Some Results about Cauchy Improper Integral

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Abstract

The objective of this paper is to estimate the value of $\int_{-1}^{1}\frac{f(x)}{(x-t)^m}dx\,; m \in N$ where f is convex from above and $\int_0^\delta \frac{w(f^{(m-1)},x)}{x}dx < \infty$; f is bounded in $C^{(m-1)}[-1,1]$.

Keywords: Improper Integral; Cauchy Principal value; Convex function

1. Introduction

The estimation of the value of on improper in integral is an interesting open problem in mathematical analysis [1-3].

In the literature, there exists many ways to compute approximations of an improper integral [4-5].

For example, in [6], we find the formula:

$$\int_a^b f(x)dx = \lim\left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx\right];\ \varepsilon \to 0$$

Where f isnited define in $C[a,b]$.

The limitation of the right side is called Cauchy principal value [6], it is denoted by V.P$\int_a^b f(x)dx$.

In this work, we get to estimate the value of $\int_{-1}^{1}\frac{f(x)}{(x-t)^m}dx\,; m \in N$ by using the modulus of continuity$w$ with some strict conditions.

Main discussion :

Definition:[1]

the integral $I(f,t) = \int_{-1}^{1}\frac{f(x)}{(x-t)^m}dx\,; -1 < t < 1, m \in N$ is called Caushy integral if $m = 1$.

Definition:[2]

The modulus of continuity of a bounded function f on $[a,b]$is defined as follows:

$$w(f,s) = sup(|f(x) - f(y)|)\,; x,y \in [a,b]\,, |x-y| \le \delta$$

Definition: [5]

Let $f$be a defined and conditions function an $[a,b]$, it is called convex from above if:

$\frac{f(t+h)-f(t)}{h} \le \frac{f(t)-f(t-h)}{h}$.

Theorem 4:

Let $f \in C[-1,1]$, $w(f,s)$ be the modulus of continuity, then if $\int_0^\delta w\left(f^{(m-1)},x\right)\frac{dx}{x} < \infty$

we get for every $0 < \delta < \frac{1}{2}\,, t \in]-1,1[$.

$\left|\int_{-1}^{1}\frac{f(x)}{(x-t)^m}dx\right| \le 2\|f\|_{C[-1,1]}.\int_\delta^1\frac{dx}{x^m} + \frac{2}{(m-1)!}\int_0^\delta w\left(f^{(m-1)},x\right)\frac{dx}{x} + \sum_{k=0}^{m-1}\frac{|f^{(k)}(t)|}{k!}.\left|\int_a^b\frac{f(x)}{(x-t)^{m-k}}dx\right|; a = max(t-\delta,-1), b = min(t+\delta,1)$.

Proof.

Let $t \in [0,1[$, we expand $f$by Taylor's series as follows:

$$f(x) = \sum_{k=0}^{m-1}\frac{f^{(k)}(t)}{k!}(x-t)^k + R_m(x) = P_{m-1}(x) + R_n(x)$$

Doi: https://doi.org/10.54216/GJMSA.010201

Received: January 10, 2022 Accepted: April 15, 2022

On the other hand, we have:
$R_m(x) = f(x) - P_{m-1}(x) = \frac{1}{(m-1)!}\int_t^c (x-t)^{m-1} f^{(m)}(t)\,dt = (x-t)^{m-1}\left[f^{(m-1)}(c) - f^{(m-1)}(t)\right];\ t < c < x.$
Now, we consider $\delta \in \left]0, \frac{1}{2}\right[$, we have two cases:
First case: if $t < 1 - \delta$, we put:
$\int_{-1}^{1} \frac{f(x)}{(x-t)^m} dx = \int_{-1}^{t-\delta} \frac{f(x)}{(x-t)^m} dx + \int_{t-\delta}^{t+\delta} \frac{f(x) - P_{m-1}(x) + P_{m-1}(x)}{(x-t)^m} dx + \int_{t+\delta}^{1} \frac{f(x)}{(x-t)^m} dx$,
Now, we estimate each term>
$\left|\int_{-1}^{t-\delta} \frac{f(x)}{(x-t)^m} dx + \int_{t+\delta}^{1} \frac{f(x)}{(x-t)^m} dx\right| \le \|f\|_{C[-1,1]} + \left(\int_{-1-t}^{-\delta} \frac{du}{u^m} + \int_{\delta}^{1+t} \frac{du}{u^m}\right),$
By substituting $u$with $-u$, we get:
$\left|\int_{-1}^{t-\delta} \frac{f(x)}{(x-t)^m} dx + \int_{t+\delta}^{1} \frac{f(x)}{(x-t)^m} dx\right| \le 2\|f(t)\| - \int_{\delta}^{1} \frac{dx}{x^m},$
Hence:
$\left|\int_{t-\delta}^{t+\delta} \frac{f(x) - P_{m-1}(x)}{(x-t)^m} dx\right| = \left|\int_{t-\delta}^{t+\delta} \frac{(x-t)^{m-1}}{(m-1)!} \cdot \frac{f^{(m-1)}(c) - f^{(m-1)}(t)}{(x-t)^m} dx\right| \le \frac{1}{(m-1)!}\left|\int_{t-\delta}^{t+\delta} \frac{f^{(m-1)}(c) - f^{(m-1)}(t)}{(x-t)^m} dx\right| \le$
$\frac{2}{(m-1)!}\int_{t-\delta}^{t+\delta} w\left(f^{(m-1)}, x\right)\frac{dx}{x},$
Also.

$$\left|\int_{t-\delta}^{t+\delta} \frac{f_{m-1}(x) - f^{(m-1)}(t)}{(x-t)^m} dx\right| \le \sum_{k=0}^{m-1} \left|\frac{f^{(k)}(t)}{k!}\right| \left|\int_{t-\delta}^{t+\delta} \frac{dx}{(x-t)^m}\right|$$

From the previous discussion, we get:

$$\left|\int_{-1}^{1} \frac{f(x)}{(x-t)^m} dx\right| \le 2\|f\|_{C[-1,1]} \cdot \int_{\delta}^{1} \frac{dx}{x^m} + \frac{2}{(m-1)!}\int_{0}^{\delta} w\left(f^{(m-1)}, x\right)\frac{dx}{x} + \sum_{k=0}^{m-1} \frac{\left|f^{(k)}(t)\right|}{k!} \cdot \int_{a}^{b} \frac{f(x)}{(x-t)^{m-k}} dx \ ; a = max(t-\delta, -1), b = min(t+\delta, 1)$$

Second case: if $t \ge 1 - \delta$, then:

$$\int_{-1}^{1} \frac{f(x)}{(x-t)^m} dx = \int_{-1}^{t-\delta} \frac{f(x)}{(x-t)^m} dx + \int_{t-\delta}^{1} \frac{f(x) - P_{m-1}(x) + P_{m+1}(x)}{(x-t)^m} dx$$

By a similar discussion to the first case

$$\left|\int_{-1}^{t-\delta} \frac{f(x)}{(x-t)^m} dx\right| \le 2\|f(t)\| \cdot \int_{\delta}^{1} \frac{dx}{x^m}$$

$$\left|\int_{t-\delta}^{1} \frac{f(x) - P_{m-1}(x)}{(x-t)^m} dx\right| = \left|\int_{t-\delta}^{1} \frac{(x-t)^{m-1}}{(m-1)!} \cdot \left[f^{(m-1)}(c) - f^{(m-1)}(t)\right] \frac{dx}{(x-t)^m}\right|$$

$$\le \frac{2}{(m-1)!} \int_{t-\delta}^{t+\delta} w\left(f^{(m-1)}, x\right)\frac{dx}{x}$$

$$\left|\int_{t-\delta}^{1} \frac{P_{m-1}(x)}{(x-t)^m} dx\right| \le \sum_{k=0}^{m-1} \frac{\left|f^{(k)}(t)\right|}{k!} \cdot \int_{t-\delta}^{1} \frac{dx}{(x-t)^{m-k}} = \sum_{k=0}^{m-1} \frac{\left|f^{(k)}(t)\right|}{k!} \cdot \left|\int_{t-\delta}^{min(t+\delta,1)} \frac{f(x)}{(x-t)^{m-k}} dx\right|$$

From the previous discussion, we get:
$\left|\int_{-1}^{1} \frac{f(x)}{(x-t)^{m-k}} dx\right| \le 2\|f\| \cdot \int_{\delta}^{1} \frac{dx}{x^m} + \frac{2}{(m-1)!}\int_{0}^{\delta} w\left(f^{(m-1)}, x\right)\frac{dx}{x} + \sum_{k=0}^{m-1} \frac{\left|f^{(k)}(t)\right|}{k!} \cdot \left|\int_{t-\delta}^{min(t+\delta,1)} \frac{dx}{(x-t)^{m-k}}\right|.$

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Doi: https://doi.org/10.54216/GJMSA.010201

Received: January 10, 2022 Accepted: April 15, 2022

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Doi: https://doi.org/10.54216/GJMSA.010201
Received: January 10, 2022 Accepted: April 15, 2022