# Examples on Some Novel Diophantine Equations Derived from the Group of Units Problem in n-Cyclic Refined Neutrosophic Rings of Integers

**A. Alrida Basheer *1, Katy D. Ahmad2, Rozina Ali3**

1 Imam Kadhum College, Iraq
2 Islamic University Of Gaza, Palestine
3 Cairo University, Egypt
Emails: basheerabdalrida66n@gmail.com ; katyon765@gmail.com ; rozyyy123n@gmail.com

**Abstract**

The objective of this paper is to present a new class of Diophantine equations derived from the group of units problem of n-cyclic refined neutrosophic rings of integers by using homomorphisms between these rings and a finite Cartesian product ring of Z with itself.

Also, this work provides many examples about this class and its solvability as a new application of neutrosophic algebraic structures in number theory.

**Keywords:** n-cyclic refined neutrosophic ring; n-cyclic refined neutrosophic integer; group of units

## 1. Introduction

The theory of neutrosophic algebraic structures was released many years ago. Many researchers around the world have presented some generalizations of classical algebraic structures by using neutrosophic and fuzzy logic [1-5]. We can see neutrosophic matrices, rings, modules, and related structures [6-13].

The concept of an n-cyclic refined neutrosophic ring was defined for the first time by M. Abobala in [1] for the first time. These rings were applied in many contexts.

In [14], the famous group of units' problem of n-cyclic refined neutrosophic rings was discussed by Sadiq and many open problems have been released. Also, Von Shtawzen [21] has found a connection between units and Diophantine equations.

The group of units' problem in these rings motivated us to study algebraic homomorphisms between some n-cyclic refined neutrosophic rings (especially integers) with the classical direct product of the ring Z with itself many times, which lead to a new class of Diophantine equations.

These equations will represent the future of studying the structure and the classification of the group of units of n-cyclic refined neutrosophic rings.

## 2. Preliminaries

**Definition 1: [1]**

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n sub-indeterminacies. We define $R_n$(I)={$a_0 + a_1 I + \cdots + a_n I_n$ ; $a_i \in R$} to be n-cyclic refined neutrosophic ring.

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Operations on $R_n$(I) are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n}(x_i + y_i)I_i$ , $\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n}(x_i \times y_j)I_i I_j = \sum_{i,j=0}^{n}(x_i \times y_j)I_{(i+j\, mod n)}$ .

× is the multiplication on the ring R.

**Example 2: [14]**

(a) The 2-cyclic refined neutrosophic ring of integers is defined as follows:

$Z_2(I) = \{t_0 + t_1 I_1 + t_2 I_2;\ t_i \in Z\}$.

(b) Addition on $Z_2(I)$ can be clarified as follows:

$(a + bI_1 + cI_2) + \ (m + nI_1 + tI_2) = (a + m) + I_1(b + n) + I_2(c + t)$.

(c) Multiplication on $Z_2(I)$ can be clarified as follows:

$(a + bI_1 + cI_2)(\, m + nI_1 + tI_2)$= $am + anI_1 + atI_2 + bmI_1 + bnI_2 + btI_1 + cmI_2 + cnI_1 + ctI_2$

= $am + I_1(an + bm + bt + cn) + I_2(at + bn + cm + ct)$ .

Where $I_1 I_1 = I_{(1+1\, mod 2)} = I_2, I_2 I_2 = I_{(2+2\, mod\, 2)} = I_2, I_1 I_2 = I_{(1+2\, mod\, 2)} = I_1$.

The elements of $Z_2$ are taken by the form $\{1,2\}$ instead of $\{0,1\}$ in the definition of indices I in sub-indeterminacies $I_i$. See [1].

**Definition 3: [15]**

Let R be any ring with unity. An arbitrary element $x \in R$ is called a unit if and only if there exists $y \in R$ such that $xy = yx = 1$. The element y is called the inverse of x.

**3. Main Discussion**

**Definition:**

Let $Z_n(I)$ be the n-cyclic refined neutrosophic ring of integers. Consider a ring homomorphism

$f: Z_n(I) \to Z \times Z \ldots \times Z\ (n + 1\ times\ at\ most)$ .

Let $U(Z_n(I))$ be the group of units of the ring $Z_n(I)$, $U(Z) \times U(Z) \times..\times U(Z)(n + 1\ times\ at\ most)$ be the group of units of $Z \times Z \ldots \times Z$.

Let $x$ be any element with the property $f(x) = (1,1 \ldots ,1)$, then $x \in Ker(f|U(Z_n(I)))$ if and only if there exists $y \in Ker(f|U(Z_n(I)))$ such that $xy = 1$ i.e. if and only if the coefficients matrix of the corresponding system of linear equations is invertible in Z.

Let M be the coefficient matrix of the previous linear systems, we define the following two Diophantine equations:

$Det(M) = 1(Equation\ I)\ or\ -1\ (Equation\ (II))$.

The previous two Diophantine equations are very important to be solved, that is because their solutions will help in determining the classification of the group of units $U(Z_n(I))$.

The following example clarifies the importance of (Equations (I) and (II)) in the study of the group of units' structure.

**Example:**

Let $Z_3(I)$ be the 3-cyclic refined neutrosophic ring of integers, consider the ring homomorphism $f: Z_3(I) \to Z \times Z; f(x + yI_1 + zI_2 + tI_3) = (x, x + y + z + t)$ .

It is clear that $g = f|U(Z_3(I))$ (the restriction of $f$) is a group homomorphism with respect to multiplication between the corresponding groups of units i.e.

$g: U(Z_3(I)) \to U(Z) \times U(Z)$.

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$Ker(g) = \{x + yI_1 + zI_2 + tI_3 \in U(Z_3(I)); f(x + yI_1 + zI_2 + tI_3) = (1,1)\}$, this implies that

$x = 1, z + y = -t$. Thus $Ker(g) = \{1 + yI_1 + zI_2 + (-y - z)I_3; y \in Z, and\ 1 + yI_1 + zI_2 + (-y - z)I_3 \in U(Z_3(I))\}$

The elements of $U(Z_3(I))$ are not known, this means that not all elements of the form $1 + yI_1 + zI_2 + (-y - z)I_3$ are units.

Now, we will find the form of equations((I), (II)). Suppose that $A = 1 + a_1I_1 + a_2I_2 + (-a_1 - a_2)I_3$, A is a unit if and only if there exists $B = 1 + b_1I_1 + b_2I_2 + (-b_1 - b_2)I_3$ such that $AB = 1$.

$AB = 1$ implies:

$1 + I_1(b_1 + a_1 - a_1b_1 - a_1b_2 + a_2b_2 - a_1b_1 - a_2b_1) + I_2(b_2 + a_2 + a_1b_1 - a_2b_1 - a_2b_2 - a_1b_2 - a_2b_2) + I_3(-b_1 - b_2 + a_1b_2 + a_2b_1 + a_2b_2 + a_1b_1 + a_2b_1 + a_1b_2) = 1, hence,\quad b_1(1 - 2a_1 - a_2) + b_2(a_2 - a_1) = -a_1\ and\ b_1(a_1 - a_2) + b_2(1 - 2a_2 - a_1) = -a_2.$

The coefficient matrix of the previous system is $M = \begin{pmatrix} 1 - 2a_1 - a_2 & a_2 - a_1 \\ -a_2 + a_1 & 1 - 2a_2 - a_1 \end{pmatrix}$, M is invertible if and only if $\det(M) = 3(a_2 + a_1)^2 - 3(a_2 + a_1 + a_1a_2) + 1 \in \{1, -1\}$

Thus, we get two corresponding Diophantine equations:

(Equation I) $3(a_2 + a_1)^2 - 3(a_2 + a_1 + a_1a_2) = 0$.

(Equation II) $3(a_2 + a_1)^2 - 3(a_2 + a_1 + a_1a_2) = -2$. (This equation is not solvable in Z, that is because the left side is equal to 0(mod3), but the right side is not).

**Remark:**

If we replaced the 3-cyclic refined neutrosophic ring of integers $Z_3$(I) with the 3-cyclic ring of real numbers $R_3(I)$, we get an equation of a surface, not a Diophantine equation as we show.

M will be invertible in R if and only if det(M) is not zero, hence we get a surface described by the following equation

$3(a_2 + a_1)^2 - 3(a_2 + a_1 + a_1a_2) + 1 = k \neq 0$.

**Example:**

Let $Z_4$(I) be the 4-cyclic refined neutrosophic ring of integers, there exists a ring homomorphism described as follows:

$f: Z_4(I) \to Z \times Z \times Z; f(a_0 + a_1I_1 + a_2I_2 + a_3I_3 + a_4I_4) = (a_0, a_1 + a_2 + a_3 + a_4 + a_5, a_0 - a_1 + a_2 - a_3 + a_4)$.

Let $x = a_0 + a_1I_1 + a_2I_2 + a_3I_3 + a_4I_4$ with $f(x) = (1,1,1)$, hence

$a_0 = 1, a_1 + a_2 + a_3 + a_4 = a_2 - a_1 + a_4 - a_3 = 0$. This implies that:

$a_4 = -a_2, a_3 = -a_1$.

So that $Ker(g) = \{x = 1 + a_1I_1 + a_2I_2 - a_1I_3 - a_2I_4; x\ is\ a\ unit\}$.

Now, we will try to find the form of the corresponding Diophantine equations.

We have $x = 1 + a_1I_1 + a_2I_2 - a_1I_3 - a_2I_4$ as a unit if and only if there exists $y = 1 + b_1I_1 + b_2I_2 - b_1I_3 - b_2I_4$ such that: $xy = 1$. This implies:

$1 + I_1(a_1 + b_1 - a_1b_2 - a_2b_1 - a_1b_2 - a_2b_1) + I_2(a_2 + b_2 + a_1b_1 - a_2b_2 - a_2b_2 + a_1b_1) + I_3(-a_1 - b_1 + a_1b_2 + a_2b_1 + a_1b_2 + a_2b_1) + I_4(-a_2 - b_2 - a_1b_1 + a_2b_2 + a_2b_2 - a_1b_1) = 1$. This means that:

$b_1(1 - 2a_2) + b_2(-2a_1) = -a_1, and\ b_1(2a_1) + b_2(1 - 2a_2) = -a_2.$

The coefficients matrix is $M = \begin{pmatrix} 1 - 2a_2 & -2a_1 \\ 2a_1 & 1 - 2a_2 \end{pmatrix}$.

M is invertible if and only if: $\det(M) = 4{a_1}^2 + 4{a_2}^2 - 4a_2 + 1 \in \{1, -1\}$.

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The corresponding Diophantine equations are:

(Equation I) $4{a_1}^2 + 4{a_2}^2 - 4a_2 = 0$.

(Equation II) $4{a_1}^2 + 4{a_2}^2 - 4a_2 = -2$.

We can see easily that Equation (II) has no integer solutions, that is because the left side is equal to 0(mod4), but the right side is not.

The equation (I) is equivalent to ${a_1}^2 + {a_2}^2 - a_2 = 0$ .

If $a_2 \geq 2$ or $a_2 \leq -2$ we get a contradiction. The only possible non-zero solutions are $a_2 = 1, which\ means\ that\ a_1 = 0, or\ a_2 = -1\ which\ is\ impossible.$

The previous discussion ensures that $Ker(g)$ has exactly two units $\{1, 1 + I_2 - I_4\}$, thus $Ker(g) \cong Z_2$.

By using the isomorphism theorem and Lagrange's theorem, we can find that $Z_4$(I) has 16 units at most.

**Remark:**

In a similar way, if we replaced Z with the real field R, we get a quadratic surface equation:

$$4{a_1}^2 + 4{a_2}^2 - 4a_2 + 1 = k \neq 0.$$

The previous equation is equivalent to:

${a_1}^2 + (a_2 - \frac{1}{2})^2 = \frac{k}{4}$.

If $k > 0$, then an n-cyclic refined neutrosophic real number with form $1 + a_1I_1 + a_2I_2 - a_1I_3 - a_2I_4$ is invertible if and only if $(a_1, a_2)$ are the points of the circle's surface with radius $\sqrt{\frac{k}{4}}$ and center (0, $\frac{1}{2}$ ).

If $k < 0$, then an n-cyclic refined neutrosophic real number with form $1 + a_1I_1 + a_2I_2 - a_1I_3 - a_2I_4$ is not invertible, that is because $(a_1, a_2)$ are the points of imaginary circle's surface with radius $\sqrt{\frac{-k}{4}}\, i$ and center (0, $\frac{1}{2}$ ), which contradicts the choice of $a_1, a_2$ as real numbers.

**Remarks and some Open problems:**

**Remark :**

For every natural number n, and a ring homomorphism

$f: Z_n(I) \rightarrow Z \times Z \ldots \times Z\ (n + 1\ times\ at\ most)$. We can get some corresponding Diophantine equations. Their solutions will be very helpful in determining the units in the corresponding n-cyclic refined neutrosophic ring of integers and then the algebraic structure of the corresponding group of units as a direct product of cyclic groups.

**Remark :**

For every natural number n, and a ring homomorphism

$f{:}\ R_n(I) \rightarrow R \times R \ldots \times R\ (n + 1\ times\ at\ most)$. We can get some corresponding real surfaces' equations. Their geometrical shape will be very helpful in determining the units in the corresponding n-cyclic refined neutrosophic ring of reals and then the algebraic structure of the corresponding group of units as a direct product of cyclic groups.

**Open problem 1:**

Find an algorithm to solve the n-cyclic refined neutrosophic Diophantine equations defined above for every natural number n.

**Open problem 2:**

Classify the group of units of the n-cyclic refined neutrosophic ring of integers as a direct product of cyclic groups.

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**Open problem 3:**

Find the geometrical shape of every n-cyclic refined neutrosophic real equation described above.

**Open problem 4:**

If we replaced the real field R with the complex field C, what are the geometrical shapes which arise from this assumption?

**Remark:**

The answers to previously open questions are very important in finding n-cyclic refined neutrosophic units, and in finding the structure of the corresponding group of units.

**Conclusion**

In this work, we have presented a novel class of Diophantine equations by homomorphisms defined over n-cyclic refined neutrosophic rings of integers and their groups of units. This work may open a new door in future studies about the applications of neutrosophic algebraic structures in classical number theory.

As a future research direction, we aim that Diophantine equations\surfaces derived from the group of units' problem in n-cyclic refined neutrosophic rings will be solvable and classified.

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