boldsymbol{groups}$

Definition 1.2 :

We call a group G $(\varphi - solvable)$ group, if it has a subnormal series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$, such that $(H_i)'_\varphi \leq H_{i-1}$ for $1 \leq i \leq n$.

-The previous condition is equivalent to $H_i/H_{i-1}\ is\ (\varphi_{H_i,H_{i-1}} - abelian)$ because if we suppose that $(H_i)'_\varphi \leq H_{i-1}$ we find : for $h\,, h' \in H_i\ we\ have\ h^{-1}\left(\varphi(h')\right)^{-1} h\varphi(h') \in H_{i-1}\ so\ h\varphi(h')H_{i-1} =\ \varphi(h')h\ H_{i-1}$.

-We remark that if $(H_i)'_\varphi \leq H_{i-1}$, then $(H_i)'_\varphi \leq H_i$, thus $\left(\varphi(h')\right)^{-1} h\varphi(h') \in H_i$.

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- It is easy to see that $(H_i)'_\varphi \leq H_i$ if $H_i \blacktriangleright G$ but the converse is not true in general.

-Let H be a subgroup of G we say it is a $(\varphi - solvable)$ subgroup with respect to $\varphi \in map(G)$, if it has a subnormal series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = H$ such $(H_i)'_\varphi \leq H_{i-1}$ for $1 \leq i \leq n$.

Theorem 2.2 :

Let G be a group, then

(a) If G is $(\varphi - abelian)$, then it is $(\varphi - solvable)$.

(b) If G is solvable, then it is $(\varphi - solvable)$ for each $\varphi \in map(G)$ such that $\varphi(H_i)$ *is contained in* H_i , where H_i is a subnormal group of any solvable series of G.

Proof:

(a) G is $(\varphi - solvable)$ by the series $\{e\} \leq G$.

(b) suppose that G has a solvable series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$ with abelian factors, then $\forall\, h, h' \in H_i$ *we have* $h\varphi(h')H_{i-1} = \varphi(h')h\,H_{i-1}$ *because* $\varphi(h') \in H_i$*, so G is* $(\varphi - solvable)$.

Theorem 3.2 :

Let G be a $(\varphi - solvable)$ group by the series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$, if $\varphi(H_i) = H_i$, then G must be solvable.

Proof: Let h, h' *be two arbitrary elements in* $H_i, \exists\, h'' \in H_i$ where $h' = \varphi(h'')$*, now we write* $h\,\varphi(h'')H_{i-1} = \varphi(h'')hH_{i-1}$*, this means* that $hh'\,H_{i-1} = h'\,hH_{i-1}$ *so* H_i/H_{i-1} *is abelian and G is solvable.*

-We remark that G is solvable if and only if it is $(I - solvable)$; I is the identity function
- It is easy to show that if $K \leq H \leq G$, then $K'_\varphi \leq H'_\varphi$.

Theorem 4.2 :
Let $[x, y]_\varphi \in G'_\varphi$ and $g \in Z_\varphi(\mathrm{G})$, then $g^{-1}[x, y]_\varphi\, g \in G'_\varphi$.
Proof :
$g^{-1}[x, y]_\varphi\, g = g^{-1}x^{-1}(\varphi(y))^{-1}x\varphi(y)g = (xg)^{-1}\big(\varphi(y)\big)^{-1}(xg)\varphi(y) = [xg, y]_\varphi \in G'_\varphi$.

3. Power maps

Definition 1.3:[7]

Let G be a group, we define $\phi_G = \{\varphi \in map(G);\ \varphi(x^{-1}) = (\varphi(x))^{-1}\ and\ \varphi \circ f = f \circ \varphi\ \ \forall f \in aut(G)\}$. We call ϕ_G the set of power maps of G.

-We denote by ϕ_l to the set of one-to-one power maps.

Lemma 2.3 :[7]

Let G be a group and $\varphi: G \to G\ such\ that\ \varphi(x) = x^n\ with\ a\ fixed\ integer\ n$, then $\varphi \in \phi_G$

Lemma 2.3 ensures that power maps generalize the classical concept of powers in groups.

Lemma 3.3 : [7]

Let G be a group, then

(a) ϕ_l is a subgroup of S_G.

(b)$K_G = aut(G) \cap \phi_l$=$Z(aut(G))$.

(c) Aut(G) is abelian if and only if Aut(G)$\blacktriangleright$ ϕ_l.

(d) If G=Z_p, then aut(G)= ϕ_l.

(e)$(\phi_l.aut(G))' = \phi_l{}'[aut(G)]'$.

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Definition 4.3 :

Let G be a group, and $\varphi \in map(G)$. We define $(G)^n_\varphi = (G^{n-1}_\varphi)'_\varphi$ and $G^1_\varphi = G'_\varphi$, where n is a positive integer greater than 1.

Theorem 5.3 :

If $\varphi \in \phi_G$ then $(G)^n_\varphi$ *is a characteristic subgroup of* G.

Proof:

We remark that $G'_\varphi \blacktriangleright G$ because for any automorphism $f \in aut(G)$ and $x \in G'_\varphi$, we have $x = [t, \varphi(y)]$ *such* $t, y \in G$ *and* $f(x) = [f(t), f(\varphi(y))] = [f(t), \varphi(f(y))] \in G'_\varphi$, so G'_φ is a characteristic subgroup and then normal.

Suppose that $(G)^i_\varphi$ *is a characteristic subgroup of* G, then $\forall\, z \in (G)^{i+1}_\varphi$ we have $z = x^{-1}(\varphi(y))^{-1}x\varphi(y)\,; x, y \in (G)^i_\varphi$ so $f(z) = [f(x), f(\varphi(y))] = [f(x), \varphi(f(y))] \in [(G)^i_\varphi, (G)^i_\varphi]_\varphi = (G)^{i+1}_\varphi$, so $(G)^{i+1}_\varphi$ *is a characteristic subgroup* by induction and then normal

-It easy to see that if φ is the identity on G then $G'_\varphi = G'$ and $(G)^n_\varphi$ is the (n-th) derived subgroup of G.

4. Under Solvable groups

Definition 1.4 :

(a) Let G be a group, we say that G is under solvable with respect to a power map $\varphi \in map(G)$ if G is $(\varphi - solvable)$.

(b) If H is a group, we say that H is under solvable with respect to a power map $\varphi \in map(G)$, if there is a subgroup K of G, such that K is under solvable, and $K \cong H$.

(C) Let G and K be two groups and S$\blacktriangleright$K , H$\blacktriangleright$G , we say that K/S is under solvable factor with respect to a power map $\varphi \in map(G)$ if there is a subgroup T/H of G/H such that $T/H \cong K/S$ and T/H is undersolvable .

Lemma 2.4 :

(a) If G is under solvable with respect to a power map φ, then G is $(\varphi - solvable)$.

(b) If G is solvable, then it is under solvable.

Proof :

(a) Holds easily from the definition

(b) If G is solvable, then G is $(I - solvable)$, hence it is under solvable since I is a power map.

Theorem 3.4 :

G is under solvable with respect to a power map φ if and only if there is a positive integer n such $(G)^n_\varphi = \{e\}$.

Proof:

We assume that there is a positive integer n such $(G)^n_\varphi = \{e\}$, then we have the series $\{e\} = G^n_\varphi \le G^{n-1}_\varphi \le \cdots \le G'_\varphi \le G^0_\varphi = G$, so G is under solvable.

Conversely suppose that G has a subnormal series $\{e\} = H_0 \le H_1 \le \cdots \le H_n = G$ such $(H_i)'_\varphi \le H_{i-1}$ for $1 \le i \le n$, we will prove $G^i_\varphi \le H_{n-i}$ for $0 \le i \le n$.

For $i = 0$, we have $G^0_\varphi = G \le H_n = G$. Assume that $G^i_\varphi \le H_{n-i}$ then $G^{i+1}_\varphi = [G^i_\varphi, G^i_\varphi]_\varphi \le [H_{n-i}, H_{n-i}]_\varphi = (H_{n-i})'_\varphi \le H_{n-i+1}$, now we get $G^n_\varphi \le \{e\} = H_0$ so $G^n_\varphi = \{e\}$.

Definition 4.4 :

Let G be a group with a power map φ, H be a subgroup of G. We say that H is under solvable with respect to φ, if there is a positive integer n such that $H^n_\varphi = \{e\}$.

Theorem 5.4 :

If G is under solvable with respect to a power map φ, and H is a subgroup of G. Then H is under solvable with respect to φ.

Proof :

Assume that G is under solvable with respect to a power map φ, then there is a positive integer n such $(G)^n_\varphi = \{e\}$, by easy induction we get that $H^i_\varphi \le G^i_\varphi$ for each $0 \le i \le n$, so $H^n_\varphi = \{e\}$ and H is undersolvable with respect to φ.

Definition 6.4 :

Let G be a group with a power map φ, and H,T be two subgroups of G with H►T. We say that T/H is under solvable subfactor with respect to φ, if there is a positive integer n such that $(T/H)^n_\varphi = \{e\}$; $(T/H)'_\varphi = \langle [xH, \varphi_{T,H}(yH)] \,; x, y \in T\rangle \; and \; (T/H)^{i+1}_\varphi = (\,(T/H)^i_\varphi)'_\varphi$.

Theorem 7.4 :

Let G be a group with a power map φ, and H,T be two subgroups of G with H►T. Then T/H is under solvable subfactor with respect to φ if and only if there is a subnormal series $\{e\} = T_0/H \le T_1/H \le \cdots \le T_{n-1}/H \le T_n/H = T/H$ such that $(T_i/H)'_\varphi \le T_{i-1}/H$.

Proof :

Assume that T/H is under solvable subfactor with respect to φ, then there is a positive integer n such that $(T/H)^n_\varphi = \{e\}$, thus we have the series (*) $\{e\} = (T/H)^n_\varphi \le (T/H)^{n-1}_\varphi \le \cdots \le (T/H)^0_\varphi = T/H$, while $(T/H)^i_\varphi = (T)^i_\varphi H/H \le (T/H)^{i-1}_\varphi$; $0 \le i \le n$.

Conversely suppose that there is a series $\{e\} = T_0/H \le T_1/H \le \cdots \le T_{n-1}/H \le T_n/H = T/H$ such that $(T_i/H)'_\varphi \le T_{i-1}/H$, then by induction we find that

$(T/H)^i_\varphi \le T_{n-i}/H$, this implies $(T/H)^n_\varphi = \{e\}$ and we get the proof .

Theorem 8.4 :

Let G be an under solvable group with respect to a power map φ, and H►G. Then G/H

is under solvable subfactor with respect to φ.

Proof :

Assume that there is a positive integer n such $(G)^n_\varphi = \{e\}$, we have the series $\{e\} = (G)^n_\varphi H/H \le (G)^{n-1}_\varphi H/H \le \cdots \le (G)^0_\varphi H/H = G/H$ and $(G)^i_\varphi H/H/(G)^{i+1}_\varphi H/H \cong (G)^i_\varphi H/(G)^{i+1}_\varphi H \cong (G)^i_\varphi/(G)^i_\varphi \cap (G)^{i+1}_\varphi H$. We remark that $((G)^i_\varphi)'_\varphi = (G)^{i+1}_\varphi \le (G)^i_\varphi \cap (G)^{i+1}_\varphi H$, so that $(G)^i_\varphi/(G)^i_\varphi \cap (G)^{i+1}_\varphi H$ is a $(\varphi - abelian)$ factor and G/H is undersolvable subfactor with respect to φ.

Theorem 9.4 :

Let G be a group with a power map φ, and H►G , if H is under solvable with respect to φ and G/H is undersolvable subfactor with respect to φ, then G is under solvable with respect to φ.

Proof :

Assume that there is a subnormal series $\{e\} = T_0/H \le T_1/H \le \cdots \le T_{n-1}/H \le T_n/H = G/H$ such that $(T_i/H)'_\varphi \le T_{i-1}/H$ and a positive integer m such $H^m_\varphi = \{e\}$. Thus we get the subnormal series $H \le T_1 \le \cdots \le T_n = G$ and $T_i/T_{i-1} \cong (T_i/H)/(T_{i-1}/H)$.

Since $(T_i/H)'_\varphi \le T_{i-1}/H$ we find that $(T_i)'_\varphi H \le T_{i-1}$ so $(T_i)'_\varphi \le T_{i-1}$ and G is undersolvable.

Conclusion

In this paper we have used power maps on groups to define the concept of under solvability as a new generalization of solvable groups. Also, we have studied some of elementary properties of this novel generalization.

In the future, we aim to find more applications of power maps in other algebraic structures such as rings, fields, and vector spaces.

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