Interval-Valued Neutrosophic Deductive Systems of Hilbert Algebras

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Abstract

Interval-valued neutrosophic sets (IVNSs) are a notion that was initially developed by Wang et al.[19] The idea of IVNSs to deductive systems (DSs) in Hilbert algebras is presented in this study. It is shown how interval-valued neutrosophic deductive systems (IVNDSs) relate to their level cuts. In addition, certain related features are examined as well as the homomorphic inverse image of IVNDSs in Hilbert algebras.

Keywords: Hilbert algebra; deductive system; interval-valued neutrosophic deductive system; level cut

1 Introduction

Zadeh[20] first developed the idea of fuzzy sets (FSs). Numerous academics have studied FS theory because it has numerous practical applications. Numerous investigations were undertaken on the generalizations of FSs after the idea of FSs was introduced. In,[1,3,6] it is explained how FSs may be integrated with various uncertainty-reduction strategies like soft sets and rough sets. The idea of intuitionistic fuzzy sets (IFSs), as out by Atanassov,[2] is one of the more beneficial extensions of FSs. Medical diagnostics, optimization problems, and multi-criteria decision-making are just a few of the areas in which IFSs are applied.[10–12] In 1999, Smarandache[16] presented the idea of neutrosophic sets, which is a broader concept that encompasses the ideas of classic sets, FSs, IFSs, and interval-valued (I)FSs (see[16,17]).

For various examinations of implication in intuitionistic and other non-classical logics, Henkin and Skolem proposed the idea of Hilbert algebras in the early 1950s. These algebras were researched in the 1960s from an algebraic perspective, in particular by Horn and Diego. Diego established that Hilbert algebras form a variety that is locally finite (cf.[7]). Busneag (cf.,[45]) and Jun (cf.[13]) both dealt with Hilbert algebras, and several of their filters that create DSs were identified. The fuzzifications of subalgebras/ideals and DSs in Hilbert algebras were taken into consideration by Dudek (cf.[8]). The intuitionistic fuzzification of the idea of DSs in Hilbert algebras was examined by Zhan and Tan.[21]

This study introduces the idea of IVNSs to DSs in Hilbert algebras. There are presented correlations between IVNDSs and their level cuts. Additionally, several features relating to the homomorphic inverse image of IVNDSs in Hilbert algebras are examined.

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2 Preliminaries

Let's examine the definition of Hilbert algebras, which was provided by Diego[7] in 1966, before we start the research.

Definition 2.1. [7] An algebra of the form $\mathfrak{H} = (\mathfrak{H}, \circledast, \iota)$ of type $(2, 0)$, where $\mathfrak{H} \neq \emptyset$, $\circledast$ is a binary operation, and ι is a fixed element of $\mathfrak{H}$, is referred to as a *Hilbert algebra* if the axioms listed below are true:

1. $(\forall \varrho, \epsilon \in \mathfrak{H})(\varrho \circledast (\epsilon \circledast \varrho) = \iota)$,
2. $(\forall \varrho, \epsilon, \nu \in \mathfrak{H})((\varrho \circledast (\epsilon \circledast \nu)) \circledast ((\varrho \circledast \epsilon) \circledast (\varrho \circledast \nu)) = \iota)$,
3. $(\forall \varrho, \epsilon \in \mathfrak{H})(\varrho \circledast \epsilon = \iota, \epsilon \circledast \varrho = \iota \Rightarrow \varrho = \epsilon)$.

From now on we define $\mathfrak{H} = (\mathfrak{H}, \circledast, \iota)$ and $\mathfrak{K} = (\mathfrak{K}, \circledast, \iota)$ as two Hilbert algebras.

In,[8] the following conclusion was established.

Lemma 2.2. *In $\mathfrak{H}$, we obtain*

1. $(\forall \varrho \in \mathfrak{H})(\varrho \circledast \varrho = \iota)$,
2. $(\forall \varrho \in \mathfrak{H})(\iota \circledast \varrho = \varrho)$,
3. $(\forall \varrho \in \mathfrak{H})(\varrho \circledast \iota = \iota)$,
4. $(\forall \varrho, \epsilon, \nu \in \mathfrak{H})(\varrho \circledast (\epsilon \circledast \nu) = \epsilon \circledast (\varrho \circledast \nu))$.

In $\mathfrak{H}$, the binary relation $\leq$ is defined by

$$(\forall \varrho, \epsilon \in \mathfrak{H})(\varrho \leq \epsilon \Leftrightarrow \varrho \circledast \epsilon = \iota).$$

Definition 2.3. [9] A subset $\emptyset \neq \mathfrak{D} \subseteq \mathfrak{H}$ is called a *deductive system* (DS) of $\mathfrak{H}$ if

1. $\iota \in \mathfrak{D}$,
2. $(\forall \varrho \in \mathfrak{D}, \forall \epsilon \in \mathfrak{H})(\varrho \circledast \epsilon \in \mathfrak{D} \Rightarrow \epsilon \in \mathfrak{D})$.

In addition, DSs were also studied in GE-algebras by Jun and Bandaru in 2022 (see[14]).

The definition of a *fuzzy set* (FS)[20] is a function $\mu : \mathfrak{H} \to [0, 1]$, where $[0, 1] \subseteq \mathbb{R}$.

Definition 2.4. A FS μ in $\mathfrak{H}$ is said to be a *fuzzy deductive system* (FDS) of $\mathfrak{H}$ if

1. $(\forall \varrho \in \mathfrak{H})(\mu(\iota) \geq \mu(\varrho))$,
2. $(\forall \varrho, \epsilon \in \mathfrak{H})(\mu(\epsilon) \geq \min\{\mu(\varrho \circledast \epsilon), \mu(\varrho)\})$

and an *anti fuzzy deductive system* (AFDS) of $\mathfrak{H}$ if

1. $(\forall \varrho \in \mathfrak{H})(\mu(\iota) \leq \mu(\varrho))$,
2. $(\forall \varrho, \epsilon \in \mathfrak{H})(\mu(\epsilon) \leq \max\{\mu(\varrho \circledast \epsilon), \mu(\varrho)\})$.

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An *interval number* we mean a close subinterval $\widehat{s} = [a^l, a^u]$ of $[0, 1]$, where $0 \leq a^l \leq a^u \leq 1$. Denote by $[[0, 1]]$ the set of all interval numbers. If $\widehat{s_1}, \widehat{s_2} \in [[0, 1]]$, we define $\text{rmin}\{\widehat{s_1}, \widehat{s_2}\}$ and $\text{rmax}\{\widehat{s_1}, \widehat{s_2}\}$, respectively, as follows:

$$\text{rmin}\{\widehat{s_1}, \widehat{s_2}\} = [\min\{s_1^l, s_2^l\}, \min\{s_1^u, s_2^u\}],$$
$$\text{rmax}\{\widehat{s_1}, \widehat{s_2}\} = [\max\{s_1^l, s_2^l\}, \max\{s_1^u, s_2^u\}].$$

Definition 2.5. [15] For the symbols $\succeq$, $\preceq$, and $=$ for $\widehat{s_1}$ and $\widehat{s_2}$ in $[[0, 1]]$, respectively, we define them as follows:

$$\widehat{s_1} \succeq \widehat{s_2} \Leftrightarrow s_1^l \geq s_2^l \text{ and } s_1^u \geq s_2^u,$$

additionally, we might have $\widehat{s_1} \preceq \widehat{s_2}$ and $\widehat{s_1} = \widehat{s_2}$.

The following claims are true in $[[0, 1]]$ (see[18]).

$$(\forall \widehat{s} \in [[0, 1]]) \left(\begin{array}{c} \text{rmax}\{\widehat{s}, \widehat{s}\} = \widehat{s} \\ \text{rmin}\{\widehat{s}, \widehat{s}\} = \widehat{s} \end{array} \right). \tag{1}$$

$$(\forall \widehat{s_1}, \widehat{s_2} \in [[0, 1]]) \left(\begin{array}{c} \text{rmax}\{\widehat{s_1}, \widehat{s_2}\} \succeq \widehat{s_1} \\ \widehat{s_1} \succeq \text{rmin}\{\widehat{s_1}, \widehat{s_2}\} \end{array} \right). \tag{2}$$

$$(\forall \widehat{s_1}, \widehat{s_2}, \widehat{s_3} \in [[0, 1]]) \left(\widehat{s_1} \succeq \widehat{s_2}, \widehat{s_3} \succeq \widehat{s_2} \Leftrightarrow \text{rmin}\{\widehat{s_1}, \widehat{s_3}\} \succeq \widehat{s_2} \right). \tag{3}$$

Definition 2.6. [19] An interval-valued neutrosophic set (IVNS) $\mathbb{N}$ in a set $\mathfrak{H} \neq \emptyset$ is defined to be a structure

$$\mathbb{N} = \{(\varrho, \mathfrak{T}_{\mathbb{N}}(\varrho), \mathfrak{I}_{\mathbb{N}}(\varrho), \mathfrak{F}_{\mathbb{N}}(\varrho)) \mid \varrho \in \mathfrak{H}\}, \tag{4}$$

where $\mathfrak{T}_{\mathbb{N}} : \mathfrak{H} \to [[0, 1]]$, $\mathfrak{I}_{\mathbb{N}} : \mathfrak{H} \to [[0, 1]]$, and $\mathfrak{F}_{\mathbb{N}} : \mathfrak{H} \to [[0, 1]]$ such that $\mathfrak{T}_{\mathbb{N}}(\varrho) = [\mathfrak{T}_{\mathbb{N}}^l(\varrho), \mathfrak{T}_{\mathbb{N}}^u(\varrho)]$, $\mathfrak{I}_{\mathbb{N}}(\varrho) = [\mathfrak{I}_{\mathbb{N}}^l(\varrho), \mathfrak{I}_{\mathbb{N}}^u(\varrho)]$, and $\mathfrak{F}_{\mathbb{N}}(\varrho) = [\mathfrak{F}_{\mathbb{N}}^l(\varrho), \mathfrak{F}_{\mathbb{N}}^u(\varrho)]$ for all $\varrho \in \mathfrak{H}$. Also note that $\overline{\mathfrak{T}_{\mathbb{N}}}(\varrho) = \mathbf{1} - \mathfrak{T}_{\mathbb{N}}(\varrho) = [1 - \mathfrak{T}_{\mathbb{N}}^u(\varrho), 1 - \mathfrak{T}_{\mathbb{N}}^l(\varrho)]$, $\overline{\mathfrak{I}_{\mathbb{N}}}(\varrho) = \mathbf{1} - \mathfrak{I}_{\mathbb{N}}(\varrho) = [1 - \mathfrak{I}_{\mathbb{N}}^u(\varrho), 1 - \mathfrak{I}_{\mathbb{N}}^l(\varrho)]$, and $\overline{\mathfrak{F}_{\mathbb{N}}}(\varrho) = \mathbf{1} - \mathfrak{F}_{\mathbb{N}}(\varrho) = [1 - \mathfrak{F}_{\mathbb{N}}^u(\varrho), 1 - \mathfrak{F}_{\mathbb{N}}^l(\varrho)]$ for all $\varrho \in \mathfrak{H}$, where $(\varrho, \overline{\mathfrak{T}_{\mathbb{N}}}(\varrho), \overline{\mathfrak{I}_{\mathbb{N}}}(\varrho), \overline{\mathfrak{F}_{\mathbb{N}}}(\varrho))$ represents the complement of ϱ in $\mathbb{N}$. We define $\overline{\mathbb{N}} = (\overline{\mathfrak{T}_{\mathbb{N}}}, \overline{\mathfrak{I}_{\mathbb{N}}}, \overline{\mathfrak{F}_{\mathbb{N}}})$ as the complement of $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$. For the purpose of convenience, we will refer to the IVNS set $\mathbb{N} = \{(\varrho, \mathfrak{T}_{\mathbb{N}}(\varrho), \mathfrak{I}_{\mathbb{N}}(\varrho), \mathfrak{F}_{\mathbb{N}}(\varrho)) \mid \varrho \in \mathfrak{H}\}$ using the notation $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$.

After this let $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ and $\mathbb{K} = (\mathfrak{T}_{\mathbb{K}}, \mathfrak{I}_{\mathbb{K}}, \mathfrak{F}_{\mathbb{K}})$ mean IVNSs.

3 Interval-valued neutrosophic deductive systems

We introduce the idea of IVNDSs of Hilbert algebras in this part and look at some related characteristics.

Definition 3.1. An IVNS $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ in $\mathfrak{H}$ is called an *interval-valued neutrosophic deductive system* (IVNDS) of $\mathfrak{H}$ if

$$(\forall \varrho \in \mathfrak{H}) \left(\begin{array}{c} \mathfrak{T}_{\mathbb{N}}(\iota) \succeq \mathfrak{T}_{\mathbb{N}}(\varrho) \\ \mathfrak{I}_{\mathbb{N}}(\iota) \preceq \mathfrak{I}_{\mathbb{N}}(\varrho) \\ \mathfrak{F}_{\mathbb{N}}(\iota) \succeq \mathfrak{F}_{\mathbb{N}}(\varrho) \end{array} \right), \tag{5}$$

$$(\forall \varrho, \epsilon \in \mathfrak{H}) \left(\begin{array}{c} \mathfrak{T}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\} \\ \mathfrak{I}_{\mathbb{N}}(\epsilon) \preceq \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\} \\ \mathfrak{F}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}(\varrho)\} \end{array} \right). \tag{6}$$

Lemma 3.2. *If $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ is an IVNDS of $\mathfrak{H}$, then*

$$(\forall \varrho, \epsilon, \nu \in \mathfrak{H}) \left(\nu \leq \varrho \circledast \epsilon \Rightarrow \left\{ \begin{array}{l} \mathfrak{T}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho), \mathfrak{T}_{\mathbb{N}}(\nu)\} \\ \mathfrak{I}_{\mathbb{N}}(\epsilon) \preceq \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho), \mathfrak{I}_{\mathbb{N}}(\nu)\} \\ \mathfrak{F}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho), \mathfrak{F}_{\mathbb{N}}(\nu)\} \end{array} \right. \right). \tag{7}$$

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Proof. Let $\varrho, \epsilon, \nu \in \mathfrak{H}$ be such that $\nu \leq \varrho \circledast \epsilon$. Then $\nu \circledast (\varrho \circledast \epsilon) = \iota$ and so

$$\begin{aligned}\mathfrak{T}_{\mathbb{N}}(\epsilon) &\succeq \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\} \\ &\succeq \text{rmin}\{\text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\nu \circledast (\varrho \circledast \epsilon)), \mathfrak{T}_{\mathbb{N}}(\nu)\}, \mathfrak{T}_{\mathbb{N}}(\varrho)\} \\ &= \text{rmin}\{\text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\iota), \mathfrak{T}_{\mathbb{N}}(\nu)\}, \mathfrak{T}_{\mathbb{N}}(\varrho)\} \\ &= \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho), \mathfrak{T}_{\mathbb{N}}(\nu)\},\end{aligned}$$

$$\begin{aligned}\mathfrak{I}_{\mathbb{N}}(\epsilon) &\preceq \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\} \\ &\preceq \text{rmax}\{\text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\nu \circledast (\varrho \circledast \epsilon)), \mathfrak{I}_{\mathbb{N}}(\nu)\}, \mathfrak{I}_{\mathbb{N}}(\varrho)\} \\ &= \text{rmax}\{\text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\iota), \mathfrak{I}_{\mathbb{N}}(\nu)\}, \mathfrak{I}_{\mathbb{N}}(\varrho)\} \\ &= \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho), \mathfrak{I}_{\mathbb{N}}(\nu)\},\end{aligned}$$

$$\begin{aligned}\mathfrak{F}_{\mathbb{N}}(\epsilon) &\succeq \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}(\varrho)\} \\ &\succeq \text{rmin}\{\text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\nu \circledast (\varrho \circledast \epsilon)), \mathfrak{F}_{\mathbb{N}}(\nu)\}, \mathfrak{F}_{\mathbb{N}}(\varrho)\} \\ &= \text{rmin}\{\text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\iota), \mathfrak{F}_{\mathbb{N}}(\nu)\}, \mathfrak{F}_{\mathbb{N}}(\varrho)\} \\ &= \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho), \mathfrak{F}_{\mathbb{N}}(\nu)\}.\end{aligned}$$

□

Lemma 3.3. *If $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ is an IVNDS of $\mathfrak{H}$, then*

$$(\forall \varrho, \epsilon \in \mathfrak{H}) \left(\varrho \leq \epsilon \Rightarrow \begin{cases} \mathfrak{T}_{\mathbb{N}}(\varrho) \preceq \mathfrak{T}_{\mathbb{N}}(\epsilon) \\ \mathfrak{I}_{\mathbb{N}}(\varrho) \succeq \mathfrak{I}_{\mathbb{N}}(\epsilon) \\ \mathfrak{F}_{\mathbb{N}}(\varrho) \preceq \mathfrak{F}_{\mathbb{N}}(\epsilon) \end{cases} \right). \tag{8}$$

Proof. Let $\varrho, \epsilon \in \mathfrak{H}$ be such that $\varrho \leq \epsilon$. Then $\varrho \circledast \epsilon = \iota$ and so

$$\mathfrak{T}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\} = \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\iota), \mathfrak{T}_{\mathbb{N}}(\varrho)\} = \mathfrak{T}_{\mathbb{N}}(\varrho),$$

$$\mathfrak{I}_{\mathbb{N}}(\epsilon) \preceq \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\} = \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\iota), \mathfrak{I}_{\mathbb{N}}(\varrho)\} = \mathfrak{I}_{\mathbb{N}}(\varrho),$$

$$\mathfrak{F}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}(\varrho)\} = \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\iota), \mathfrak{F}_{\mathbb{N}}(\varrho)\} = \mathfrak{F}_{\mathbb{N}}(\varrho).$$

□

Theorem 3.4. *An IVNS $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ in $\mathfrak{H}$ is an IVNDS of $\mathfrak{H}$ if and only if $\mathfrak{T}^l_{\mathbb{N}}, \mathfrak{T}^u_{\mathbb{N}}, \mathfrak{F}^l_{\mathbb{N}}$, and $\mathfrak{F}^u_{\mathbb{N}}$ are FDSs of $\mathfrak{H}$ and $\mathfrak{I}^l_{\mathbb{N}}$ and $\mathfrak{I}^u_{\mathbb{N}}$ are AFDSs of $\mathfrak{H}$.*

Proof. Since $\mathfrak{T}^l_{\mathbb{N}}(\iota) \succeq \mathfrak{T}^l_{\mathbb{N}}(\varrho)$, $\mathfrak{T}^u_{\mathbb{N}}(\iota) \succeq \mathfrak{T}^u_{\mathbb{N}}(\varrho)$, $\mathfrak{I}^l_{\mathbb{N}}(\iota) \preceq \mathfrak{I}^l_{\mathbb{N}}(\varrho)$, $\mathfrak{I}^u_{\mathbb{N}}(\iota) \preceq \mathfrak{I}^u_{\mathbb{N}}(\varrho)$, and $\mathfrak{F}^l_{\mathbb{N}}(\iota) \succeq \mathfrak{F}^l_{\mathbb{N}}(\varrho)$, $\mathfrak{F}^u_{\mathbb{N}}(\iota) \succeq \mathfrak{F}^u_{\mathbb{N}}(\varrho)$, we obtain $\mathfrak{T}_{\mathbb{N}}(\iota) \succeq \mathfrak{T}_{\mathbb{N}}(\varrho)$, $\mathfrak{I}_{\mathbb{N}}(\iota) \preceq \mathfrak{I}_{\mathbb{N}}(\varrho)$, and $\mathfrak{F}_{\mathbb{N}}(\iota) \succeq \mathfrak{F}_{\mathbb{N}}(\varrho)$. Let $\mathfrak{T}^l_{\mathbb{N}}$ and $\mathfrak{T}^u_{\mathbb{N}}$ be FDSs of $\mathfrak{H}$ and $\varrho, \epsilon \in \mathfrak{H}$. Then

$$\begin{aligned}\mathfrak{T}_{\mathbb{N}}(\epsilon) &= [\mathfrak{T}^l_{\mathbb{N}}(\epsilon), \mathfrak{T}^u_{\mathbb{N}}(\varrho)] \\ &\succeq [\min\{\mathfrak{T}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^l_{\mathbb{N}}(\varrho)\}, \min\{\mathfrak{T}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{N}}(\varrho)\}] \\ &= \text{rmin}\{[\mathfrak{T}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{N}}(\varrho \circledast \epsilon)], [\mathfrak{T}^l_{\mathbb{N}}(\varrho), \mathfrak{T}^u_{\mathbb{N}}(\varrho)]\} \\ &= \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\}.\end{aligned}$$

Let $\mathfrak{I}^l_{\mathbb{N}}$ and $\mathfrak{I}^u_{\mathbb{N}}$ be AFDSs of $\mathfrak{H}$ and $\varrho, \epsilon \in \mathfrak{H}$. Then

$$\begin{aligned}\mathfrak{I}_{\mathbb{N}}(\epsilon) &= [\mathfrak{I}^l_{\mathbb{N}}(\epsilon), \mathfrak{I}^u_{\mathbb{N}}(\varrho)] \\ &\preceq [\max\{\mathfrak{I}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^l_{\mathbb{N}}(\varrho)\}, \max\{\mathfrak{I}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{N}}(\varrho)\}] \\ &= \text{rmax}\{[\mathfrak{I}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{N}}(\varrho \circledast \epsilon)], [\mathfrak{I}^l_{\mathbb{N}}(\varrho), \mathfrak{I}^u_{\mathbb{N}}(\varrho)]\} \\ &= \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\}.\end{aligned}$$

Let $\mathfrak{F}^l_{\mathbb{N}}$ and $\mathfrak{F}^u_{\mathbb{N}}$ be FDSs of $\mathfrak{H}$ and $\varrho, \epsilon \in \mathfrak{H}$. Then

$$\begin{aligned}\mathfrak{F}_{\mathbb{N}}(\epsilon) &= [\mathfrak{F}^l_{\mathbb{N}}(\epsilon), \mathfrak{F}^u_{\mathbb{N}}(\varrho)] \\ &\succeq [\min\{\mathfrak{F}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^l_{\mathbb{N}}(\varrho)\}, \min\{\mathfrak{F}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{N}}(\varrho)\}] \\ &= \text{rmin}\{[\mathfrak{F}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{N}}(\varrho \circledast \epsilon)], [\mathfrak{F}^l_{\mathbb{N}}(\varrho), \mathfrak{F}^u_{\mathbb{N}}(\varrho)]\} \\ &= \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}(\varrho)\}.\end{aligned}$$

So we conclude that $\mathbb{N}$ is an IVNDS of $\mathfrak{H}$.

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Assume, on the other hand, that $\mathbb{N}$ is an IVNDS of $\mathfrak{H}$. Let $\varrho \in \mathfrak{H}$. Then

$$[\mathfrak{T}^l_{\mathbb{N}}(\iota), \mathfrak{T}^u_{\mathbb{N}}(\iota)] = \mathfrak{T}_{\mathbb{N}}(\iota) \succeq \mathfrak{T}_{\mathbb{N}}(\varrho) = [\mathfrak{T}^l_{\mathbb{N}}(\varrho), \mathfrak{T}^u_{\mathbb{N}}(\varrho)] \Rightarrow \mathfrak{T}^l_{\mathbb{N}}(\iota) \succeq \mathfrak{T}^l_{\mathbb{N}}(\varrho), \mathfrak{T}^u_{\mathbb{N}}(\iota) \succeq \mathfrak{T}^u_{\mathbb{N}}(\varrho),$$

$$[\mathfrak{I}^l_{\mathbb{N}}(\iota), \mathfrak{I}^u_{\mathbb{N}}(\iota)] = \mathfrak{I}_{\mathbb{N}}(\iota) \preceq \mathfrak{I}_{\mathbb{N}}(\varrho) = [\mathfrak{I}^l_{\mathbb{N}}(\varrho), \mathfrak{I}^u_{\mathbb{N}}(\varrho)] \Rightarrow \mathfrak{I}^l_{\mathbb{N}}(\iota) \preceq \mathfrak{I}^l_{\mathbb{N}}(\varrho), \mathfrak{I}^u_{\mathbb{N}}(\iota) \preceq \mathfrak{I}^u_{\mathbb{N}}(\varrho),$$

$$[\mathfrak{F}^l_{\mathbb{N}}(\iota), \mathfrak{F}^u_{\mathbb{N}}(\iota)] = \mathfrak{F}_{\mathbb{N}}(\iota) \succeq \mathfrak{F}_{\mathbb{N}}(\varrho) = [\mathfrak{F}^l_{\mathbb{N}}(\varrho), \mathfrak{F}^u_{\mathbb{N}}(\varrho)] \Rightarrow \mathfrak{F}^l_{\mathbb{N}}(\iota) \succeq \mathfrak{F}^l_{\mathbb{N}}(\varrho), \mathfrak{F}^u_{\mathbb{N}}(\iota) \succeq \mathfrak{F}^u_{\mathbb{N}}(\varrho).$$

Let $\varrho, \epsilon \in \mathfrak{H}$. Then

$$\begin{aligned}
[\mathfrak{T}^l_{\mathbb{N}}(\epsilon), \mathfrak{T}^u_{\mathbb{N}}(\epsilon)] &= \mathfrak{T}_{\mathbb{N}}(\epsilon) \\
&\succeq \mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\} \\
&= \mathrm{rmin}\{[\mathfrak{T}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{N}}(\varrho \circledast \epsilon)], [\mathfrak{T}^l_{\mathbb{N}}(\varrho), \mathfrak{T}^u_{\mathbb{N}}(\varrho)]\} \\
&= [\min\{\mathfrak{T}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^l_{\mathbb{N}}(\varrho)\}, \min\{\mathfrak{T}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{N}}(\varrho)\}].
\end{aligned}$$

So we conclude that $\mathfrak{T}^l_{\mathbb{N}}(\epsilon) \geq \min\{\mathfrak{T}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^l_{\mathbb{N}}(\varrho)\}$ and $\mathfrak{T}^u_{\mathbb{N}}(\epsilon) \geq \min\{\mathfrak{T}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{N}}(\varrho)\}$. Now,

$$\begin{aligned}
[\mathfrak{I}^l_{\mathbb{N}}(\epsilon), \mathfrak{I}^u_{\mathbb{N}}(\epsilon)] &= \mathfrak{I}_{\mathbb{N}}(\epsilon) \\
&\preceq \mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\} \\
&= \mathrm{rmax}\{[\mathfrak{I}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{N}}(\varrho \circledast \epsilon)], [\mathfrak{I}^l_{\mathbb{N}}(\varrho), \mathfrak{I}^u_{\mathbb{N}}(\varrho)]\} \\
&= [\max\{\mathfrak{I}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^l_{\mathbb{N}}(\varrho)\}, \max\{\mathfrak{I}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{N}}(\varrho)\}].
\end{aligned}$$

So we conclude that $\mathfrak{I}^l_{\mathbb{N}}(\epsilon) \leq \max\{\mathfrak{I}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^l_{\mathbb{N}}(\varrho)\}$ and $\mathfrak{I}^u_{\mathbb{N}}(\epsilon) \leq \max\{\mathfrak{I}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{N}}(\varrho)\}$. Also,

$$\begin{aligned}
[\mathfrak{F}^l_{\mathbb{N}}(\epsilon), \mathfrak{F}^u_{\mathbb{N}}(\epsilon)] &= \mathfrak{F}_{\mathbb{N}}(\epsilon) \\
&\succeq \mathrm{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}(\varrho)\} \\
&= \mathrm{rmin}\{[\mathfrak{F}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{N}}(\varrho \circledast \epsilon)], [\mathfrak{F}^l_{\mathbb{N}}(\varrho), \mathfrak{F}^u_{\mathbb{N}}(\varrho)]\} \\
&= [\min\{\mathfrak{F}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^l_{\mathbb{N}}(\varrho)\}, \min\{\mathfrak{F}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{N}}(\varrho)\}].
\end{aligned}$$

So we conclude that $\mathfrak{F}^l_{\mathbb{N}}(\epsilon) \geq \min\{\mathfrak{F}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^l_{\mathbb{N}}(\varrho)\}$ and $\mathfrak{F}^u_{\mathbb{N}}(\epsilon) \geq \min\{\mathfrak{F}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{N}}(\varrho)\}$. Therefore, $\mathfrak{T}^l_{\mathbb{N}}, \mathfrak{T}^u_{\mathbb{N}}, \mathfrak{F}^l_{\mathbb{N}}$, and $\mathfrak{F}^u_{\mathbb{N}}$ are FDSs of $\mathfrak{H}$ and $\mathfrak{I}^l_{\mathbb{N}}$ and $\mathfrak{I}^u_{\mathbb{N}}$ are AFDSs of $\mathfrak{H}$. □

Theorem 3.5. *If $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ and $\mathbb{K} = (\mathfrak{T}_{\mathbb{K}}, \mathfrak{I}_{\mathbb{K}}, \mathfrak{F}_{\mathbb{K}})$ are two IVNDSs of $\mathfrak{H}$, then $\mathbb{N} \cap \mathbb{K} = (\mathfrak{T}_{\mathbb{N}\cap\mathbb{K}}, \mathfrak{I}_{\mathbb{N}\cap\mathbb{K}}, \mathfrak{F}_{\mathbb{N}\cap\mathbb{K}})$ is an IVNDS of $\mathfrak{H}$, where*

$$(\forall \varrho \in \mathfrak{H}) \begin{pmatrix} \mathfrak{T}_{\mathbb{N}\cap\mathbb{K}}(\varrho) = [\min\{\mathfrak{T}^l_{\mathbb{N}}(\varrho), \mathfrak{T}^l_{\mathbb{K}}(\varrho)\}, \min\{\mathfrak{T}^u_{\mathbb{N}}(\varrho), \mathfrak{T}^u_{\mathbb{K}}(\varrho)\}] \\ \mathfrak{I}_{\mathbb{N}\cap\mathbb{K}}(\varrho) = [\max\{\mathfrak{I}^l_{\mathbb{N}}(\varrho), \mathfrak{I}^l_{\mathbb{K}}(\varrho)\}, \max\{\mathfrak{I}^u_{\mathbb{N}}(\varrho), \mathfrak{I}^u_{\mathbb{K}}(\varrho)\}] \\ \mathfrak{F}_{\mathbb{N}\cap\mathbb{K}}(\varrho) = [\min\{\mathfrak{F}^l_{\mathbb{N}}(\varrho), \mathfrak{F}^l_{\mathbb{K}}(\varrho)\}, \min\{\mathfrak{F}^u_{\mathbb{N}}(\varrho), \mathfrak{F}^u_{\mathbb{K}}(\varrho)\}] \end{pmatrix}. \tag{9}$$

Proof. Let $\varrho \in \mathfrak{H}$. Then

$$\begin{aligned}
\mathfrak{T}_{\mathbb{N}\cap\mathbb{K}}(\iota) &= [\mathfrak{T}^l_{\mathbb{N}\cap\mathbb{K}}(\iota), \mathfrak{T}^u_{\mathbb{N}\cap\mathbb{K}}(\iota)] \\
&= [\min\{\mathfrak{T}^l_{\mathbb{N}}(\iota), \mathfrak{T}^l_{\mathbb{K}}(\iota)\}, \min\{\mathfrak{T}^u_{\mathbb{N}}(\iota), \mathfrak{T}^u_{\mathbb{K}}(\iota)\}] \\
&\succeq [\min\{\mathfrak{T}^l_{\mathbb{N}}(\varrho), \mathfrak{T}^l_{\mathbb{K}}(\varrho)\}, \min\{\mathfrak{T}^u_{\mathbb{N}}(\varrho), \mathfrak{T}^u_{\mathbb{K}}(\varrho)\}] \\
&= [\mathfrak{T}^l_{\mathbb{N}\cap\mathbb{K}}(\varrho), \mathfrak{T}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)] \\
&= \mathfrak{T}_{\mathbb{N}\cap\mathbb{K}}(\varrho),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{I}_{\mathbb{N}\cap\mathbb{K}}(\iota) &= [\mathfrak{I}^l_{\mathbb{N}\cap\mathbb{K}}(\iota), \mathfrak{I}^u_{\mathbb{N}\cap\mathbb{K}}(\iota)] \\
&= [\max\{\mathfrak{I}^l_{\mathbb{N}}(\iota), \mathfrak{I}^l_{\mathbb{K}}(\iota)\}, \max\{\mathfrak{I}^u_{\mathbb{N}}(\iota), \mathfrak{I}^u_{\mathbb{K}}(\iota)\}] \\
&\preceq [\max\{\mathfrak{I}^l_{\mathbb{N}}(\varrho), \mathfrak{I}^l_{\mathbb{K}}(\varrho)\}, \max\{\mathfrak{I}^u_{\mathbb{N}}(\varrho), \mathfrak{I}^u_{\mathbb{K}}(\varrho)\}] \\
&= [\mathfrak{I}^l_{\mathbb{N}\cap\mathbb{K}}(\varrho), \mathfrak{I}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)] \\
&= \mathfrak{I}_{\mathbb{N}\cap\mathbb{K}}(\varrho),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{F}_{\mathbb{N}\cap\mathbb{K}}(\iota) &= [\mathfrak{F}^l_{\mathbb{N}\cap\mathbb{K}}(\iota), \mathfrak{F}^u_{\mathbb{N}\cap\mathbb{K}}(\iota)] \\
&= [\min\{\mathfrak{F}^l_{\mathbb{N}}(\iota), \mathfrak{F}^l_{\mathbb{K}}(\iota)\}, \min\{\mathfrak{F}^u_{\mathbb{N}}(\iota), \mathfrak{F}^u_{\mathbb{K}}(\iota)\}] \\
&\succeq [\min\{\mathfrak{F}^l_{\mathbb{N}}(\varrho), \mathfrak{F}^l_{\mathbb{K}}(\varrho)\}, \min\{\mathfrak{F}^u_{\mathbb{N}}(\varrho), \mathfrak{F}^u_{\mathbb{K}}(\varrho)\}] \\
&= [\mathfrak{F}^l_{\mathbb{N}\cap\mathbb{K}}(\varrho), \mathfrak{F}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)] \\
&= \mathfrak{F}_{\mathbb{N}\cap\mathbb{K}}(\varrho).
\end{aligned}$$

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Let $\varrho, \epsilon \in \mathfrak{H}$. Then

$$\begin{array}{rcl} \mathfrak{T}_{\mathbb{N}\cap\mathbb{K}}(\epsilon) & = & [\mathfrak{T}^l_{\mathbb{N}\cap\mathbb{K}}(\epsilon), \mathfrak{T}^u_{\mathbb{N}\cap\mathbb{K}}(\epsilon)] \\ & = & [\min\{\mathfrak{T}^l_{\mathbb{N}}(\epsilon), \mathfrak{T}^l_{\mathbb{K}}(\epsilon)\}, \min\{\mathfrak{T}^u_{\mathbb{N}}(\epsilon), \mathfrak{T}^u_{\mathbb{K}}(\epsilon)\}] \\ & \succeq & \left[\begin{array}{l} \min\{\min\{\mathfrak{T}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^l_{\mathbb{N}}(\varrho)\}, \min\{\mathfrak{T}^l_{\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{T}^l_{\mathbb{K}}(\varrho)\}\}, \\ \min\{\min\{\mathfrak{T}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{N}}(\varrho)\}, \min\{\mathfrak{T}^u_{\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{K}}(\varrho)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \min\{\min\{\mathfrak{T}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^l_{\mathbb{K}}(\varrho \circledast \epsilon)\}, \min\{\mathfrak{T}^l_{\mathbb{N}}(\varrho), \mathfrak{T}^l_{\mathbb{K}}(\varrho)\}\}, \\ \min\{\min\{\mathfrak{T}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{K}}(\varrho \circledast \epsilon)\}, \min\{\mathfrak{T}^u_{\mathbb{N}}(\varrho), \mathfrak{T}^u_{\mathbb{K}}(\varrho)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \min\{\mathfrak{T}^l_{\mathbb{N}\cap\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)\}, \\ \min\{\mathfrak{T}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{T}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)\} \end{array}\right], \end{array}$$

$$\begin{array}{rcl} \mathfrak{I}_{\mathbb{N}\cap\mathbb{K}}(\epsilon) & = & [\mathfrak{I}^l_{\mathbb{N}\cap\mathbb{K}}(\epsilon), \mathfrak{I}^u_{\mathbb{N}\cap\mathbb{K}}(\epsilon)] \\ & = & [\max\{\mathfrak{I}^l_{\mathbb{N}}(\epsilon), \mathfrak{I}^l_{\mathbb{K}}(\epsilon)\}, \max\{\mathfrak{I}^u_{\mathbb{N}}(\epsilon), \mathfrak{I}^u_{\mathbb{K}}(\epsilon)\}] \\ & \preceq & \left[\begin{array}{l} \max\{\max\{\mathfrak{I}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^l_{\mathbb{N}}(\varrho)\}, \max\{\mathfrak{I}^l_{\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{I}^l_{\mathbb{K}}(\varrho)\}\}, \\ \max\{\max\{\mathfrak{I}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{N}}(\varrho)\}, \max\{\mathfrak{I}^u_{\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{K}}(\varrho)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \max\{\max\{\mathfrak{I}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^l_{\mathbb{K}}(\varrho \circledast \epsilon)\}, \max\{\mathfrak{I}^l_{\mathbb{N}}(\varrho), \mathfrak{I}^l_{\mathbb{K}}(\varrho)\}\}, \\ \max\{\max\{\mathfrak{I}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{K}}(\varrho \circledast \epsilon)\}, \max\{\mathfrak{I}^u_{\mathbb{N}}(\varrho), \mathfrak{I}^u_{\mathbb{K}}(\varrho)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \max\{\mathfrak{I}^l_{\mathbb{N}\cap\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)\}, \\ \max\{\mathfrak{I}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{I}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)\} \end{array}\right], \end{array}$$

$$\begin{array}{rcl} \mathfrak{F}_{\mathbb{N}\cap\mathbb{K}}(\epsilon) & = & [\mathfrak{F}^l_{\mathbb{N}\cap\mathbb{K}}(\epsilon), \mathfrak{F}^u_{\mathbb{N}\cap\mathbb{K}}(\epsilon)] \\ & = & [\min\{\mathfrak{F}^l_{\mathbb{N}}(\epsilon), \mathfrak{F}^l_{\mathbb{K}}(\epsilon)\}, \min\{\mathfrak{F}^u_{\mathbb{N}}(\epsilon), \mathfrak{F}^u_{\mathbb{K}}(\epsilon)\}] \\ & \succeq & \left[\begin{array}{l} \min\{\min\{\mathfrak{F}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^l_{\mathbb{N}}(\varrho)\}, \min\{\mathfrak{F}^l_{\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{F}^l_{\mathbb{K}}(\varrho)\}\}, \\ \min\{\min\{\mathfrak{F}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{N}}(\varrho)\}, \min\{\mathfrak{F}^u_{\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{K}}(\varrho)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \min\{\min\{\mathfrak{F}^l_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^l_{\mathbb{K}}(\varrho \circledast \epsilon)\}, \min\{\mathfrak{F}^l_{\mathbb{N}}(\varrho), \mathfrak{F}^l_{\mathbb{K}}(\varrho)\}\}, \\ \min\{\min\{\mathfrak{F}^u_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{K}}(\varrho \circledast \epsilon)\}, \min\{\mathfrak{F}^u_{\mathbb{N}}(\varrho), \mathfrak{F}^u_{\mathbb{K}}(\varrho)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \min\{\mathfrak{F}^l_{\mathbb{N}\cap\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)\}, \\ \min\{\mathfrak{F}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho \circledast \epsilon), \mathfrak{F}^u_{\mathbb{N}\cap\mathbb{K}}(\varrho)\} \end{array}\right]. \end{array}$$

So we conclude that $\mathbb{N} \cap \mathbb{K}$ is an IVNDS of $\mathfrak{H}$. □

Corollary 3.6. *If $\{\mathbb{N}_i \mid i \in \Delta\}$ is a nonempty family of IVNDSs of $\mathfrak{H}$, then $\bigcap_{i\in\Delta} \mathbb{N}_i$ is an IVNDS of $\mathfrak{H}$.*

Corollary 3.7. *If $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ is an IVNDS of $\mathfrak{H}$, then $\overline{\mathbb{N}}$ is also an IVNDS of $\mathfrak{H}$.*

Definition 3.8. Let $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ be an IVNS in $\mathfrak{H}$. The IVNSs $\oplus\mathbb{N}, \otimes\mathbb{N}$, and $\odot\mathbb{N}$ in $\mathfrak{H}$ are defined as follows: $\oplus\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \overline{\mathfrak{T}_{\mathbb{N}}}, \mathfrak{F}_{\mathbb{N}}), \otimes\mathbb{N} = (\overline{\mathfrak{I}_{\mathbb{N}}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$, and $\odot\mathbb{N} = (\overline{\mathfrak{I}_{\mathbb{N}}}, \mathfrak{I}_{\mathbb{N}}, \overline{\mathfrak{I}_{\mathbb{N}}})$.

Theorem 3.9. *If $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ is an IVNDS of $\mathfrak{H}$, then $\oplus\mathbb{N}, \otimes\mathbb{N}$, and $\odot\mathbb{N}$ are IVNDSs of $\mathfrak{H}$.*

Proof. Let $\varrho \in \mathfrak{H}$. Then $\overline{\mathfrak{T}_{\mathbb{N}}}(\iota) = \mathbf{1} - \mathfrak{T}_{\mathbb{N}}(\iota) \preceq \mathbf{1} - \mathfrak{T}_{\mathbb{N}}(\varrho) \preceq \overline{\mathfrak{T}_{\mathbb{N}}}(\varrho)$. Let $\varrho, \epsilon \in \mathfrak{H}$. Then

$$\begin{array}{rcl} \overline{\mathfrak{T}_{\mathbb{N}}}(\epsilon) & = & \mathbf{1} - \mathfrak{T}_{\mathbb{N}}(\epsilon) \\ & \preceq & \mathbf{1} - \mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\} \\ & = & \mathrm{rmax}\{\mathbf{1} - \mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathbf{1} - \mathfrak{T}_{\mathbb{N}}(\varrho)\} \\ & = & \mathrm{rmax}\{\overline{\mathfrak{T}_{\mathbb{N}}}(\varrho \circledast \epsilon), \overline{\mathfrak{T}_{\mathbb{N}}}(\varrho)\}. \end{array}$$

So we conclude that $\oplus\mathbb{N}$ is an IVNDS of $\mathfrak{H}$.

Let $\varrho \in \mathfrak{H}$. Then $\overline{\mathfrak{I}_{\mathbb{N}}}(\iota) = \mathbf{1} - \mathfrak{I}_{\mathbb{N}}(\iota) \succeq \mathbf{1} - \mathfrak{I}_{\mathbb{N}}(\varrho) \succeq \overline{\mathfrak{I}_{\mathbb{N}}}(\varrho)$. Let $\varrho, \epsilon \in \mathfrak{H}$. Then

$$\begin{array}{rcl} \overline{\mathfrak{I}_{\mathbb{N}}}(\epsilon) & = & \mathbf{1} - \mathfrak{I}_{\mathbb{N}}(\epsilon) \\ & \succeq & \mathbf{1} - \mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\epsilon)\} \\ & = & \mathrm{rmin}\{\mathbf{1} - \mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathbf{1} - \mathfrak{I}_{\mathbb{N}}(\epsilon)\} \\ & = & \mathrm{rmin}\{\overline{\mathfrak{I}_{\mathbb{N}}}(\varrho \circledast \epsilon), \overline{\mathfrak{I}_{\mathbb{N}}}(\epsilon)\}. \end{array}$$

So we conclude that $\otimes\mathbb{N}$ is an IVNDS of $\mathfrak{H}$. Similar arguments may be made for $\odot\mathbb{N}$. □

Theorem 3.10. *An IVNS $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ in $\mathfrak{H}$ is an IVNDS of $\mathfrak{H}$ if and only if for every $[s_1, s_2], [t_1, t_2], [u_1, u_2] \in [[0, 1]]$, the sets $U(\mathfrak{T}_{\mathbb{N}} : [t_1, t_2])$, $L(\mathfrak{I}_{\mathbb{N}} : [s_1, s_2])$, and $U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$ are DSs of $\mathfrak{H}$ if they are nonempty.*

Proof. Let $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ be an IVNDS of $\mathfrak{H}$ and let $[s_1, s_2]$, $[t_1, t_2]$, $[u_1, u_2] \in [[0, 1]]$ be such that $U(\mathfrak{T}_{\mathbb{N}} : [t_1, t_2])$, $L(\mathfrak{I}_{\mathbb{N}}, [s_1, s_2])$, and $U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$ are nonempty sets of $\mathfrak{H}$. Clearly, $\iota \in U(\mathfrak{T}_{\mathbb{N}} : [t_1, t_2]) \cap L(\mathfrak{I}_{\mathbb{N}} : [s_1, s_2]) \cap U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$ since $\mathfrak{T}_{\mathbb{N}}(\iota) \succeq [t_1, t_2]$, $\mathfrak{I}_{\mathbb{N}}(\iota) \preceq [s_1, s_2]$, and $\mathfrak{F}_{\mathbb{N}}(\iota) \succeq [u_1, u_2]$. Let $\varrho, \epsilon \in \mathfrak{H}$ be such that $\varrho, \varrho \circledast \epsilon \in U(\mathfrak{T}_{\mathbb{N}} : [t_1, t_2])$. Then $\mathfrak{T}_{\mathbb{N}}(\varrho) \succeq [t_1, t_2]$ and $\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon) \succeq [t_1, t_2]$. So we conclude that $\mathfrak{T}_{\mathbb{N}}(\epsilon) \succeq \min\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\} \succeq [t_1, t_2]$ so that $\epsilon \in U(\mathfrak{T}_{\mathbb{N}} : [t_1, t_2])$. So we conclude that $U(\mathfrak{T}_{\mathbb{N}} : [t_1, t_2])$ is a DS of $\mathfrak{H}$. Let $\varrho, \epsilon \in \mathfrak{H}$ be such that $\varrho, \varrho \circledast \epsilon \in L(\mathfrak{I}_{\mathbb{N}} : [s_1, s_2])$. Then $\mathfrak{I}_{\mathbb{N}}(\varrho) \preceq [s_1, s_2]$ and $\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon) \preceq [s_1, s_2]$. So we conclude that $\mathfrak{I}_{\mathbb{N}}(\epsilon) \preceq \max\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\} \preceq [s_1, s_2]$ so that $\epsilon \in L(\mathfrak{I}_{\mathbb{N}} : [s_1, s_2])$. So we conclude that $L(\mathfrak{I}_{\mathbb{N}} : [s_1, s_2])$ is a DS of $\mathfrak{H}$. Let $\varrho, \epsilon \in \mathfrak{H}$ be such that $\varrho, \varrho \circledast \epsilon \in U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$. Then $\mathfrak{F}_{\mathbb{N}}(\varrho) \succeq [u_1, u_2]$ and $\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon) \succeq [u_1, u_2]$. So we conclude that $\mathfrak{F}_{\mathbb{N}}(\epsilon) \succeq \min\{\mathfrak{F}_{\mathbb{N}}(x \cot y), \mathfrak{F}_{\mathbb{N}}(\varrho)\} \succeq [u_1, u_2]$ so that $\epsilon \in U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$. So we conclude that $U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$ is a DS of $\mathfrak{H}$.

Assume now that every nonempty sets $U(\mathfrak{T}_{\mathbb{N}} : [t_1, t_2])$, $L(\mathfrak{I}_{\mathbb{N}}, [s_1, s_2])$, and $U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$ are DSs in $\mathfrak{H}$. If $\mathfrak{T}_{\mathbb{N}}(\iota) \succeq \mathfrak{T}_{\mathbb{N}}(\varrho)$ is not true for each $\varrho \in \mathfrak{H}$, then there is $\varrho_0 \in \mathfrak{H}$ such that $\mathfrak{T}_{\mathbb{N}}(\iota) \prec \mathfrak{T}_{\mathbb{N}}(\varrho_0)$. But in this case for $[s_1, s_2] = \frac{1}{2}(\mathfrak{T}_{\mathbb{N}}(\iota) + \mathfrak{T}_{\mathbb{N}}(\varrho_0))$. Then $\varrho_0 \in U(\mathfrak{T}_{\mathbb{N}} : [s_1, s_2])$, that is, $U(\mathfrak{T}_{\mathbb{N}} : [s_1, s_2]) \neq \emptyset$. Because of the assumption, $U(\mathfrak{T}_{\mathbb{N}} : [s_1, s_2])$ is a DS of $\mathfrak{H}$, then $\mathfrak{T}_{\mathbb{N}}(\iota) \succeq [s_1, s_2]$, which is impossible. So we conclude that $\mathfrak{T}_{\mathbb{N}}(\iota) \succeq \mathfrak{T}_{\mathbb{N}}(\varrho)$ for all $\varrho \in \mathfrak{H}$. If $\mathfrak{I}_{\mathbb{N}}(\iota) \preceq \mathfrak{I}_{\mathbb{N}}(\varrho)$ is not true for each $\varrho \in \mathfrak{H}$, then there is $\epsilon_0 \in \mathfrak{H}$ such that $\mathfrak{I}_{\mathbb{N}}(\iota) \succ \mathfrak{I}_{\mathbb{N}}(\epsilon_0)$. But in this case for $[s_0', s_0''] = \frac{1}{2}(\mathfrak{I}_{\mathbb{N}}(\iota) + \mathfrak{I}_{\mathbb{N}}(\epsilon_0))$. Then $\epsilon_0 \in L(\mathfrak{I}_{\mathbb{N}} : [s_0', s_0''])$, that is, $L(\mathfrak{I}_{\mathbb{N}} : [s_0', s_0'']) \neq \emptyset$. Because of the assumption, $L(\mathfrak{I}_{\mathbb{N}} : [s_0', s_0''])$ is a DS of $\mathfrak{H}$, then $\mathfrak{I}_{\mathbb{N}}(\iota) \preceq [s_0', s_0'']$, which is impossible. So we conclude that $\mathfrak{I}_{\mathbb{N}}(\iota) \preceq \mathfrak{I}_{\mathbb{N}}(\varrho)$ for all $\varrho \in \mathfrak{H}$. If $\mathfrak{F}_{\mathbb{N}}(\iota) \succeq \mathfrak{F}_{\mathbb{N}}(\varrho)$ is not true for each $\varrho \in \mathfrak{H}$, then there is $\varrho_0 \in \mathfrak{H}$ such that $\mathfrak{F}_{\mathbb{N}}(\iota) \prec\prec \mathfrak{F}_{\mathbb{N}}(\varrho_0)$. But in this case for $[u_1, u_2] = \frac{1}{2}(\mathfrak{F}_{\mathbb{N}}(\iota) + \mathfrak{F}_{\mathbb{N}}(\varrho_0))$. Then $\varrho_0 \in U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$, that is, $U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2]) \neq \emptyset$. Because of the assumption, $U(\mathfrak{F}_{\mathbb{N}} : [u_1, u_2])$ is a DS of $\mathfrak{H}$, then $\mathfrak{F}_{\mathbb{N}}(\iota) \succeq [u_1, u_2]$, which is impossible. So we conclude that $\mathfrak{F}_{\mathbb{N}}(\iota) \succeq \mathfrak{F}_{\mathbb{N}}(\varrho)$ for all $\varrho \in \mathfrak{H}$. Suppose $\mathfrak{T}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\epsilon)\}$ is not true for each $\varrho, \epsilon \in \mathfrak{H}$. Then there are $u_0, v_0 \in \mathfrak{H}$ such that $\mathfrak{T}_{\mathbb{N}}(v_0) \prec \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{T}_{\mathbb{N}}(u_0)\}$. Taking $[p', p''] = \frac{1}{2}(\mathfrak{T}_{\mathbb{N}}(v_0) + \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{T}_{\mathbb{N}}(u_0)\})$. Then $\mathfrak{T}_{\mathbb{N}}(u_0) \prec [p', p''] \prec \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{T}_{\mathbb{N}}(u_0)\}$, that is, $u_0, v_0 \in U(\mathfrak{T}_{\mathbb{N}} : [p', p''])$. Since $U(\mathfrak{T}_{\mathbb{N}} : [p', p''])$ is a DS of $\mathfrak{H}$, $u_0 \in U(\mathfrak{T}_{\mathbb{N}} : [p', p''])$, a contradiction. Thus $\mathfrak{T}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\}$ is true for all $\varrho, \epsilon \in \mathfrak{H}$. Suppose $\mathfrak{I}_{\mathbb{N}}(\epsilon) \preceq \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\}$ is not true for each $\varrho, \epsilon \in \mathfrak{H}$. Then there are $u_0, v_0 \in \mathfrak{H}$ such that $\mathfrak{I}_{\mathbb{N}}(v_0) \succ \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{I}_{\mathbb{N}}(u_0)\}$. Taking $[p_0', p_0''] = \frac{1}{2}(\mathfrak{I}_{\mathbb{N}}(v_0) + \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{I}_{\mathbb{N}}(u_0)\})$. Then $\mathfrak{I}_{\mathbb{N}}(v_0) \succ [p_0', p_0''] \succ \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{I}_{\mathbb{N}}(u_0)\}$, that is, $u_0, v_0 \in L(\mathfrak{I}_{\mathbb{N}} : [p_0', p_0''])$. Since $L(\mathfrak{I}_{\mathbb{N}} : [p_0', p_0''])$ is a DS of $\mathfrak{H}$, $u_0 \circledast v_0 \in L(\mathfrak{I}_{\mathbb{N}} : [p_0', p_0''])$, a contradiction. Thus $\mathfrak{I}_{\mathbb{N}}(\epsilon) \preceq \text{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\}$ is true for all $\varrho, \epsilon \in \mathfrak{H}$. Suppose $\mathfrak{F}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}(\epsilon)\}$ is not true for each $\varrho, \epsilon \in \mathfrak{H}$. Then there are $u_0, v_0 \in \mathfrak{H}$ such that $\mathfrak{F}_{\mathbb{N}}(v_0) \prec \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{F}_{\mathbb{N}}(u_0)\}$. Taking $[q', q''] = \frac{1}{2}(\mathfrak{F}_{\mathbb{N}}(v_0) + \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{F}_{\mathbb{N}}(u_0)\})$. Then $\mathfrak{F}_{\mathbb{N}}(u_0) \prec [q', q''] \prec \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(u_0 \circledast v_0), \mathfrak{F}_{\mathbb{N}}(u_0)\}$, that is, $u_0, v_0 \in U(\mathfrak{F}_{\mathbb{N}} : [q', q''])$. Since $U(\mathfrak{F}_{\mathbb{N}} : [q', q''])$ is a DS of $\mathfrak{H}$, $u_0 \in U(\mathfrak{F}_{\mathbb{N}} : [q', q''])$, a contradiction. Thus $\mathfrak{F}_{\mathbb{N}}(\epsilon) \succeq \text{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}(\varrho)\}$ is true for all $\varrho, \epsilon \in \mathfrak{H}$. So we conclude that $\mathbb{N}$ is an IVNDS of $\mathfrak{H}$. □

Theorem 3.11. *Let $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{H}$ and $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ be an IVNS in $\mathfrak{H}$ defined by*

$$\mathfrak{T}_{\mathbb{N}}(\varrho) = \begin{cases} \alpha_0 & \text{if } \varrho \in \mathfrak{B} \\ \alpha_1 & \text{otherwise,} \end{cases}$$

$$\mathfrak{I}_{\mathbb{N}}(\varrho) = \begin{cases} \beta_0 & \text{if } \varrho \in \mathfrak{B} \\ \beta_1 & \text{otherwise,} \end{cases}$$

$$\mathfrak{F}_{\mathbb{N}}(\varrho) = \begin{cases} \theta_0 & \text{if } \varrho \in \mathfrak{B} \\ \theta_1 & \text{otherwise} \end{cases}$$

for all $\varrho \in \mathfrak{H}$ and $\alpha_0, \alpha_1, \beta_0, \beta_1, \theta_0, \theta_1 \in [[0, 1]]$ with $\alpha_0 \succ \alpha_1$, $\beta_0 \prec \beta_1$, $\theta_0 \succ \theta_1$. Then $\mathbb{N}$ is an IVNDS of $\mathfrak{H}$ if and only if $\mathfrak{B}$ is a DS of $\mathfrak{H}$.

Proof. Let's assume that $\mathbb{N}$ is an IVNDS of $\mathfrak{H}$. Since $\mathfrak{T}_{\mathbb{N}}(\iota) \succeq \mathfrak{T}_{\mathbb{N}}(\varrho)$, $\mathfrak{I}_{\mathbb{N}}(\iota) \preceq \mathfrak{I}_{\mathbb{N}}(\varrho)$, and $\mathfrak{F}_{\mathbb{N}}(\iota) \succeq \mathfrak{F}_{\mathbb{N}}(\varrho)$ for all $\varrho \in \mathfrak{H}$, we obtain $\mathfrak{T}_{\mathbb{N}}(\iota) = \alpha_1$, $\mathfrak{I}_{\mathbb{N}}(\iota) = \beta_1$, and $\mathfrak{F}_{\mathbb{N}}(\iota) = \theta_1$ and so $\iota \in \mathfrak{B}$. Let $\varrho, \epsilon \in \mathfrak{H}$ be such that $\varrho, \varrho \circledast \epsilon \in \mathfrak{B}$. Then $\mathfrak{T}_{\mathbb{N}}(\epsilon) \succeq \text{rmax}\{\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}(\varrho)\} = \alpha_1$ and then $\mathfrak{T}_{\mathbb{N}}(\epsilon) = \alpha_1$. Also, $\mathfrak{I}_{\mathbb{N}}(\epsilon) \preceq \text{rmin}\{\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}(\varrho)\} = \beta_1$ and then $\mathfrak{I}_{\mathbb{N}}(\epsilon) = \beta_1$, and $\mathfrak{F}_{\mathbb{N}}(\epsilon) \succeq \text{rmax}\{\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}(\varrho)\} = \theta_1$; hence $\mathfrak{F}_{\mathbb{N}}(\epsilon) = \theta_1$. So we conclude that $\epsilon \in \mathfrak{B}$. Therefore, $\mathfrak{B}$ is a DS of $\mathfrak{H}$.

Assume, on the other hand, that $\mathfrak{B}$ is a DS of $\mathfrak{H}$. Since $\iota \in \mathfrak{B}$, we obtain $\mathfrak{T}_{\mathbb{N}}(\iota) = \alpha_1 \succeq \mathfrak{T}_{\mathbb{N}}(\varrho)$, $\mathfrak{I}_{\mathbb{N}}(\iota) = \beta_1 \preceq \mathfrak{I}_{\mathbb{N}}(\varrho)$, and $\mathfrak{F}_{\mathbb{N}}(\iota) = \theta_1 \succeq \mathfrak{F}_{\mathbb{N}}(\varrho)$ for all $\varrho \in \mathfrak{H}$. Let $\varrho, \epsilon \in \mathfrak{H}$. If $\epsilon \in \mathfrak{B}$, then $\varrho \circledast \epsilon \in \mathfrak{B}$ and so $\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon) = \alpha_1 = \mathfrak{T}_{\mathbb{N}}(\epsilon)$, $\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon) = \beta_1 = \mathfrak{I}_{\mathbb{N}}(\epsilon)$, and $\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon) = \theta_1 = \mathfrak{F}_{\mathbb{N}}(\epsilon)$. If $\epsilon \in \mathfrak{H} \backslash \mathfrak{B}$, then $\mathfrak{T}_{\mathbb{N}}(\epsilon) = \alpha_2$, $\mathfrak{I}_{\mathbb{N}}(\epsilon) = \beta_2$, and $\mathfrak{F}_{\mathbb{N}}(\epsilon) = \theta_2$, and hence $\mathfrak{T}_{\mathbb{N}}(\varrho \circledast \epsilon) \succeq \alpha_2 = \mathfrak{T}_{\mathbb{N}}(\epsilon)$, $\mathfrak{I}_{\mathbb{N}}(\varrho \circledast \epsilon) \preceq \beta_2 = \mathfrak{I}_{\mathbb{N}}(\epsilon)$, and $\mathfrak{F}_{\mathbb{N}}(\varrho \circledast \epsilon) \succeq \theta_2 = \mathfrak{F}_{\mathbb{N}}(\epsilon)$. So we conclude that $\mathbb{N}$ is an IVNDS of $\mathfrak{H}$. □

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Definition 3.12. Let $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ and $\mathbb{K} = (\mathfrak{T}_{\mathbb{K}}, \mathfrak{I}_{\mathbb{K}}, \mathfrak{F}_{\mathbb{K}})$ be IVNSs in $\mathfrak{H}$ and $\mathfrak{K}$, respectively. The cartesian product $\mathbb{N} \times \mathbb{K} = \{((\varrho, \epsilon), (\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})(\varrho, \epsilon), (\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})(\varrho, \epsilon), (\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})(\varrho, \epsilon)) \mid \varrho \in \mathfrak{H}, \epsilon \in \mathfrak{K}\}$ is defined by

$$(\forall (\varrho, \epsilon) \in \mathfrak{H} \times \mathfrak{K}) \left(\begin{array}{l} (\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})(\varrho, \epsilon) = \mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho), \mathfrak{T}_{\mathbb{K}}(\epsilon)\} \\ (\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})(\varrho, \epsilon) = \mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho), \mathfrak{I}_{\mathbb{K}}(\epsilon)\} \\ (\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})(\varrho, \epsilon) = \mathrm{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho), \mathfrak{F}_{\mathbb{K}}(\epsilon)\} \end{array} \right),$$

where $\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}} : \mathfrak{H} \times \mathfrak{K} \to [[0, 1]]$, $\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}} : \mathfrak{H} \times \mathfrak{K} \to [[0, 1]]$, and $\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}} : \mathfrak{H} \times \mathfrak{K} \to [[0, 1]]$.

Remark 3.13. We define the binary operation $\circledast$ on $\mathfrak{H} \times \mathfrak{K}$ for $\mathfrak{H}$ and $\mathfrak{K}$ as $(\varrho, \epsilon) \circledast (\varsigma, \vartheta) = (\varrho \circledast \varsigma, \epsilon \circledast \vartheta)$ for each $(\varrho, \epsilon), (\varsigma, \vartheta) \in \mathfrak{H} \times \mathfrak{K}$. Then clearly $(\mathfrak{H} \times \mathfrak{K}, \circledast, (\iota, \iota))$ is a Hilbert algebra.

Proposition 3.14. *If $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ and $\mathbb{K} = (\mathfrak{T}_{\mathbb{K}}, \mathfrak{I}_{\mathbb{K}}, \mathfrak{F}_{\mathbb{K}})$ are IVNDSs of $\mathfrak{H}$ and $\mathfrak{K}$, respectively, then $\mathbb{N} \times \mathbb{K}$ is also an IVNDS of $\mathfrak{H} \times \mathfrak{K}$.*

Proof. Let $(\varrho, \epsilon) \in \mathfrak{H} \times \mathfrak{K}$. Then

$$\begin{array}{rcl} (\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})(\iota, \iota) & = & \mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\iota), \mathfrak{T}_{\mathbb{K}}(\iota)\} \\ & \succeq & \mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho), \mathfrak{T}_{\mathbb{K}}(\epsilon)\} \\ & = & (\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})(\varrho, \epsilon), \end{array}$$

$$\begin{array}{rcl} (\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})(\iota, \iota) & = & \mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\iota), \mathfrak{I}_{\mathbb{K}}(\iota)\} \\ & \preceq & \mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho), \mathfrak{I}_{\mathbb{K}}(\epsilon)\} \\ & = & (\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})(\varrho, \epsilon), \end{array}$$

$$\begin{array}{rcl} (\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})(\iota, \iota) & = & \mathrm{rmin}\{\mathfrak{F}_{\mathbb{N}}(\iota), \mathfrak{F}_{\mathbb{K}}(\iota)\} \\ & \succeq & \mathrm{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho), \mathfrak{F}_{\mathbb{K}}(\epsilon)\} \\ & = & (\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})(\varrho, \epsilon). \end{array}$$

Let $(\varrho_1, \varrho_2), (\epsilon_1, \epsilon_2) \in \mathfrak{H} \times \mathfrak{K}$. Then

$$\begin{array}{l} (\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})(\epsilon_1, \epsilon_2) \\ = \mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\epsilon_1), \mathfrak{T}_{\mathbb{K}}(\epsilon_2)\} \\ \succeq \mathrm{rmin}\{\mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho_1 \circledast \epsilon_1), \mathfrak{T}_{\mathbb{N}}(\varrho_1)\}, \mathrm{rmin}\{\mathfrak{T}_{\mathbb{K}}(\varrho_2 \circledast \epsilon_2), \mathfrak{T}_{\mathbb{K}}(\varrho_2)\}\} \\ = \mathrm{rmin}\{\mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho_1 \circledast \epsilon_1), \mathfrak{T}_{\mathbb{K}}(\varrho_2 \circledast \epsilon_2)\}, \mathrm{rmin}\{\mathfrak{T}_{\mathbb{N}}(\varrho_1), \mathfrak{T}_{\mathbb{K}}(\varrho_2)\}\} \\ = \mathrm{rmin}\{(\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})((\varrho_1 \circledast \epsilon_1), (\varrho_2 \circledast \epsilon_2)), (\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})(\varrho_1, \varrho_2)\} \\ = \mathrm{rmin}\{(\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})((\varrho_1, \varrho_2) \circledast (\epsilon_1, \epsilon_2)), (\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}})(\varrho_1, \varrho_2)\}, \end{array}$$

$$\begin{array}{l} (\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})(\epsilon_1, \epsilon_2) \\ = \mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\epsilon_1), \mathfrak{I}_{\mathbb{K}}(\epsilon_2)\} \\ \preceq \mathrm{rmax}\{\mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho_1 \circledast \epsilon_1), \mathfrak{I}_{\mathbb{N}}(\varrho_1)\}, \mathrm{rmax}\{\mathfrak{I}_{\mathbb{K}}(\varrho_2 \circledast \epsilon_2), \mathfrak{I}_{\mathbb{K}}(\varrho_2)\}\} \\ = \mathrm{rmax}\{\mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho_1 \circledast \epsilon_1), \mathfrak{I}_{\mathbb{K}}(\varrho_2 \circledast \epsilon_2)\}, \mathrm{rmax}\{\mathfrak{I}_{\mathbb{N}}(\varrho_1), \mathfrak{I}_{\mathbb{K}}(\varrho_2)\}\} \\ = \mathrm{rmax}\{(\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})((\varrho_1 \circledast \epsilon_1), (\varrho_2 \circledast \epsilon_2)), (\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})(\varrho_1, \varrho_2)\} \\ = \mathrm{rmax}\{(\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})((\varrho_1, \varrho_2) \circledast (\epsilon_1, \epsilon_2)), (\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}})(\varrho_1, \varrho_2)\}, \end{array}$$

$$\begin{array}{l} (\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})(\epsilon_1, \epsilon_2) \\ = \mathrm{rmin}\{\mathfrak{F}_{\mathbb{N}}(\epsilon_1), \mathfrak{F}_{\mathbb{K}}(\epsilon_2)\} \\ \succeq \mathrm{rmin}\{\mathrm{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho_1 \circledast \epsilon_1), \mathfrak{F}_{\mathbb{N}}(\varrho_1)\}, \mathrm{rmin}\{\mathfrak{F}_{\mathbb{K}}(\varrho_2 \circledast \epsilon_2), \mathfrak{F}_{\mathbb{K}}(\varrho_2)\}\} \\ = \mathrm{rmin}\{\mathrm{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho_1 \circledast \epsilon_1), \mathfrak{F}_{\mathbb{K}}(\varrho_2 \circledast \epsilon_2)\}, \mathrm{rmin}\{\mathfrak{F}_{\mathbb{N}}(\varrho_1), \mathfrak{F}_{\mathbb{K}}(\varrho_2)\}\} \\ = \mathrm{rmin}\{(\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})((\varrho_1 \circledast \epsilon_1), (\varrho_2 \circledast \epsilon_2)), (\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})(\varrho_1, \varrho_2)\} \\ = \mathrm{rmin}\{(\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})((\varrho_1, \varrho_2) \circledast (\epsilon_1, \epsilon_2)), (\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}})(\varrho_1, \varrho_2)\}. \end{array}$$

So we conclude that $\mathbb{N} \times \mathbb{K}$ is an IVNDS of $\mathfrak{H} \times \mathfrak{K}$. □

The following theorem is derived from Theorem 3.9 and Proposition 3.14.

Theorem 3.15. *If $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ and $\mathbb{K} = (\mathfrak{T}_{\mathbb{K}}, \mathfrak{I}_{\mathbb{K}}, \mathfrak{F}_{\mathbb{K}})$ are IVNDSs of $\mathfrak{H}$ and $\mathfrak{K}$, respectively, then $\oplus(\mathbb{N} \times \mathbb{K}), \otimes(\mathbb{N} \times \mathbb{K})$, and $\odot(\mathbb{N} \times \mathbb{K})$ are IVNDSs of $\mathfrak{H} \times \mathfrak{K}$.*

The following theorem can be derived from Theorem 3.10.

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Theorem 3.16. *Let $\mathbb{N} = (\mathfrak{T}_{\mathbb{N}}, \mathfrak{I}_{\mathbb{N}}, \mathfrak{F}_{\mathbb{N}})$ and $\mathbb{K} = (\mathfrak{T}_{\mathbb{K}}, \mathfrak{I}_{\mathbb{K}}, \mathfrak{F}_{\mathbb{K}})$ be IVNSs in $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Then $\mathbb{N} \times \mathbb{K}$ is an IVNDS of $\mathfrak{H} \times \mathfrak{K}$ if and only if for each $[s_1, s_2], [t_1, t_2], [u_1, u_2] \in [[0,1]]$, $U(\mathfrak{T}_{\mathbb{N}} \times \mathfrak{T}_{\mathbb{K}} : [s_1, s_2])$, $L(\mathfrak{I}_{\mathbb{N}} \times \mathfrak{I}_{\mathbb{K}} : [t_1, t_2])$, and $U(\mathfrak{F}_{\mathbb{N}} \times \mathfrak{F}_{\mathbb{K}} : [u_1, u_2])$ are DSs of $\mathfrak{H} \times \mathfrak{K}$ if they are nonempty.*

For any $\widehat{s}^+, \widehat{s}^-, \widehat{t}^+, \widehat{t}^-, \widehat{q}^+, \widehat{q}^- \in [[0,1]]$ such that $\widehat{s}^+ \succ \widehat{s}^-$, $\widehat{t}^+ \succ \widehat{t}^-$, $\widehat{q}^+ \succ \widehat{q}^-$ and $\emptyset \neq \mathfrak{G} \subseteq \mathfrak{H}$, the IVNS

$$\mathbb{N}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+, & \widehat{t}^-, & \widehat{q}^+ \\ \widehat{s}^-, & \widehat{t}^+, & \widehat{q}^- \end{bmatrix} = \left(\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix}, \mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{t}^- \\ \widehat{t}^+ \end{bmatrix}, \mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{q}^+ \\ \widehat{q}^- \end{bmatrix} \right)$$

in $\mathfrak{H}$ is defined by for all $\varrho \in \mathfrak{H}$,

$$\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\varrho) = \begin{cases} \widehat{s}^+ & \text{if } \varrho \in \mathfrak{G} \\ \widehat{s}^- & \text{otherwise,} \end{cases}$$

$$\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{t}^- \\ \widehat{t}^+ \end{bmatrix} (\varrho) = \begin{cases} \widehat{t}^- & \text{if } \varrho \in \mathfrak{G} \\ \widehat{t}^+ & \text{otherwise,} \end{cases}$$

$$\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{q}^+ \\ \widehat{q}^- \end{bmatrix} (\varrho) = \begin{cases} \widehat{q}^+ & \text{if } \varrho \in \mathfrak{G} \\ \widehat{q}^- & \text{otherwise.} \end{cases}$$

Lemma 3.17. *If $\iota \in \mathfrak{G} \subseteq \mathfrak{H}$, then $\mathbb{N}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+, & \widehat{t}^-, & \widehat{q}^+ \\ \widehat{s}^-, & \widehat{t}^+, & \widehat{q}^- \end{bmatrix}$ in $\mathfrak{H}$ satisfies* (5).

Proof. If $\iota \in \mathfrak{G}$, then $\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\iota) = \widehat{s}^+$, $\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{t}^- \\ \widehat{t}^+ \end{bmatrix} (\iota) = \widehat{t}^-$, and $\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{q}^+ \\ \widehat{q}^- \end{bmatrix} (\iota) = \widehat{q}^+$. Thus

$$(\forall \varrho \in \mathfrak{H}) \begin{pmatrix} \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\iota) = \widehat{s}^+ \succeq \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\varrho) \\ \mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{t}^- \\ \widehat{t}^+ \end{bmatrix} (\iota) = \widehat{t}^- \preceq \mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{t}^- \\ \widehat{t}^+ \end{bmatrix} (\varrho) \\ \mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{q}^+ \\ \widehat{q}^- \end{bmatrix} (\iota) = \widehat{q}^+ \succeq \mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{q}^+ \\ \widehat{q}^- \end{bmatrix} (\varrho) \end{pmatrix}.$$

So we conclude that $\mathbb{N}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+, & \widehat{t}^-, & \widehat{q}^+ \\ \widehat{s}^-, & \widehat{t}^+, & \widehat{q}^- \end{bmatrix}$ satisfies (5). □

Lemma 3.18. *If $\mathbb{N}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+, & \widehat{t}^-, & \widehat{q}^+ \\ \widehat{s}^-, & \widehat{t}^+, & \widehat{q}^- \end{bmatrix}$ in $\mathfrak{H}$ satisfies* (5)*, then $\iota \in \mathfrak{G} \subseteq \mathfrak{H}$.*

Proof. Let's assume that $\mathbb{N}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+, & \widehat{t}^-, & \widehat{q}^+ \\ \widehat{s}^-, & \widehat{t}^+, & \widehat{q}^- \end{bmatrix}$ satisfies (5). Then $\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\iota) \succeq \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\varrho)$ for all $\varrho \in \mathfrak{H}$. Since $\mathfrak{G} \neq \subseteq$, there is $g \in \mathfrak{G}$. Thus $\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (g) = \widehat{s}^+$ and so $\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\iota) \succeq \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (g) = \widehat{s}^+ \succeq \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\iota)$, that is, $\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\iota) = \widehat{s}^+$. So we conclude that $\iota \in \mathfrak{G}$. □

Theorem 3.19. $\mathbb{N}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+, & \widehat{t}^-, & \widehat{q}^+ \\ \widehat{s}^-, & \widehat{t}^+, & \widehat{q}^- \end{bmatrix}$ *in $\mathfrak{H}$ is an IVNDS of $\mathfrak{H}$ if and only if $\emptyset \neq \mathfrak{G} \subseteq \mathfrak{H}$ is a DS of $\mathfrak{H}$.*

Proof. Let's assume that $\mathbb{N}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+, & \widehat{t}^-, & \widehat{q}^+ \\ \widehat{s}^-, & \widehat{t}^+, & \widehat{q}^- \end{bmatrix}$ is an IVNDS of $\mathfrak{H}$. Since $\mathbb{N}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+, & \widehat{t}^-, & \widehat{q}^+ \\ \widehat{s}^-, & \widehat{t}^+, & \widehat{q}^- \end{bmatrix}$ satisfies (5) and by Lemma 3.18, we get $\iota \in \mathfrak{G}$. Suppose $\varrho, \epsilon \in \mathfrak{H}$ is such that $\varrho, \varrho \circledast \epsilon \in \mathfrak{G}$. Then $\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}} \begin{bmatrix} \widehat{s}^+ \\ \widehat{s}^- \end{bmatrix} (\varrho) = \widehat{s}^+ =$

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$\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho \circledast \epsilon)$. So we conclude that

$$\begin{aligned}\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\epsilon) &\succeq \operatorname{rmin}\left\{\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho)\right\}\\ &= \operatorname{rmin}\{\widehat{s}^{+}, \widehat{s}^{+}\}\\ &= \widehat{s}^{+}\\ &\succeq \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\epsilon)\end{aligned}$$

and so $\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\epsilon) = \widehat{s}^{+}$. Thus $\epsilon \in \mathfrak{G}$. So we conclude that $\mathfrak{G}$ is a DS of $\mathfrak{H}$.

Assume, on the other hand, that $\mathfrak{G}$ is a DS of $\mathfrak{H}$. Since $\iota \in \mathfrak{G}$, it follows from Lemma 3.17 that $\mathbb{N}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}, & \widehat{t}^{-}, & \widehat{q}^{+}\\ \widehat{s}^{-}, & \widehat{t}^{+}, & \widehat{q}^{-}\end{bmatrix}$ satisfies (5). Next, let $\varrho, \epsilon \in \mathfrak{H}$.

Case 1: Suppose $\varrho \circledast \epsilon \in \mathfrak{G}$ and $\varrho \in \mathfrak{G}$. Then

$$\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho \circledast \epsilon) = \widehat{s}^{+} = \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho),$$

$$\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho \circledast \epsilon) = \widehat{t}^{-} = \mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho),$$

$$\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho \circledast \epsilon) = \widehat{q}^{+} = \mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho).$$

Since $\mathfrak{G}$ is a DS of $\mathfrak{H}$, we obtain $\epsilon \in \mathfrak{G}$ and so

$$\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\epsilon) = \widehat{s}^{+}, \mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\epsilon) = \widehat{t}^{-}, \mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\epsilon) = \widehat{q}^{+}.$$

It follows from (1) that

$$\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\epsilon) = \widehat{s}^{+} \succeq \widehat{s}^{+} = \operatorname{rmin}\{\widehat{s}^{+}, \widehat{s}^{+}\} = \operatorname{rmin}\left\{\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho)\right\},$$

$$\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\epsilon) = \widehat{t}^{-} \preceq \widehat{t}^{-} = \operatorname{rmax}\{\widehat{t}^{-}, \widehat{t}^{-}\} = \operatorname{rmax}\left\{\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho)\right\},$$

$$\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\epsilon) = \widehat{q}^{+} \succeq \widehat{q}^{+} = \operatorname{rmin}\{\widehat{q}^{+}, \widehat{q}^{+}\} = \operatorname{rmin}\left\{\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho \circledast \epsilon), \mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho)\right\}.$$

Case 2: Suppose $\varrho \circledast \epsilon \notin \mathfrak{G}$ or $x \notin \mathfrak{G}$. Then

$$\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho \circledast \epsilon) = \widehat{s}^{-} \text{ or } \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho) = \widehat{s}^{-},$$

$$\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho \circledast \epsilon) = \widehat{t}^{+} \text{ or } \mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho) = \widehat{t}^{+},$$

$$\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho \circledast \epsilon) = \widehat{q}^{-} \text{ or } \mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho) = \widehat{q}^{-}.$$

It follows from (1) that

$$\operatorname{rmin}\left\{\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho \circledast \epsilon), \mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\epsilon)\right\} = \operatorname{rmin}\{\widehat{s}^{-}, \widehat{s}^{-}\} = \widehat{s}^{-},$$

$$\operatorname{rmax}\left\{\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho \circledast \epsilon), \mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\epsilon)\right\} = \operatorname{rmax}\{\widehat{t}^{-}, \widehat{t}^{-}\} = \widehat{t}^{-},$$

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$$\operatorname{rmin}\left\{\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho\circledast\epsilon),\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\epsilon)\right\}=\operatorname{rmin}\{\widehat{q}^{-},\widehat{q}^{-}\}=\widehat{q}^{-}.$$

Therefore,

$$\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\epsilon)\succeq\widehat{s}^{-}=\operatorname{rmin}\left\{\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho\circledast\epsilon),\mathfrak{T}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}\\ \widehat{s}^{-}\end{bmatrix}(\varrho)\right\},$$

$$\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\epsilon)\preceq\widehat{t}^{-}=\operatorname{rmax}\left\{\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho\circledast\epsilon),\mathfrak{I}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{t}^{-}\\ \widehat{t}^{+}\end{bmatrix}(\varrho)\right\},$$

$$\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\epsilon)\succeq\widehat{q}^{-}=\operatorname{rmin}\left\{\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho\circledast\epsilon),\mathfrak{F}_{\mathbb{N}}^{\mathfrak{G}}\begin{bmatrix}\widehat{q}^{+}\\ \widehat{q}^{-}\end{bmatrix}(\varrho)\right\}.$$

So we conclude that $\mathbb{N}^{\mathfrak{G}}\begin{bmatrix}\widehat{s}^{+}, & \widehat{t}^{+}, & \widehat{q}^{+}\\ \widehat{s}^{-}, & \widehat{t}^{-}, & \widehat{q}^{-}\end{bmatrix}$ is an IVNDS of $\mathfrak{H}$. □

A function $f:\mathfrak{H}\to\mathfrak{K}$ of $\mathfrak{H}$ and $\mathfrak{K}$ is called a *homomorphism* if $f(\varrho\circledast\epsilon)=f(\varrho)\circledast f(\epsilon)$ for all $\varrho,\epsilon\in\mathfrak{H}$. Note that if $f:\mathfrak{H}\to\mathfrak{K}$ is a homomorphism of $\mathfrak{H}$ and $\mathfrak{K}$, then $f(\iota)=\iota$. Let $f:\mathfrak{H}\to\mathfrak{K}$ be a homomorphism of $\mathfrak{H}$ and $\mathfrak{K}$. For an IVNS $\mathbb{N}=(\mathfrak{T}_{\mathbb{N}},\mathfrak{I}_{\mathbb{N}},\mathfrak{F}_{\mathbb{N}})$ in $\mathfrak{K}$, we define the IVNS $f^{-1}(\mathbb{N})=(\mathfrak{T}_{f^{-1}(\mathbb{N})},\mathfrak{I}_{f^{-1}(\mathbb{N})},\mathfrak{F}_{f^{-1}(\mathbb{N})})$ in $\mathfrak{H}$ by

$$(\forall\varrho\in\mathfrak{H})\left(\begin{array}{l}\mathfrak{T}_{f^{-1}(\mathbb{N})}(\varrho)=\mathfrak{T}_{\mathbb{N}}(f(\varrho))\\ \mathfrak{I}_{f^{-1}(\mathbb{N})}(\varrho)=\mathfrak{I}_{\mathbb{N}}(f(\varrho))\\ \mathfrak{F}_{f^{-1}(\mathbb{N})}(\varrho)=\mathfrak{F}_{\mathbb{N}}(f(\varrho))\end{array}\right).$$

Theorem 3.20. *Give us a homomorphism $f:\mathfrak{H}\to\mathfrak{K}$. If $\mathbb{N}=(\mathfrak{T}_{\mathbb{N}},\mathfrak{I}_{\mathbb{N}},\mathfrak{F}_{\mathbb{N}})$ is an IVNDS of $\mathfrak{K}$, then $f^{-1}(\mathbb{N})$ of $\mathbb{N}$ is an IVNDS of $\mathfrak{H}$.*

Proof. Let $\varrho\in\mathfrak{H}$. Then

$$\mathfrak{T}_{f^{-1}(\mathbb{N})}(\varrho)=\mathfrak{T}_{\mathbb{N}}(f(\varrho))\preceq\mathfrak{T}_{\mathbb{N}}(\iota)=\mathfrak{T}_{\mathbb{N}}(f(\iota))=\mathfrak{T}_{f^{-1}(\mathbb{N})}(\iota),$$

$$\mathfrak{I}_{f^{-1}(\mathbb{N})}(\varrho)=\mathfrak{I}_{\mathbb{N}}(f(\varrho))\succeq\mathfrak{I}_{\mathbb{N}}(\iota)=\mathfrak{I}_{\mathbb{N}}(f(\iota))=\mathfrak{I}_{f^{-1}(\mathbb{N})}(\iota),$$

$$\mathfrak{F}_{f^{-1}(\mathbb{N})}(\varrho)=\mathfrak{F}_{\mathbb{N}}(f(\varrho))\preceq\mathfrak{F}_{\mathbb{N}}(\iota)=\mathfrak{F}_{\mathbb{N}}(f(\iota))=\mathfrak{F}_{f^{-1}(\mathbb{N})}(\iota).$$

Let $\varrho,\epsilon\in\mathfrak{H}$. Then

$$\begin{array}{rcl}\operatorname{rmin}\{\mathfrak{T}_{f^{-1}(\mathbb{N})}(\varrho\circledast\epsilon),\mathfrak{T}_{f^{-1}(\mathbb{N})}(\varrho)\} &=& \operatorname{rmin}\{\mathfrak{T}_{\mathbb{N}}(f(\varrho\circledast\epsilon)),\mathfrak{T}_{\mathbb{N}}(f(\varrho))\}\\ &=& \operatorname{rmin}\{\mathfrak{T}_{\mathbb{N}}(f(\varrho)\circledast f(\epsilon)),\mathfrak{T}_{\mathbb{N}}(f(\varrho))\}\\ &\preceq& \mathfrak{T}_{\mathbb{N}}(f(\epsilon))\\ &=& \mathfrak{T}_{f^{-1}(\mathbb{N})}(\epsilon),\end{array}$$

$$\begin{array}{rcl}\operatorname{rmax}\{\mathfrak{I}_{f^{-1}(\mathbb{N})}(\varrho\circledast\epsilon),\mathfrak{I}_{f^{-1}(\mathbb{N})}(\varrho)\} &=& \operatorname{rmax}\{\mathfrak{I}_{\mathbb{N}}(f(\varrho\circledast\epsilon)),\mathfrak{I}_{\mathbb{N}}(f(\varrho))\}\\ &=& \operatorname{rmax}\{\mathfrak{I}_{\mathbb{N}}(f(\varrho)\circledast f(\epsilon)),\mathfrak{I}_{\mathbb{N}}(f(\varrho))\}\\ &\succeq& \mathfrak{I}_{\mathbb{N}}(f(\epsilon))\\ &=& \mathfrak{I}_{f^{-1}(\mathbb{N})}(\epsilon),\end{array}$$

$$\begin{array}{rcl}\operatorname{rmin}\{\mathfrak{F}_{f^{-1}(\mathbb{N})}(\varrho\circledast\epsilon),\mathfrak{F}_{f^{-1}(\mathbb{N})}(\varrho)\} &=& \operatorname{rmin}\{\mathfrak{F}_{\mathbb{N}}(f(\varrho\circledast\epsilon)),\mathfrak{F}_{\mathbb{N}}(f(\varrho))\}\\ &=& \operatorname{rmin}\{\mathfrak{F}_{\mathbb{N}}(f(\varrho)\circledast f(\epsilon)),\mathfrak{F}_{\mathbb{N}}(f(\varrho))\}\\ &\preceq& \mathfrak{F}_{\mathbb{N}}(f(\epsilon))\\ &=& \mathfrak{F}_{f^{-1}(\mathbb{N})}(\epsilon).\end{array}$$

So we conclude that $f^{-1}(\mathbb{N})$ is an IVNDS of $\mathfrak{H}$. □

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