A Study of Novel Algebraic Game Over Some Finite Groups and Open Problems

Mohammad Abobala

Tishreen University, Department of Mathematics, Latakia, Syria
Email: Mohammadabobala777@gmail.com

Abstract

This paper uses the algebraic structure of the group to introduce a novel algebraic game with three players that occurs over finite groups.

Also, it analyzes this game over some finite groups with orders up to ten.

Keywords: Algebraic game; Winning strategy; MH-game**.**

1. Introduction

Game theory as a branch of applied mathematics played an important role in the study of real actions, and many examples of game theory applications were introduced, see [6,7]. In particular, Game Theory was used widely in algebraic studies. In[1], authors created two games played over finite groups. They analyzed these games in some special cases, especially abelian groups and dihedral groups. In[3,4,5], we find other games played over algebraic rings and infinite groups.

One of the most interesting applications of games is to get some algebraic properties from a game.

In this paper, we try to define and study a new non-combinatorial game over finite groups. We study this game over some finite groups with order $n \leq 10$. This work may be very interesting and useful, it provides us with more comprehension of some group-theoretic properties related to this game. In particular, we introduce an interesting open problem related to this game.

2. Preliminaries

Definition 2.2 :[1]

Let (F,*) be a finite group then the (Avoid the identity) game [ID-Game] can be defined as :

One and Two alternately pick unchosen members of F.

The player whose selection causes the group product of all members chosen thus far to be the group identity, loses.

Doi : https://doi.org/10.54216/GJMSA.010104
Received: February 25, 2022 Accepted: April 15, 2022

Example 2.3:[1]

Let G=Z_5 then One and Two plays ID(G,+) like

Player One	Player Two	Sum mod 5
2		2
	4	1
3		4
	0	4
1		0

Player Two wins the game.

Theorem 2.3 :[1]

If (F,*) is an abelian group with odd order, then player Two has a winning strategy.

Theorem 2.4 :[1]

If (F,*) is abelian with even order, then player one has a winning strategy if and only if G has no subgroups which are isomorphic to $Z_2 \times Z_2$.

3. Rules of MH-game over finite groups

Definition 3.1:

Let G be a finite group, suppose that A, B, and C are three players. They doing their choices alternatively. Player A makes the first choice, B makes the second, and so on. This game ends when there is not any element to choose from.

In each step i, we compute the group product of chosen elements, say it is s_i.

If s_i is an element of the first player's choices, then A gets a point, if it is in the second player's choices, then B gets a point, and so on. The player with the biggest number of points wins the game.

If any two players have the same number of points. The winner will be the player who picked the identity of the group.

Example 3.2:

Let G=Z_7 be the additive group of integers modulo 7, we summarize MH-game between three players A, B, and C as follows:

A	B	C	Group product
0	3	2	5
5	1	4	3
6			6

We can see that player A has two points because he owns {5,6}, B has one point because he owns {3}, and C has no points. Thus A is the winner.

Theorem 3.3:

Let G be a finite abelian group with order $n \in \{2, 3, 4, 5, 6\}$. The first player A has a winning strategy.

Proof:

(a) If O(G)=2, then $G \cong Z_2$. The winning strategy can be described as follows:

A begins with 1, B is forced to choose 0, and C has no choices. The group product in this case is equal to 1, hence A is the winner since he had 1 in his choices.

Doi : https://doi.org/10.54216/GJMSA.010104
Received: February 25, 2022 Accepted: April 15, 2022

(b) If O(G)=3, then $G \cong Z_3$. The winning strategy is:

A begins with 0, B should pick 1 or 2, C has to pick 1 if B picked 2, and 2 if B picked 1. In all cases, the group product will be equal to 0, and A is the winner.

(c) If O(G)=5, then $G \cong Z_5$. The winning strategy is:

A begins with 0, then B and C pick two elements $x \text{ and } y$. If $x + y = 0$ then A should pick any unchosen element z, B has an additional choice t with property $t + z = 0$ (because y is the inverse of x, so t is the inverse of z). In this case, A will has two points and it is the winner.

If $x + y = m \neq 0$, then A should pick m in the next step. In this case, A will be the winner since he has one point (because he owns m) and he has the identity 0.

(d) If O(G)=6, then $G \cong Z_6$. We describe the winning strategy as follows:

A begins with 0, then B and C pick two elements $x \text{ and } y$. If $x + y = 0$ then A should pick any unchosen element z, B has an additional choice t, and C has a choice k.

We remark that A has one point and identity too. Thus A is the winner.

If $x + y = m \neq 0$, then A should pick m in the next step. In this case, A will be the winner since he has one point (because he owns m) and he has the identity 0.

(e) If O(G)=4, then $G \cong Z_4 \text{ or } G \cong Z_2 \times Z_2$. Firstly, we assume that $G \cong Z_4$. In this case A should begin with 0, if B and C picked 1 and 3, A will be the winner because he owns their group product 0. If B and C picked 2 and 3, A will be the winner because in the next step he is forced to choose their group product 1. If B and C picked 1 and 2, A will be the winner because he owns their product 3.

Now we assume that $G \cong Z_2 \times Z_2$. In this case, A should begin with (0,0), if B and C picked (1,0) and (0,1), A will be the winner because he is forced to choose (1,1), which is their product. If B and C picked (1,0) and (1,1), A will be the winner because he owns their product (0,1). If B and C picked (1,1) and (0,1), A will be the winner because he owns their product (1,0).

Example 3.4:

In this example, we clarify the winning strategy with O(G)=6.

Suppose that A begins with 0 (as we suggested in the previous theorem), B picked 2, and C picked 3. We can see that 2+3=5 which is not zero, so A should pick 5.

Now player B has to choose between 4 and 1, suppose that he picked 1, and C is forced to pick 4. The following table clarifies these actions.

A	**B**	**C**	**Group product**
0	**2**	**3**	**5**
5	**1**	**4**	**4**

We find that A has one point, and C has one point, hence A is the winner since he has 0.

Theorem 3.5:

Let G be a finite abelian group with order 10. The first player A has a winning strategy.

Proof:

We have O(G)=10, so $G \cong Z_{10}$. We describe the winning strategy as follows:

A should begin with 0, suppose that B and C picked x and y respectively.

If $x + y = m \neq 0$, then A should take m in the second step. Now A has a point and identity in its choices. Since we have 10 elements in G, we find that after three steps, player A will be forced to choose the only element which has not been chosen in the first three steps, so it will get an additional easy point. In this case, A is the winner since he has two points with identity too.

Doi : https://doi.org/10.54216/GJMSA.010104
Received: February 25, 2022 Accepted: April 15, 2022

If B and C picked x and y, while $x + y = 0$, then A gets a point. By following the same argument we find that A has a certain point as a result of the fourth step. In this case, A is the winner because he has two points with identity too.

Example 3.6:

We clarify MH-game over Z_{10}. We describe it in the following table:

A	**B**	**C**	**Group product**
0	**4**	**5**	**9**
9	**2**	**3**	**4**
1	**7**	**6**	**4**
8			**8**

We can see that A has two points, B has two points, and C has no points.

A is the winner since he owns 0.

It is known that non-abelian groups with order less than ten is S_3 *and* D_4.

Theorem 3.7:

For the symmetric group G=S_3. A has a winning strategy.

Proof:

A should begin with I (Identity map). If C and B took x *and* $y; xy = I$, then A gets a point. In this case, A will be the winner since he has a point and identity in his choices.

If C and B took x *and* $y; xy = z \neq I$, A should pick z in the second step. Now A is the winner since he has a point with identity in his choices.

Example 3.8:

We assume that three players A, B, and C play MH-game over S_3 by the following table:

A	**B**	**C**	**Group Product**
I	**(1 2)**	**(2 3)**	**(2 3 1)**
(2 3 1)	**(1 3)**	**(3 2 1)**	**(2 3 1)**

It is easy to see that according to our algorithm, player A is the winner.

Remark 3.9:

In Example 3.8, we wrote each map as the second line in Cayley's representation of S_n.

Theorem 3.12:

Let $G = Z_9$ be the group of integers modulo 9. If A began the game by taking 0, then he has not a winning strategy.

Proof:

We will find a scenario of an MH-game leads A to a loser position in all possible choices.

First step: Under the theorem's condition, A must begin with 0. We assume that B and C took 1, 3 respectively.

A	**B**	**C**	**Group product**
0	**1**	**3**	**4**

In the second step, if A picked 4, then B and C should pick 7, 8.

A	**B**	**C**	**Group product**
4	**7**	**8**	**1**

Doi : https://doi.org/10.54216/GJMSA.010104
Received: February 25, 2022 Accepted: April 15, 2022

In the third step, player A is forced to pick 2 or 5. In both cases, B will be the winner.

If A picked 2 in the second step, B and C should pick 4, 5.

A	B	C	Group product
2	4	5	2

In the third step, A is forced to pick 6 or 7. In both cases, B will be the winner.

If A picked 5 in the second step, B and C should pick 4,2.

A	B	C	Group product
5	4	2	2

In the third step, A is forced to pick 6 or 7. In both cases, B will be the winner.

If A picked 7, B and C can take 4, 8. If A picked 8, B and C can take 4,7. In both cases we get a situation that is similar to the first one, thus A is a loser.

By the previous argument, A has no winning strategy if he began with 0.

Theorem 3.13:

Let G= Z_7, then A has a winning strategy.

Proof:

A should begin with 0. If B and C picked two elements $x, and\ y$ with $x + y = 0$, then A gets a point. Also, player A has an easy point because in the third step he will be the only player who has choices, hence A is the winner in this case.

If B and C picked two elements $x, and\ y$ with $x + y \neq 0$, then A should pick $x + y$ in the second step. According to the first argument, A will be the winner because he owns two points.

Definition 3.14:

Let G be a finite group, we call it an MH group if and only if there is a winning strategy for at least one player in the corresponding MH game.

Result 3.15:

According to previous theorems, we can see that

$Z_2,$ $Z3, Z4, Z5, Z6, Z7, Z_8,\ Z_2 \times Z_2,\ Z_{10},\ and\ S_3$, are MH-groups; but Z_9 maybe not.

Results and recommendations

In this paper, we have defined the MH-game over finite groups. Also, we have found a winning strategy with respect to first player A in the following cases:

$Z_2, Z_3, Z_4, Z_5, Z_6, S_3, Z_{10}, Z_7$. And we have proved that A has no winning strategy with respect to Z_9, under the assumption that he began his choices with 0.

The authors hope that MH-game will be analyzed over $Z_2 \times Z_2 \times Z_2, Z_8,\ Z_4 \times Z_2,\ and\ D_4$ in future research.

Open Problems

1-) Describe all MH-abelian groups.

2-) Describe all MH-non abelian groups.

References

[1] Babinkostova, L; Cosket, S ; Kontradyuk, D; Navert, S; Potter, S; and Scheepers, M, "A Study of Games Over Finite Groups", Publication at www.researchgate.net, Boise State university, 2015, pp13.

[2] Bal, M., and Abobala, M., " On The Representation Of Winning Strategies Of Finite Games By Groups and Neutrosophic Groups", Journal Of Neutrosophic and Fuzzy Systems, 2022.

[3] Selvakumari, K; and Lavanya, S, "Neutrosophic Fuzzy Soft Game", International Journal of Engineering and Technology, vol 7, 2018, pp. 667-669.

Doi : https://doi.org/10.54216/GJMSA.010104

Received: February 25, 2022 Accepted: April 15, 2022

[4] Benesh, B.J; Ernest, D . C; and Sieben, N, "Impartial Avoidance Games for Generating Finite Groups", North-Western European Journal of Mathematics, 2016, pp. 89-103.
[5] Ahmad Dar, Y; and Sharma, Y, "Algebraic Method in Game Theory", International Journal of Multidisciplinary Research and Development, vol. 4, 2017, pp. 368-372.
[6] Gavel, H; and Strimling, P, "Nim with a modular Muller Twist", Integers, 2004.
[7] John Milnor, "Sums of positional games", Ann. Of Math. Stud. (Contributions to the Theory of Games, H. W. Kuhn and A. W. Tucker, eds.), Princeton, 2(28):291-301, 1953.

Doi : https://doi.org/10.54216/GJMSA.010104
Received: February 25, 2022 Accepted: April 15, 2022