On Some Applications and Open Problems about (m-Groups)

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Abstract

The generalizations of abelian groups have been studied widely because of their importance in classification theorem and representation. A group G is called an m-power closed group or (m-group) if and only if it has the following property $x^m y^m = z^m \ \forall x, y \in G$ and for $z \in G$. This paper studies a special case of m-groups, when G is a finite m-group and n-group at the same time with relatively prime integers m and n, which is called a Monic group. It presents the necessary and sufficient conditions for a monic group G to be cyclic, abelian, nilpotent, and solvable by the corresponding property of its power subgroups $G_m,\ G_n$. Also, this work introduces three open problems in the theory of finite groups.

Keywords: m-power closed group; m-abelian group; monic group.

1. Introduction

Abelian groups were the most applicative concept in Group Theory since they were used to define rings, modules, and vector spaces, and to classify other kinds of groups [9-10].

Some interesting generalizations of abelian groups such as m-groups and n-abelian groups were defined in [1,4-7].

Through this paper, we study a special case of m-groups (Monic groups), where G is an m-group and n-group at the same time, with relatively prime integers m and n. We determine the necessary and sufficient conditions for the cyclicity, nilpotency, and solvability of Monic groups, as well as, we list three open questions concerning the relationship between the properties of a Monic group G and its power subgroups $G_m,\ G_n$.

Monic groups

Definition [1] :

Let G be a finite (m-group) with m/$|G|$ then we say that it is a monic group if and only if G is an (n-group) with n/$|G|$ and gcd (n,m)=1

Lemma [1] :

Let G be a monic group then :

(a) G=$G_m G_n$

(b) The homomorphic image of G is also monic

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Proof :

(a) Holds directly.

(b) Since the homomorphic image of (m-group) is also (m-group) the proof is complete

Lemma [1]:

Let G be a group and H▶G then G/H is monic if and only if $H \blacktriangleright_m G$ and $H \blacktriangleright_n G$

Proof :

Since G/H is (m-group) if and only if $H \blacktriangleright_m G$ then the proof is complete

Theorem[1]:

Let G be a monic group then :

(a) G is solvable if and only if G_m , G_n are solvable

(b) If G_m, and G_n are nilpotent groups then G is nilpotent

(c) If G_m, G_n have an abelian automorphism group then G has an abelian automorphism group

Theorem:

Let G be a monic group then G is cyclic if and only if G_m and G_n are cyclic

Proof: If G is cyclic then it is monic with cyclic G_m, G_n. Conversely, suppose that G_m and G_n are cyclic then they are nilpotent so G is nilpotent and G is a direct product of its Sylow subgroups, let P_i be the (i-th) Sylow subgroup of this product with order p^s then P_i is (m-cyclic) and (n-cyclic) because p is a prime we find that gcd (m,p)=1 or gcd (n,p)=1, without affecting the generality we assume that gcd (m,p)=1 so $(P_i)_m = P_i$ and P_i must be cyclic. By cyclicity of G_m, G_n we find that they have an abelian automorphism group so G is a direct product of cyclic groups with abelian automorphism group then G must be cyclic.

Theorem :

Let G be a finite nilpotent group then G is monic.

Proof: Assume that G is nilpotent then $G = P_1 \times P_2 \times \dots . P_n$ where P_i *is a Sylow subgroup with order* ${p_i}^{k_i}$, we put m= ${p_1}^{k_1}$ and n= ${p_2}^{k_2}$ then $G_m = P_2 \times \dots . P_n$ and $G_n = P_1 \times P_3 \times \dots . \times P_n$ so G is a monic group since gcd(n,m)=1

Theorem :
The direct product of two monic groups is also a monic group
Proof: Holds directly.

Meta properties of Monic Groups

A famous result in group theory ensures that if G is a finite group and H, K are normal subgroups with $G = HK$, and S is another finite group with two normal subgroups M, N with $S = MN$, then

If $H \cong M$ and $K \cong N$, it is not true that $S \cong G$ in general.

Also, if H and K are meta abelian, or meta nilpotent then G is not supposed to be the same in general.

Since the only known example of a non-abelian monic group is a nilpotent group or direct product of two groups with relatively prime orders, we can suggest the following conjectures:

Conjecture :

Let G, and H be two monic groups, we have:

If $G_m \cong H_m, G_n \cong H_n$ then $G \cong H$.

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Conjecture:

Let G be a monic group then G is meta abelian if and only if G_m *and* G_n are meta abelian.

Conjecture :

Let G be a monic group then G is meta nilpotent if and only if G_m *and* G_n are meta nilpotent.

Example :

Let G =D_4 the dihedral group, G is monic with respect to $m = 2, n = 3$;

$G_2 \cong Z_2,\ G_3 = G$.

We can find that G_2, $G3$, *and* G are meta abelian, meta cyclic, and meta nilpotent at the same time.

Example:

Let $G = Z_6 \times D_4$ be a monic group with order 48, $G_2 \cong Z_3 \times Z_2$, $G_3 \cong Z_2 \times D_4$, we can see that G_2, $G3, G$ are meta abelian, meta cyclic and meta nilpotent at the same time.

We discuss some special cases and interesting properties.

Lemma:

Let G be a monic group with respect to m, n, suppose that G_m is abelian, then$G' \leq G_n$.

Proof:

We know that $G_m/(G_n \cap G_m) \cong G/G_n$ see [1], so $[G_m/(G_n \cap G_m)]' = [G/G_n]'$, since G_m is abelian we find that $G_m/(G_n \cap G_m)$ is abelian and its derived subgroup is equal to the identity group, thus $[G/G_n]' = G'G_n/G_n = G_n/G_n$ which means that $G' \leq G_n$.

Theorem:

Let G be a monic group with respect to m, n. If G_m is abelian and G_n is meta abelian then G' is meta abelian.

Proof:

We can find that $G' \leq G_n$. Hence G' is Meta abelian since it is a subgroup of the meta abelian subgroup.

Theorem:

Let G be a finite nilpotent monic group. Then previous conjectures are true.

Proof :

G is monic, so G can be written as a direct product of its Sylow subgroups such

$G = P_1 \times P_2 \times \ldots \times P_s$.

G is monic with respect to $m = O(P_1), n = O(P_2)$;

$G_m = P_2 \times \ldots \times P_s, G_n = P_1 \times P_3 \times \ldots \times P_n$.

By the previous aspect, we find easily that if G_m *and* G_n are meta abelian/nilpotent, then G is the same, hence conjectures hold in this case.

Now assume that $H = Q_1 \times Q_2 \times \ldots \times Q_l = H_m H_n$ is a finite nilpotent monic group with property $G_n \cong H_n$, $G_m \cong H_m$, we get that $l = s$ *and* $Q_i \cong P_i$ for each i.

This implies that $H \cong G$.

Conclusion

In this article we have studied the relationship between powers' subgroups of a monic group and we have listed the following three conjectures:

Conjecture 1:

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Let G, and H be two monic groups we have:

If $G_m \cong H_m, G_n \cong H_n$ then $G \cong H$.

Conjecture 2:

Let G be a monic group then G is meta abelian if and only if G_m *and* G_n are meta abelian.

Conjecture 3:

Let G be a monic group then G is meta nilpotent if and only if G_m *and* G_n are meta nilpotent.

We have proved the validity of previous conjectures in the case of nilpotent monic groups as a partial solution of them.

Also, we have proved the following results:

1) Proving that a monic group has an abelian automorphisms group if G_m, G_n has abelian automorphisms groups.

2) A monic group G is solvable/nilpotent if and only if G_m *and* G_n are the same.

3) A monic group G is cyclic/abelian if and only if G_m *and* G_n are the same.

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