On A Novel Group Derived From a Generalization of Integer Exponents and Open Problems

Oliver V. Shtawzen

University Of Nizwa, Sultanate Of Oman
Emails: Vonshtawzen1970abc@gmail.com

Abstract

This paper is devoted to defining a new generalization of m-power closed groups by using some special mappings defined on a group (power maps). Also, a new group will be presented according to these mappings which will be related to other algebraic structures.

Keywords: Power map; m-power closed group; integer exponent; MA-group.

1. Introduction

Since the concept of the abelian group has played a perfect role in the theory of groups (solvability, nilpotency,..etc) and other algebraic structures, it was very important to study the generalizations of this concept. Many generalizations came to light such as n-abelian groups [2], m-power closed groups [1,3], m-Abelian Solvable groups [8], and other related concepts about solvability. See [6,7].

The concept of the (n-power closed) group had been introduced in [3] as a generalization of abelian groups, and (m-abelian) groups [8] respectively.

(n-Power closed) groups were defined by the condition $\forall\, \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{G}\ \exists \boldsymbol{z} \in \boldsymbol{G}\ \boldsymbol{such\ that}\quad \boldsymbol{x^n y^n = z^n}$**.** By considering a generalization of the mapping $\boldsymbol{f(x) = x^n}$ which we call (power map), we will be able to generalize those groups into $\boldsymbol{\varphi - group}$s.

In this paper, we define for the first time the concept of $\boldsymbol{\varphi - group}$. Also, we study some basic group theoretical properties of this concept, and we construct some interesting examples.

On the other hand, we will study the algebraic structure of power maps in some other algebraic structures such as rings, vector spaces, and modules. They will form a group related to these structures.

This new group will lead to many interesting open questions in the theory of groups and the algebraic theory of neutrosophic structures

2. $\varphi -$ groups and power maps

We denote by map(G) to the set of all mappings $\varphi: G \to G$.

Doi : https://doi.org/10.54216/GJMSA.010102
Received: March 12, 2022 Accepted: May 03, 2022

Definition 1:

Let G be a group, we define $\phi_G = \{\varphi \in map(G);\ \varphi(x^{-1}) = (\varphi(x))^{-1}\ and\ \varphi \circ f = f \circ \varphi\ \ \forall f \in aut(G)\}$, we call ϕ_G the set of power maps of G.

-We denote by ϕ_l to the set of one-to-one power maps.

Example 2 :

Let G be a group and $\varphi: G \to G\ such\ \varphi(x) = x^n\ with\ a\ fixed\ integer\ n$ then $\varphi \in \phi_G$.

The proof is obvious.

The previous example shows that the concept of power maps is considered a generalization of integer exponents in groups.

. **Lemma 3** :

Let G be a group, then

(a) ϕ_l is a subgroup of S_G.

(b)$K_G = Aut(G) \cap \phi_l = Z(aut(G))$.

(c) Aut(G) is abelian if and only if aut(G)► ϕ_l.

(d)If G=Z_p then aut(G)= ϕ_l.

(e)$(\phi_l.aut(G))' = \phi_l'[aut(G)]'$.

Proof :

(a) Let $\varphi, h \in \phi_l$, then for each $x \in G\ and\ f \in aut(G)$ we have $\varphi \circ h(x^{-1}) = (\varphi \circ h(x))^{-1}$ and $(\varphi \circ h) \circ f = f \circ (\varphi \circ h)$ so $\varphi \circ h \in \phi_l$. Since $\varphi \circ f = f \circ \varphi$ we find that $\varphi^{-1} \circ f = f \circ \varphi^{-1}$ so $\varphi^{-1} \in \phi_l$ and ϕ_l is a subgroups of S_G.

(b) It is clear that $Z(aut(G)) \leq K_G$, now assume that $f \in K_G$, then $f\ is\ an\ automorphism\ and\ f \circ g = g \circ f\ for\ each\ automorphism\ g\ so\ f \in Z(aut(G))$ and we get the desired proof.

(c) It is easy to show that if aut(G) is abelian then aut(G)≤ ϕ_l, we have to show the normality condition. For this goal, we suppose that $\varphi \in \phi_l\ and\ f \in aut(G)$ then we have $\varphi^{-1} \circ f \circ \varphi = f \circ \varphi^{-1} \circ \varphi = f \in aut(G)$ which means that aut(G)► ϕ_l.

The converse is clear.

(d) If G=Z_p then aut(G)=$\{f\ ; f(x) = x^s\ ; 1 \leq s \leq p\}$is abelian, hence aut(G)≤ ϕ_l .$\forall \varphi \in \phi_l$, we have

$\varphi \circ f = f \circ \varphi\ so\ \varphi(x^s) = (\varphi(x))^s$. Now, let x, y be two arbitrary elements in G, then $\varphi(x + y) = \varphi(1^{x+y}) = [\varphi(1)]^{x+y} = \varphi(1^x) + \varphi(1^y) = \varphi(x) + \varphi(y)$ so $\varphi \in aut(G)$ and aut(G)= ϕ_l.

(e) $\forall\ x, y \in \phi_l.aut(G)\ then\ x = \varphi_1 \circ f_1,\ y = \varphi_2 \circ f_2$ so $[x, y] = x^{-1}y^{-1}xy = f_1^{-1} \circ \varphi_1^{-1} \circ f_2^{-1} \circ \varphi_2^{-1} \circ \varphi_1 \circ f_1 \circ \varphi_2 \circ f_2 = f_1^{-1} o f_2^{-1} o f_1 o f_2 o \varphi_1^{-1} o \varphi_2^{-1} o \varphi_1 o \varphi_2 = [f_1, f_2][\varphi_1, \varphi_2] \in \phi_l'[aut(G)]'$.

The inverse inclusion can be proved in the same way.

Doi : https://doi.org/10.54216/GJMSA.010102
Received: March 12, 2022 Accepted: May 03, 2022

Definition 4 :

Let φ be a power map on the group G, we define $G_\varphi = \{\varphi(x), x \in G\}$, and we say that G is a $\varphi - group$ if and only if G_φ is a subgroup of G.

By the previous definition and Example (2) we can see that the concept of $\varphi - group$ is a generalization of the (m-power closed) group.

Theorem 5:

Let G be a group and φ be a power map G, then

(a) G is a $\varphi - group$ if and only if $\varphi(x)\varphi(y) = \varphi(z)$ for each x,y $\in G$ and

some z$\in G$.

(b) G_φProof :

(a) φ is a power map so $\varphi(x^{-1}) = [\varphi(x)]^{-1}$ then $G_\varphi = \{\varphi(x), x \in G\}$ is a subgroup of G if and only if $\varphi(x)\varphi(y) = \varphi(z)$ for each x,y $\in G$ and some z$\in G$.

(b) Let f $\in aut(G)$ and $\varphi(x) \in G_\varphi$ so $f\big(\varphi(x)\big) = \varphi(f(x)) \in G_\varphi$ thus G_φ is a characteristic subgroup and then normal.

is a normal subgroup.

Doi : https://doi.org/10.54216/GJMSA.010102
Received: March 12, 2022 Accepted: May 03, 2022

Theorem 6:

Let G be a group and φ be a power map on G, then

(a) $[\varphi(e)]^2 = e$.

(b) If G is finite with odd order then $\varphi(e) = e$.

Proof :

(a) We have : $\varphi(e^{-1}) = \varphi(e) = [\varphi(e)]^{-1}$ so $[\varphi(e)]^2 = e$.

(b)It holds directly from (a) and from the fact that the odd-order finite group has no nontrivial elements of order two.

Definition 7 :

Let G be a $\varphi - group$, we say that G is a $\varphi_A - group$ if G_φ is abelian, and

$\varphi_C - group$ if G_φ is cyclic.

Theorem 8:

Let G be a φ −group and H be a normal subgroup of G, then G/H is a $\varphi_{G,H} - group$, where $\varphi_{G,}\ H: G/H \to G/H$ with $\varphi_{G,H}(xH) = \varphi(x)H$.

Proof :

Since φ is a power map on G, then $\varphi_{G,H}(xH)\varphi_{G,H}(yH) = \varphi(x)H\,\varphi(y)H = \varphi(x)\,\varphi(y)H = \varphi(z)H = \varphi_{G,H}(zH)$ for all x,y $\in G$ and for some z$\in G$, so that G/H is a $\varphi_{G,H} - group$.

Theorem 9 :

Let G be a $\varphi_A - group$ and be a normal subgroup of G then G/H is a

$\varphi_{(G,H)_A} - group$.

Proof:

It can be proved by a similar argument to the previous theorem.

Definition 10:

Let G be a group, and x, y $\in$G, we define the $\varphi - commutator$ $[x,y]_\varphi = \varphi(x^{-1})\varphi(y^{-1})\varphi(x)\varphi(y)$. G'_φ is the subgroup of G generated by all $\varphi - commutators$.

Theorem 11:

Let G be a group and φ be a power map on G, then

(a) If G is a φ −group then $G'_\varphi = \{e\}$ if and only if G is a $\varphi_A - group$.

(b) G'_φ is normal.

(c) If H is a normal subgroup of G and G/H is a $\varphi_{G,H} - group$, then G/H is a $\varphi_{(G,H)_A} - group$ if and only if $G'_\varphi \leq H$.

Proof :

(a) It is clear.

(b) Let f be an arbitrary automorphism on G and for any x, y $\in$G, we have :

$f([x,y]_\varphi) = f\left(\varphi(x^{-1})\right)f\left(\varphi(y^{-1})\right)f\left(\varphi(x)\right)f\left(\varphi(y)\right) = \varphi(f(x^{-1}))\varphi(f(y^{-1}))\varphi(f(x))\varphi(f(y)) = [f(x),f(y)]_\varphi$ so G'_φ is characteristic subgroup and then normal.

Doi : https://doi.org/10.54216/GJMSA.010102

Received: March 12, 2022 Accepted: May 03, 2022

(c) It is easy to check the proof.

Definition 12:

Let G be a group and φ be a power map on G, let H be a normal subgroup of G, and we say that H$\blacktriangleright_{\varphi}$G if and only if G/H is a $\varphi_{G,H} - group$.

Theorem 13:

Let H be a normal subgroup of G, then H$\blacktriangleright_{\varphi}$G if and only if for each x, y $\in G$ there is z $\in G$ such $\varphi(z)\varphi(x)\varphi(y) \in H$.

Proof :

Suppose that H$\blacktriangleright_{\varphi}$G then for each x, y $\in G$ there is z $\in G$ such $\varphi(x)\varphi(y)H = \varphi(z)H$ so $\varphi(z^{-1})\varphi(x)\varphi(y) \in H$. The converse is similar.

Theorem 14:

Let G be a group and φ be a power map on G, then

(a) If H$\blacktriangleright_{\varphi}$G and K$\blacktriangleright$G then HK$\blacktriangleright_{\varphi}$G.

(b) If G is a simple φ −group then φ is surjective or trivial.

Proof :

(a) For each x, y $\in G$ there is z $\in G$ such $\varphi(z)\varphi(x)\varphi(y) \in H \leq HK$ so HK$\blacktriangleright_{\varphi}$G

(b) Since G_{φ} is a normal subgroup, then $G_{\varphi} = G \ or \ G_{\varphi} = \{e\}$, thus φ is surjective or trivial.

Theorem 15 :

If G is a $\varphi - group$ and H is a $\sigma - group$ then $G \times H \ is \ (\varphi, \sigma) - group$.

Proof:

We define the map $(\varphi, \sigma): G \times H \ \to G \times H; (\varphi, \sigma)(x, y) = (\varphi(x), \sigma(y))$, it is easy to see that $G \times H_{(\varphi,\sigma)} = (G_{\varphi} \times H_{\sigma})$, hence the proof holds.

Theorem 16 :

If G is a $\varphi_A - group$ and H is $a \ \sigma_A - group$ then $G \times H \ is \ (\varphi, \sigma)_A - group$.

Proof :

It holds directly from the previous theorem and from the fact that the direct product of two abelian groups is an abelian group.

Definition 17:

Let G be a group and φ be a power map on G and H is a subgroup of G, we say that H is a $\varphi - characteristic$ subgroup if $\varphi(H) = H$.

Theorem 18:

If H is a $\varphi - characteristic$ subgroup then $\varphi(f(H)) = f(H)$ for any $f \in autG$.

Proof :

We have $\varphi(f(H)) = f(\varphi(H)) = f(H)$.

Example 19:

The following example ensures that there are power maps that are different from the classical powers map $\boldsymbol{f(x) = x^n}$ and different from homomorphisms.

Doi : https://doi.org/10.54216/GJMSA.010102

Received: March 12, 2022 Accepted: May 03, 2022

We define the map $f : Z \to Z$ with $f(x) = 2$ if x is positive integer, $f(x) = -2$ if x is negative integer , f(0)=0 , it is clear that f is not a homomorphism , and $f(-x) = -f(x)$.

It is known that aut(Z) contains two automorphisms. The identity map I and the map $g(x) = -x$. It is clear that $foI = Iof$. Now, we must prove that $fog = gof$.

Let x be any positive integer, we have $fog(x) = f(-x) = 2, gof(x) = g(-2) = 2, thus\ fog(x) = gof(x)$. If x is a negative integer, the proof will be the same.

On the other hand, $fog(0) = f(0) = 0 = gof(0)$. This implies that $fog(x) = gof(x)$ for all $x \in Z$. So, f is a power map on the additive group $(Z, +)$.

It is clear that (Z,+) is not an $f - group$, that is because $f(Z) = \{0,2,-2\}$ is not a subgroup of Z.

3. Power Maps Structure

Definition :

Let (G,+) be an additive abelian group and F_G is the set of all power maps on G, we define an addition operation on F_G by :

$+ : F_G \times F_G \to F_G$ with $(\varphi + \sigma)(x) = \varphi(x) + \sigma(x)$ for $\varphi, \sigma \in F_G$

Theorem :

$(F_G, +)$ is abelian group

Proof :

The zero map $f(x) = 0$ is a power map and for each $g \in F_G$ then $(f + g) = (g + f) = g$

Also, we have $-g: G \to G\ such\ (-g)(x) = -g(x)$ is a power map and

$(g + (-g)) = (-g + g) = 0$, and it is easy to see that the addition is associative and commutative

Now we prove that F_G is closed under addition , let f be an automorphism of G and g , h be two power maps then $(g + h)(-x) = g(-x) + h(-x) = -(g + h)(x)$ and $fo(g + h)(x) = f[g(x) + h(x)] = f(g(x)) + f(h(x)) = g(f(x)) + h(f(x)) = (g + h)of(x)\ thus\ g + h \in F_G$

Definition :

We call F_G the MA-group of the abelian group G

Definition :

We denote by F_G^c to the set of all bijective power maps on G, (F_G^c, o) is a subgroup of S_G.

Theorem :

$(F_G, +, .)$ is a module over the ring of integers with (.): $Z \times F_G \to F_G$

Defined as $(m.f)(x) = mf(x)$ for a power map f and integer m and arbitrary x in G

Proof :

It is easy to see that for a power map f and an integer m, $(m.f)$ is a power map

The whole conditions of the module are easy to check

Definition :

We call the previous module by MA-module of G

Definition :

Let W_G= { $f_a \in F_G$; f_a (x) = $a.x\ where\ a \in Z$ } , we call W_G the weak power maps set on G

Theorem :

Doi : https://doi.org/10.54216/GJMSA.010102
Received: March 12, 2022 Accepted: May 03, 2022

$(W_G, +, .)$ a commutative submodule of $(F_G, +, .)$

Proof :

It is clear.

Cyclic groups

We say that g is a P-homomorphism if g is a homomorphism and $gof = fog$ for each automorphism f, we denote the set of all P-homomorphisms by P-hom(G)

It is clear that P-hom(G)≤ F_G

Theorem :

If $G = Z_p$ then P-hom(G)= F_G

Proof :

aut(G)={f ; $f(x) = x^s$; $1 \leq s \leq p$} so $\varphi \circ f = f \circ \varphi$ so $\varphi(x^s) = (\varphi(x))^s$ for each $\varphi \in F_G$

let x, y be two arbitrary elements in G, then $\varphi(x + y) = \varphi(1^{x+y}) = [\varphi(1)]^{x+y} = \varphi(1^x) + \varphi(1^y) = \varphi(x) + \varphi(y)$ so φ is a homomorphism thus P-hom(G)= F_G

Theorem :

If $G = Z$ then hom(G)≤ F_G

Proof :

Since aut(Z) contains two automorphisms, the identity map I and the map $f(x) = -x$ then each homomorphism g on Z has the following property $fog = gof, goI = Iog$ so that

g ∈ F_G

Remark: hom(Z) is a proper subgroup of F_Z, we give an example :

We define the map $f : Z \to Z$ with $f(a) = 2$ if a is positive integer, $f(a) = -2$ if a is negative integer , f(0)=0 , it is clear that f is not a homomorphism , and $f(-a) = -f(a)$, thus f ∈ F_Z

Theorem :

If G = Z_m then hom(G)≤ F_G

Proof :

aut(G)={f ; $f(x) = x^s$; $\gcd(m, s) = 1$} , let g∈ hom(G) then for each automorphism f we find that fog=gof thus g∈ F_G

it is easy to see that f(s.a)=s.f(a) if (s,m)=1 by the previous argument for each f∈ F_G

we give an example that shows that hom(G) is a proper subgroup of F_G

let G = Z_6 and f : $Z_6 \to Z_6$ with f(0)=0 , f(2)=f(3)=f(4)=0 , f(1)=5 , f(5)=1

f is not a homomorphism and f(-2)=f(4)=-f(2) , f(-4)=f(2)=-f(4) , f(3)=f(-3)=-f(3)

f(-1)=f(5)=-f(1) , f(-5)=f(1)=-f(5)

now assume that gcd(s,6)=1 then s=1 or s=5 , it is easy to check that $f(2s) = 2f(s), f(3s) = 3f(s)$, f(4s)=4f(s) , f(s.1)=s.f(1) , f(s.5)=s.f(5) for each possible s , so f∈ F_G , thus the proof is complete.

Theorem :

If G=Z_m; m=$p_1 p_2 .. p_k$, p_i *are distinct primes for all* i then

$F_G \cong$ P-hom(Z_{p_1})×…×P-hom(Z_{p_k})

Doi : https://doi.org/10.54216/GJMSA.010102
Received: March 12, 2022 Accepted: May 03, 2022

Proof :

Since aut(G)$\cong aut(Z_{p_1}) \times \ldots \times aut(Z_{p_k})$ then we get the proof

Open problems :

Describe MA- group F_G for any cyclic group G

Describe MA- group F_G for any abelian group G

Vector spaces :

Let (V,+, .) be a vector space over the field K, then (F_V,+) is the abelian group as we saw, we define the set F_V^A= $\{\varphi: V \to V;\ \varphi(-x) = -\varphi(x), and\ \varphi o f = f o \varphi\ \forall f \in GL(V)\}$

We define the following operations :

(+): $F_V^A \times F_V^A \to F_V^A$ with $(f+g)(x) = f(x) + g(x)\ for\ each\ f, g\ \in F_V^A\ and\ x \in V$

$(.):\ K \times F_V^A \to F_V^A\ with\ (k.f)(x) = k.f(x)$.

Theorem :

$(F_V^A, +, .)$ is a vector space over K

Proof :

Suppose that $f, g \in F_V^A, and\ h \in GL(V), x \in V, k \in K$ we have that :

$(f+g)(-x) = f(-x) + g(-x) = -(f+g)(x)$ and $(k.f)(-x) = -(k.f)(x)$

$$ho(f+g)(x) = h[f(x) + g(x)] = hof(x) + hog(x) = foh(x) + goh(x) = (f+g)oh(x)$$

$(k.f)oh(x) = k.(foh)(x) = k.(hof)(x) = ho(k.f)(x).$

So $f + g, k.f\ \in F_V^A$

It is easy to see that $(F_V^A, +)$ is the abelian group and the identity is the zero map 0:V$\to V$ such $0(x) = x$ for each x$\in V$

Let $a, b\ \in K$ then : $(1.f) = f$, $(a+b).f = a.f + b.f$, $a.(f+g) = a.f + a.g$

$(a.b).f = a.(b.f)$ thus the proof is complete

Definition:

We call $(F_V^A, +, .)$ the MA-space of the vector space V.

Conclusion

In this work, we have generalized m-power closed groups by using a novel class of functions (power maps). Also, we have studied some of their elementary properties in terms of theorems.

In the future, we aim to find more applications of these mappings in the study of group theory.

References

[1] Abobala, M. and Sankari, H. A Contribution to (m-Power Closed) Groups, UMM Al-Qura University Journal of Applied Science (UQUJAS), KSA, (2020).

[2] Levi, F. Notes on Group Theory, J. Indian Math. Soc. 8, (1944), pp.1-7.

Doi : https://doi.org/10.54216/GJMSA.010102
Received: March 12, 2022 Accepted: May 03, 2022

[3] Kappe, L. and Ying, Y. On Exact Power Margin Groups, Rend. Sem. Mat. University of Padova. Vol. 87, (1992), pp.245-265.

[4] Haushi, M. The Algebraic Structures 1, Tishreen University Press, (2004).

[5] Rotman, Joseph J. An introduction to the theory of groups. Fourth edition. Graduate Texts in Mathematics, 148. Springer-Verlag, New York, 1995.

[6] Arad, Z. and Ward, M. New Criteria for the Solvability of Finite Groups, Journal of Algebra, 77, (1982), pp. 234-246.

[7] Dolfi, S. Guralnik, R. Herzog, M. and Praeger, C. A New Solvability Criterion for Finite Groups, ArXiv:1007.5394, (2011).

[8] Sankari, H., and Abobala, M., On A New Criteria For the Solvability of Non-Simple Finite Groups and m-Abelian Solvability, Journal of Mathematics, Hindawi, 2021.

[9] Hatip, A., Alhamido, R., and Abobala, M., "A Contribution to Neutrosophic Groups", International Journal of Neutrosophic Science", Vol. 0, pp. 67-76. 2019.

[10] Abobala, M., " *n*-Refined Neutrosophic Groups I", International Journal of Neutrosophic Science, Vol. 0, pp. 27-34. 2020.

Doi : https://doi.org/10.54216/*GJMSA*.010102
Received: March 12, 2022 Accepted: May 03, 2022