



## A Short Contribution to the Extension of the Group $C(n)$

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### Abstract

In this work, we study the extension problem of a group with type  $C(n) = \sum_{i \in I} C_p^\infty$  by using a cyclic group of order  $p$ . Also, we prove the following two results: 1-) The group  $C(n)$  has only one extension which is compatible with  $R_3$ . 2-) The group  $C(n)$  has two extensions which are compatible with  $R_2$ .

**Keywords:** Group; Extension; Group representation.

### 1. Introduction

In the theory of algebraic structures, the extension of these structures plays a basic role in the generalizing and improving algorithms.

In the literature, we find many concepts and results that expand our view [1].

On the other hand, an extension of the group  $C(n) = C_p^\infty \oplus C_p^\infty \oplus \dots \oplus C_p^\infty$  by a cyclic group  $A$  with prime order  $P$  is another group  $G$  such that  $G / C(n) \cong A [2]$ .

The group  $G$  is defined as a homomorphism  $R: A \rightarrow \text{Aut} C(n)$  with a system  $M_{a,b}$ ;  $a, b \in A$ .

In this paper, we define some extensions of  $C(n)$  by  $A$  which is compatible with  $Z_p$ - irreducible representation  $R$  of the cyclic group  $A$ .

### Main Discussion

#### Theorem 1:[ ]

If  $G / C(n) \cong A$  and  $\bar{a} \in A$  denotes to the equivalence classes, then:

1). the equivalence are represented  $1, a, a^2, \dots, a^{P-1}$ ;  $a^P = \alpha \in C(n)$

2).  $M_{a,b} = \begin{cases} 1; & i + j < P \\ \alpha; & i + j \geq P \end{cases}$

#### Definition 2:[ ]

Let  $A$  be an extension of  $C(n)$  by  $A$ , then we denote by  $C(R, \alpha)$ ;  $\alpha \in C(n)$ .

#### Theorem 3: [ ]

Every extension  $C(n)$  by  $A$  which is represented by an irreducible  $Z_p$ -representation has the property  $R(\alpha) = \alpha^{-1} \alpha a = \alpha$ .

#### Definition 4:[ ]

1). We define  $A(R) = \{x \in C(n); R(x) = x\}$ .

2).  $Z_p$ - irreducible representation of  $A$  :

$R_1: A \rightarrow GL(1, Z_p); a \rightarrow 1$ .

$R_2: A \rightarrow GL(P-1, Z_p); a \rightarrow \varepsilon$ .

$R_3: A \rightarrow GL(P, Z_p); a \rightarrow \begin{pmatrix} \varepsilon & \langle 1 \rangle \\ 0 & 1 \end{pmatrix}; \langle 1 \rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \varepsilon = \begin{pmatrix} 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ 0 & \dots & 0 & 0 & -1 \end{pmatrix}$

#### Theorem 5:

Let  $C(n)$  be the group described previously, hence the extension which is compatible with  $R_1$  is  $C(R_1, 0)$ .

**Proof.**

Let's compute  $A(R_1)$ ,  $B(R_1)$  as follows:

$$A(R_1) = \{x \in C(n) ; 1.x\} = C(1) = C_P^\infty.$$

$$B(R_1) = \{Px ; x \in C(1)\} = C(1) = C_P^\infty.$$

Thus  $A(R_1)/B(R_1) = 0$ , hence  $\alpha = 0$ .

**Theorem 6:**

The group  $C(n)$  has two extensions which are compatible with  $R_2$ .

**Proof.**

We have  $A(R_1) = \{x \in C(P-1) ; R_2(x) = x\}$ .

$$x \in C(P-1) \Rightarrow x = (x_1, x_2, \dots, x_{P-1}) ; x_i \in C_P^\infty.$$

$$\text{This implies: } \begin{cases} -x_{P-1} = x_1 \\ x_1 - x_{P-1} = x_2 \\ \vdots \\ \vdots \\ x_{P-2} - x_{P-1} = x_{P-1} \end{cases}$$

Thus  $x_{P-1} \equiv 0 \pmod{P}$  and  $x_{P-1} = \lambda a_0$ , where  $\lambda = 0, 1, \dots, P-1$ .

$A(R_2) = \{\lambda(a_0, 2a_0, \dots, (P-1)a_0), \lambda = 0, 1, \dots, P-1\}$  which is a  $P$ -order cyclic group.

On the other hand, we have.

$$B(R_1) = \{(I + R_2(a) + R_2(a^2) + \dots + R_2(a^{P-1}))x ; x \in C(P-1)\}.$$

This implies  $A(R_2) / B(R_2) = A(R_2)$ .

It is an easy task to check that.

The extensions  $G(R_2, 2\beta), \dots, G(R_2, (P-1)P)$  are isomorphic.

So that, we get two non-isomorphic extensions  $G(R_2, 0), G(R_2, \beta) ; \beta = (a_0, 2a_0, \dots, (P-1)a_0)$

**Theorem 7:**

The group  $C(n)$  has only one extension which is compatible with  $R_3$ .

**Proof.**

$$\text{We have } A(R_3) = \{(P-1)x, (P-1)x, \dots, 2x, x, Px ; x \in C_P^\infty.\}$$

$$B(R_3) = \{(I + R_3(a) + \dots + R_3(a^{P-1}))x ; x \in C(P)\}$$

This means that  $A(R_3) / B(R_3) = 0$ , thus  $\alpha = 0$ . So that  $G(R_3, 0)$  is the only extension.

**Conclusion**

In this article, we have presented two results about the extension problem of the group  $C(n)$  by using cyclic groups, where we have proved:

- 1-) The group  $C(n)$  has only one extension which is compatible with  $R_3$ .
- 2-) The group  $C(n)$  has two extensions which are compatible with  $R_2$ .

**References**

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