An Introduction to Neutrosophic Real Banach and Hillbert Spaces

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Abstract

Banach and Hillbert spaces are the main important concepts in the study of classical functional analysis. This paper generalizes these two kinds of functional spaces into neutrosophic systems, where the concept of neutrosophic Banach space and neutrosophic Hillbert space will be defined and discussed for the first time over partial ordered neutrosophic spaces. Also, many related concepts such as neutrosophic Cauchy sequence, neutrosophic Bessel's inequality, and neutrosophic Parseval's identity will be established and proved.

Keywords: Neutrosophic Banach space; neutrosophic Hillbert space; neutrosophic inner product.

Introduction

Neutrosophy is a new branch of philosophy established by F.Smarandache [8,9] to deal with indeterminacy in all fields of science and reality. The concept of neutrosophic set and its generalizations have been used widely in pure mathematical studies such as neutrosophic number theory, algebraic structures, matrices, and topology. See [2-7,11-16].

Neutrosophic vector spaces were defined firstly in [6] with many interesting substructures like weak/strong subspaces and weak/strong basis.

Neutrosophic functional analysis theory has been released recently in [5], with some basic concepts such as neutrosophic inner products and neutrosophic Cauchy-Schwartz inequality.

This work is concerned with extending classical concepts in functional analysis such as Banach and Hillbert spaces into neutrosophic systems, where the concept of neutrosophic Banach space and neutrosophic Hillbert space will be defined and handled for the first time. Also, we prove that Bessel's inequality and Parseval's famous identity are still true in neutrosophic functional analysis theory.

All neutrosophic spaces V(I) are considered strong over the real neutrosophic field R(I), that is because weak spaces are classified completely in [9] as a direct product of a classical vector space with itself.

The motivation of this work is to examine the validity of functional analysis tools in neutrosophic space theory.

This work will present a strong result about remaining of two classical famous inequalities into neutrosophic spaces, that is because neutrosophic vector spaces actually they are not vector spaces in the algebraic meaning (they are modules). Also, they do not have a total order relations, but only a partial order relation. This makes the remaining of the classical famous inequalities hard in general, and this paper proves this result for two famous classical inequalities.

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2. Preliminaries

Definition 1 [6]

Let $(V,+,.)$ be a vector space over the field K, $(V(I),+,.)$ is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Elements of V(I) have the form $x+yI\,;x,y\in V$, i.e V(I) can be written as $V(I)=V+VI$.

Definition 2 [6]

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I), then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition 3 [3]

Let $R(I)=\{a+bI;\ a,b\in R\}$ be the real neutrosophic field, we say that $a+bI\le c+dI$ if and only if $a\le c\ and\ a+b\le c+d$.

Theorem 1 [3]

The relation defined in the previous definition is a partial order relation.

Remark [3]

According to the previous theorem , we are able to define positive neutrosophic real numbers as follows:

$a+bI\ge 0=0+0.I$ implies that $a\ge 0, a+b\ge 0$.

Absolute value on R(I) can be defined as follows:

$|a+bI|=|a|+I[|a+b|-|a|]$, we can see that $|a+bI|\ge 0$.

Example 1 [5]

$x=2-I$ is a neutrosophic positive real number, since $2\ge 0\ and\ (2-1)=1\ge 0$.

$2+I\ge 2$, that is because $2\ge 2\ and\ (2+1)=3\ge(2+0)=2$.

Theorem 2 [5]

Let V be any inner product space over R, consider $g:V\times V\to R$ as its inner product. Then the corresponding neutrosophic strong vector space $V(I)$ has a neutrosophic real inner product as follows: $f:V(I)\times V(I)\to R(I); f(a+bI,c+dI)=g(a,c)+I[g(a+b,c+d)-g(a,c)]$

Theorem 3 [5]

Let V be any vector space over R, with a classical real inner product g, V(I) be its corresponding neutrosophic strong vector space, let f be the canonical inner product generated by g, we have

(a) $\|x\|=\|a\|+I[\|a+b\|-\|a\|]$ for all $x=a+bI\in V(I)$.

$(b) For\ x=a+bI, y=c+dI, x\perp y$ if and only if $a\perp$c, and $a+b\perp c+d$.

3. Main concepts and results

Definition 3.1:

Let V be a vector space over the field R, V(I) be its corresponding neutrosophic strong vector space over the neutrosophic field R(I). Consider an arbitrary sequence of its points

$x_1=a_1+b_1I, x_2=a_2+b_2I$, ..., it is called a neutrosophic Cauchy sequence if and only if:

For each $\varepsilon=\varepsilon_1+\varepsilon_2I>0, \exists\ N=N(\varepsilon)\in N; \forall\ m,r\ge N$, we have $\|x_m-x_r\|<\varepsilon$.

Example 3.2:

Let $V(I)=R(I)$ be the strong real neutrosophic vector space defined over itself, consider the following neutrosophic sequence $x_n=\frac{1}{n}+\frac{2}{n}I$. We can see

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$\|x_m - x_r\| = \left\|\left(\frac{1}{m}-\frac{1}{r}\right)+\left(\frac{2}{m}-\frac{2}{r}\right)I\right\| = \left|\frac{1}{m}-\frac{1}{r}\right| + I[\left|\frac{3}{m}-\frac{3}{r}\right| - \left|\frac{1}{m}-\frac{1}{r}\right|]$, without affecting the generality, we can assume that $m \le r$, thus $\left|\frac{1}{m}-\frac{1}{r}\right| = \frac{1}{m}-\frac{1}{r}, \left|\frac{3}{m}-\frac{3}{r}\right| = \frac{3}{m}-\frac{3}{r}$, so that

$\|x_m - x_r\| = \frac{1}{m}-\frac{1}{r}+I[\frac{2}{m}-\frac{2}{r}]$.

Since $x'_n = \frac{1}{n}, y'_n = \frac{2}{n}$ are two Cauchy sequences in the classical space V=R, then for each $\varepsilon_1 > 0, \varepsilon_2 > 0$ there are $N = N(\varepsilon_1, \varepsilon_2) \in N; \frac{1}{m}-\frac{1}{r} < \varepsilon_1, and \ \left(\frac{2}{m}-\frac{2}{r}\right) < \varepsilon_2$, thus $\|x_m - x_r\| = \frac{1}{m}-\frac{1}{r}+I\left[\frac{2}{m}-\frac{2}{r}\right] < \varepsilon_1 + \varepsilon_2 I = \varepsilon$, thus $x_n = \frac{1}{n}+\frac{2}{n}I$ is a neutrosophic Cauchy sequence.

Theorem 3.3:

Let x_n, y_n be two Cauchy sequences in the normed space V. Then $z_n = x_n + (y_n - x_n)I$ is a neutrosophic Cauchy sequence in V(I).

Proof:

For each $\varepsilon_1 > 0, \varepsilon_2 > 0$, there are $N_1 = N_1(\varepsilon_1), N_2 = N_2(\varepsilon_2) \in N; \forall m, r \ge N_1, N_2$, we have

$\|x_m - x_r\| < \varepsilon_1, \|y_m - y_r\| < \varepsilon_1 + \varepsilon_2$.

Now, we get $\varepsilon = \varepsilon_1 + \varepsilon_2 I > 0, and \ m, r \ge \ S = \max(N_1, N_2)$. On the other hand we have:

$\|z_m - z_r\| = \|(x_m - x_r) + I(y_m - y_r - x_m + x_r)\| = \|x_m - x_r\| + I[\|y_m - y_r\| - \|x_m - x_r\|]$.

Thus $\|z_m - z_r\| < \varepsilon$, and that is because $\|x_m - x_r\| < \varepsilon_1, \|y_m - y_r\| < \varepsilon_1 + \varepsilon_2$.

Hence $z_n = x_n + (y_n - x_n)I$ is a neutrosophic Cauchy sequence in V(I).

Theorem 3.4:

Let $z_n = x_n + y_n I$ be any neutrosophic Cauchy sequence in V(I). Then $x_n, y_n + x_n$ are two classical Cauchy sequences in V.

Proof:

For each $\varepsilon_1 > 0, \varepsilon_2 > 0$, we have $\varepsilon_1 + (\varepsilon_2 - \varepsilon_1)I = \varepsilon > 0$, and by the assumption, we have $S = S(\varepsilon) \in N; \forall m, r \ge S$: $\|z_m - z_r\| = \|(x_m - x_r) + (y_m - y_r)I\| = \|x_m - x_r\| + I[\|y_m - y_r + x_m - x_r\| - \|x_m - x_r\|] < \varepsilon$, thus $\|x_m - x_r\| < \varepsilon_1$, and $\|y_m - y_r + x_m - x_r\| < (\varepsilon_2 - \varepsilon_1) + \varepsilon_1 = \varepsilon_2$.

$\|(y_m + x_m) - (y_r + x_r)\| = \|y_m - y_r + x_m - x_r\| < \varepsilon_2$. Hence $x_n, y_n + x_n$ are two classical Cauchy sequences in V.

Theorem 3.5:

There is a corresponding one-to-one between Neutrosophic Cauchy sequences in V(I) and classical Cauchy sequences in $V \times V$.

Proof:

As a result of Theorems 3.4 and Theorem 3.3, we can find that map $f: C\big(V(I)\big) \to C(V \times V); f(z_n = x_n + y_n I) = (x_n, y_n + x_n)$ is a correspondence between the set of Neutrosophic Cauchy sequences $C\big(V(I)\big)$ and the set of classical Cauchy sequences in $V \times V$, which is $C(V \times V)$.

Example 3.6:

We have $x_n = \frac{1}{n}+\frac{2}{n}I$ as a Cauchy sequence in the neutrosophic space R(I), hence

$\frac{1}{n}, \frac{3}{n}$ are two Cauchy sequences in R.

Example 3.7:

(1)- We have $\frac{1}{n}, \frac{2}{n}$ two Cauchy sequences in R, hence $\frac{1}{n}+I\left[\frac{2}{n}-\frac{1}{n}\right] = \frac{1}{n}+I\frac{1}{n}$ is a Cauchy sequence in R(I).

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(2)- We have $\frac{n}{n-3}, \frac{1}{n^2+1}$ two Cauchy sequences in R, hence $\frac{n}{n-3} + I[\frac{1}{n^2+1} - \frac{n}{n-3}]$

Is Cauchy sequence in R(I).

This sequence converges to 1-I.

Definition 3.8:

Let V(I) be a real strong neutrosophic vector space defined over R(I), we say that it is a neutrosophic Banach space if and only if every neutrosophic Cauchy sequence of its points is convergent to a point in V(I).

Theorem 3.9:

Let V be any normed vector space over R, V(I) be its corresponding strong neutrosophic vector space. Then V(I) is a neutrosophic Banach space if and only if V is a classical Banach space.

Proof:

By Theorem 3.5, we get that any neutrosophic Cauchy sequence in V(I) such as $z_n = x_n + y_n I$ is convergent if and only if its corresponding image $(x_n, y_n + x_n)$ is convergent in $V \times V$, thus V(I) is a neutrosophic Banach space if and only if V is a classical Banach space.

Definition 3.10:

Let V(I) be any real strong neutrosophic inner product vector space, we say that V(I) is a neutrosophic Hillbert space if and only if it is a neutrosophic Banach space with respect to the norm generated by its inner product.

Theorem 3.11:

Let V be any classical Hillbert space. Then the corresponding neutrosophic strong vector space V(I) is a neutrosophic Hillbert space with respect to the canonical norm generated by the inner product on V.

Proof:

Suppose that V is a classical Hillbert space, then it is a Banach inner product space. By Theorem 3, there exists a norm on V(I) generated by the inner product on V. This means that V(I) is a neutrosophic inner product space. On the other hand, we have that V(I) is a Banach space according to Theorem 3.5, this implies that V(I) is a neutrosophic Hillbert space with respect to the canonical norm generated by the inner product on V.

Theorem 3.12: (Neutrosophic Bessel's inequality)

Let $V(I)$ be a neutrosophic strong real inner product space and $E = \{a_n + b_n I;\ a_i, b_i \in V\}$ be an orthogonal normed subset of $V(I)$. For any $x \in V(I)$, the series $\sum_{n=1}^{\infty} |f(x, e_n)|^2$ is convergent and $\sum_{n=1}^{\infty} |f(x, e_n)|^2 \leq \|x\|^2$.

Proof.

Let $x = x_1 + x_2 I\ \in V$, we have:

$f(x, e_n) = g(x_1, a_n) + I[g(x_1 + x_2, a_n + b_n) - g(x_1, a_n)]$, thus.

$\sum_{n=1}^{\infty} |f(x, e_n)|^2 = \sum_{n=1}^{\infty} |g(x_1, a_n)|^2 + I \sum_{n=1}^{\infty} (|g(x_1 + x_2, a_n + b_n)|^2 - |g(x_1, a_n)|)$.

By the Bessel's inequality in the classical space V, we can write:

$\sum_{n=1}^{\infty} |g(x_1, a_n)|^2 \leq \|x_1\|^2$ and $\sum_{n=1}^{\infty} |g(x_1, a_n)|^2$ is convergent.

Also, $\sum_{n=1}^{\infty} |g(x_1 + x_2, a_n + b_n)|^2 \leq \|x_1 + x_2\|^2$ and it is convergent too.

Thus, $\sum_{n=1}^{\infty} |f(x, e_n)|^2 \leq \|x_1\|^2 + I[\|x_1 + x_2\|^2 - \|x_1\|^2] = \|x\|^2$.

According to the definition of neutrosophic ordering partial relation.

Theorem 3.13: (Neutrosophic Parseval's identity)

Let $V(I)$ be a neutrosophicHillbert space, $E = \{e_n = a_n + b_n I;\ a_i, b_i \in V\}$ be an orthogonal normed subset of $V(I)$. Assume that $A = \{\alpha_n + \beta_n I;\ \alpha_n, \alpha_n \in R\}$ is a subset of neutrosophic real numbers field $R(I)$.

If $\sum_{n=1}^{\infty} |\alpha_n + \beta_n|^2$ is convergent, then $\|\sum_{n=1}^{\infty} A_n e_n\|^2 = \sum_{n=1}^{\infty} |A_n|^2$.

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Proof.

Firstly. We compute the left side as follows:

$\|\sum_{n=1}^{\infty} A_n\, e_n\|^2 = \|\sum_{n=1}^{\infty} \alpha_n\, a_n + I[\sum_{n=1}^{\infty}(\alpha_n + \beta_n)(a_n + b_n) - \sum_{n=1}^{\infty} \alpha_n\, a_n]\|^2 = \|\sum_{n=1}^{\infty} \alpha_n\, a_n\|^2 + I[\|\sum_{n=1}^{\infty}(\alpha_n + \beta_n)(a_n + b_n)\|^2 - \|\sum_{n=1}^{\infty} \alpha_n\, a_n\|^2]$,

By the classical Parseval's identity in V, we get: $\|\sum_{n=1}^{\infty} \alpha_n\, a_n\|^2 = \sum_{n=1}^{\infty}|\alpha_n|^2$ and

$\|\sum_{n=1}^{\infty}(\alpha_n + \beta_n)(a_n + b_n)\|^2 = \sum_{n=1}^{\infty}|\alpha_n + \beta_n|^2$.

this implies that,

$\|\sum_{n=1}^{\infty} A_n\, e_n\|^2 = \sum_{n=1}^{\infty}|\alpha_n|^2 + I[\sum_{n=1}^{\infty}|\alpha_n + \beta_n|^2 - \sum_{n=1}^{\infty}|\alpha_n|^2] = \sum_{n=1}^{\infty}|\alpha_n + \beta_n I|^2$.

Conclusion

In this paper, we have defined for the first time the concept of neutrosophic Banach space and neutrosophic Hillbert space. Also, we have studied some of their elementary properties, where we have proved that Parseval's identity and Bessel's inequality are still true in neutrosophic functional analysis according to the partial order relation defined on these spaces, although these structures are modules in the algebraic meaning, which represents a bridge between algebra and functional analysis.

According to this work, some open questions are coming to light.

Open question 1: Is Parseval's identity still true in the case of refined\n-refined neutrosophic vector spaces.

Open question 2: Is Bessel's inequality still true in the case of refined\n-refined neutrosophic vector spaces.

Open question 3: How to define refined\n-refined neutrosophic Banach/Hillbert spaces.

Open question 4: Is there an equivalence between Cauchy sequences in classical spaces and refined neutrosophic spaces.

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