



The Representation of Refined Neutrosophic Matrices By Refined Neutrosophic Linear Functions

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Abstract

The aim of this paper is to solve the problem of representing refined neutrosophic matrices by linear functions, where it describes the structure of refined neutrosophic linear transformations that represent refined neutrosophic matrices. On the other hand, this work introduces a novel algorithm to compute a basis of any refined neutrosophic vector space depending on the classical basis of its corresponding classical vector space.

Keywords: Refined neutrosophic space; refined neutrosophic matrix; Linear transformation.

1. Introduction

Neutrosophy is a new generalization of fuzzy ideas founded by Smarandache. Neutrosophic logic has found its way to many mathematical sections such as topology [26], statistics [28], decision making [16,27], number theory [17,19], and commutative algebra [6,9,11,13,14,15,23].

Neutrosophic matrix theory was studied by many researchers, where we find conditions for invertibility [4], operations [5], representations [2], and many applications in real-life problems [7,8,10].

In the literature, we find two generalizations of neutrosophic matrices. The first one is refined neutrosophic matrices [1], and the other is n-refined matrices [22].

The algebraic properties of refined neutrosophic matrices were studied in [1], especially the invertibility problem.

Through this work, we continue the previous efforts presented in [2], in the study of neutrosophic linear transformations and their relationship with neutrosophic matrices to the case of refined neutrosophic matrices and refined neutrosophic linear transformations.

All refined neutrosophic matrices are defined over the neutrosophic field of real numbers $R(I)$.

Preliminaries

Definition: [1]

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}$ be an $n \times m$ matrix, if $a_{ij} = x + yI_1 + zI_2 \in R_2(I)$, then it is called a refined neutrosophic matrix. Where $R_2(I)$ is a refined neutrosophic field.

Theorem: [1]

Let $X = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix, then it is invertible if and only if $A, A + C$, and $A + B + C$ are invertible. The inverse of X is $X^{-1} = A^{-1} + ((A + B + C)^{-1} - (A + C)^{-1})I_1 + ((A + C)^{-1} - A^{-1})I_2$.

Definition : [1]

We defined the determinant of a square $n \times n$ refined neutrosophic matrix as $\det X = \det A + [\det(A + B + C) - \det(A + C)]I_1 + [\det(A + C) - \det A]I_2$.

Theorem [2]:

Let V , and W be two vector spaces over the field F with $\dim(V) = n$, $\dim(W) = m$, $V(I)$, and $W(I)$ be the corresponding neutrosophic vector spaces over the corresponding neutrosophic field $F(I)$. Let $g, h: V \rightarrow W$ be two linear transformations, then there exists a neutrosophic linear transformation $f = g + hI: V(I) \rightarrow W(I)$, where f is defined as follows: $f(x + yI) = g(x) + [(g + h)(x + y) - g(x)]I$.

Definition [2] :

The neutrosophic linear transformation f defined in the previous Theorem is called a full AH- linear transformation.

Definition [2] :

Let $f = g + hI: V(I) \rightarrow W(I)$ be a full AH- linear transformation, $M = A + BI$ be an $n \times m$ neutrosophic matrix over $F(I)$, we call M the neutrosophic matrix of f if and only if $f(x + yI) = M(x + yI)$ for every $x + yI \in V(I)$.

Theorem [2] :

Let $f = g + hI: V(I) \rightarrow W(I)$ be any full AH- linear transformation, then $M = A + BI$ is the corresponding neutrosophic matrix if and only if A is the matrix of g , B is the matrix of h .

Main Discussion

Definition 1:

Let V, W be two vector spaces over R , and consider any three linear transformations $g, h, q: V \rightarrow W$. We define the corresponding full AH-refined linear transformation as follows:

$$f: V(I_1, I_2) \rightarrow W(I_1, I_2),$$

$$f(x + yI_1 + zI_2) = g(x) + I_1[(g + h + q)(x + y + z) - (g + q)(x + z)] + I_2[(g + q)(x + z) - g(x)].$$

We denote it by $f = g + hI_1 + qI_2$

Theorem 2:

Let $f = g + hI_1 + qI_2: V(I_1, I_2) \rightarrow W(I_1, I_2)$ be any full AH-refined linear transformation, then f is linear by classical meaning.

Proof:

$\forall X = x + yI_1 + zI_2, Y = x_1 + y_1I_1 + z_1I_2 \in V(I_1, I_2)$ we have:

$$f(X + Y) = f[(x + x_1) + I_1(y + y_1) + I_2(z + z_1)] = g(x + x_1) + I_1[(g + h + q)(x + x_1 + y + y_1 + z + z_1) - (g + q)(x + x_1 + z + z_1)] + I_2[(g + q)(x + x_1 + z + z_1) - g(x + x_1)] = g(x) + g(x_1) +$$

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$$\begin{aligned}
& I_1[(g+h+q)(x+y+z) + (g+h+q)(x_1+y_1+z_1) - (g+q)(x+z) - (g+q)(x_1+z_1)] + \\
& I_2[(g+q)(x+z) + (g+q)(x_1+z_1) - g(x) - g(x_1)] = [g(x) + I_1[(g+h+q)(x+y+z) - \\
& (g+q)(x+z)] + I_2[(g+q)(x+z) - g(x)] + [g(x_1) + I_1[(g+h+q)(x_1+y_1+z_1) - (g+q)(x_1+z_1)] \\
& + I_2[(g+q)(x_1+z_1) - g(x_1)], \\
& = f(X) + f(Y)
\end{aligned}$$

$\forall r = r_0 + r_1I_1 + r_2I_2 \in V(I_1, I_2)$, we have:

$$\begin{aligned}
r \cdot X &= r_0x + I_1[(r_0 + r_1 + r_2)(x + y + z) - (r_0 + r_2)(x + z)] + I_2[(r_0 + r_2)(x + z) - r_0x] \\
f(r \cdot X) &= g(r_0x) + I_1[(g+h+q)[r_0 + r_1 + r_2](x + y + z) - (g+q)[(r_0 + r_2)(x + z)] + I_2[(g+q)[(r_0 + r_2)(x + z)] - g(r_0x)],
\end{aligned}$$

Now we compute $r \cdot f(X)$:

$$\begin{aligned}
r \cdot f(X) &= (r_0 + r_1I_1 + r_2I_2) \cdot [g(x) + I_1[(g+h+q)(x+y+z) - (g+q)(x+z)] + I_2[(g+q)(x+z) - g(x)], \\
&= r_0g(x) + I_1[(r_0 + r_1 + r_2)(g+h+q)(x+y+z) - (r_0 + r_2)(g+q)(x+z)] + I_2[(r_0 + r_2) \cdot (g+q)(x+z) - r_0g(x)] = f(rX).
\end{aligned}$$

Definition 3:

Let $M = A + BI_1 + CI_2$ be a matrix over $R(I_1, I_2)$. We say that M is the matrix of $f = g + hI_1 + qI_2$ iff $f(x + yI_1 + zI_2) = M \cdot (x + yI_1 + zI_2)$.

Theorem 4:

Let $f = g + hI_1 + qI_2: V(I_1, I_2) \rightarrow W(I_1, I_2)$ be any full AH-refined linear transformation, then $M = A + BI_1 + CI_2$ is the matrix of f iff A is the matrix of g , B is the matrix of h , C is the matrix of q .

Proof:

Firstly, we suppose that A is the matrix of g , B is the matrix of h , and C is the matrix of q , where $g, h, q: V \rightarrow W$ are classical linear transformations.

Consider the refined neutrosophic matrices $M = A + BI_1 + CI_2$, we have for each $x + yI_1 + zI_2 \in V(I_1, I_2)$:

$$\begin{aligned}
M(x + yI_1 + zI_2) &= (A + BI_1 + CI_2)(x + yI_1 + zI_2) = A \cdot x + I_1[(A + B + C)(x + y + z) - (A + C)(x + z)] + I_2[(A + C)(x + z) - A \cdot x], \\
&= g(x) + I_1[(g+h+q)(x+y+z) - (g+q)(x+z)] + I_2[(g+q)(x+z) - g(x)] = f(x + yI_1 + zI_2), \\
&\text{hence } M \text{ is the matrix of } f.
\end{aligned}$$

Conversely, we assume that $M = A + BI_1 + CI_2$ is the matrix of f , we must prove that A is the matrix of g , B is the matrix of h , and C is the matrix of q .

$$\text{We have } f(x + yI_1 + zI_2) = M(x + yI_1 + zI_2), \text{ thus: } g(x) + I_1[(g+h+q)(x+y+z) - (g+q)(x+z)] + I_2[(g+q)(x+z) - g(x)] = Ax + I_1[(A+B+C)(x+y+z) - (A+C)(x+z)] + I_2[(A+C)(x+z) - Ax]$$

$$\text{So that, } g(x) = Ax, (A+C)(x+z) = (g+q)(x+z), (A+B+C)(x+y+z) = (g+h+q)(x+y+z)$$

This implies that A is the matrix of g , $A+C$ is the matrix of $g+q$, thus C is the matrix of q and $A+B+C$ is the matrix of $g+h+q$, thus B is the matrix of h .

In order to prove that every refined neutrosophic matrix can be represented by a unique AH-refined linear transformation, we introduce the following algorithm to derive a basis of $V(I_1, I_2)$ from any classical basis of V .

Theorem 5: Let $S = \{V_1, V_2, \dots, V_n\}$ be a basis of V over R , then:

$$L_{i,j,k} = \{V_i + (V_j - V_k)I_1 + (V_k - V_i)I_2\}$$

Is a basis of $V(I_1, I_2)$ over $R(I_1, I_2)$.

Proof: Firstly, we prove that $L_{i,j,k}$ generates $V(I_1, I_2)$ over $R(I_1, I_2)$.

$$\forall r = x + r_1 I_1 + r_2 I_2 \in V(I_1, I_2); x, r_1, r_2 \in V$$

We can write $r = x + (y - z)I_1 + (z - x)I_2$; $z - x = r_1, y - z = r_2 \in V$ and $y, z \in V$

By the assumption, we have S as a basis of V over R , so:

$$x = \sum_{i=1}^n a_i V_i, y = \sum_{j=1}^n b_j V_j, z = \sum_{k=1}^n c_k V_k; a_i, b_j, c_k \in R$$

So that:

$$r = x + (y - z)I_1 + (z - x)I_2 = \sum_{i=1}^n a_i V_i + I_1 \left[\sum_{j=1}^n (b_j - c_j) V_j \right] + I_2 \left[\sum_{k=1}^n (c_k - a_k) V_k \right]$$

Now, we put $m_{i,j,k} = a_i + (b_j - c_j)I_1 + (c_k - a_k)I_2 \in R(I_1, I_2)$

We must prove that $\sum_{i,j,k=1}^n m_{i,j,k} L_{i,j,k} = r$

$$\sum_{i,j,k=1}^n m_{i,j,k} L_{i,j,k} = \sum_{i,j,k=1}^n [a_i V_i + (b_j V_j - c_k V_k)I_1 + (c_k V_k - a_i V_i)I_2] =$$

$$\sum_{i=1}^n a_i V_i + (\sum_{j=1}^n b_j V_j - \sum_{k=1}^n c_k V_k)I_1 + (\sum_{k=1}^n c_k V_k - \sum_{i=1}^n a_i V_i)I_2 = x + (y - z)I_1 + (z - x)I_2 = r.$$

Now we must prove that $L_{i,j,k}$ is linearly independent.

Suppose that $\sum_{i,j,k=1}^n m_{i,j,k} L_{i,j,k} = 0$

Hence,

$$\sum_{i=1}^n a_i V_i = 0, \sum_{j=1}^n b_j V_j - \sum_{k=1}^n c_k V_k = 0, \sum_{k=1}^n c_k V_k - \sum_{i=1}^n a_i V_i = 0$$

This implies that $a_i = 0, \sum_{k=1}^n c_k V_k = \sum_{j=1}^n b_j V_j = 0$, so that $c_k = b_j = 0$ for all i, j, k .

Theorem 6: Let $f: V(I_1, I_2) \rightarrow W(I_1, I_2)$ be a linear function, then f must be a full AH-refined linear transformation.

Proof: we must find three classical linear transformations $g, h, q: V \rightarrow W$ with the property $f = g + hI_1 + qI_2$.

It is known that the direct image of a basis like $L_{i,j,k}$ with a linear transformation must be a basis again, thus we get:

$$f(L_{i,j,k}) = \{w_i + (w_j - w_k)I_1 + (w_k - w_i)I_2\}$$

Is a basis of $W(I_1, I_2)$, where $w_i, w_j, w_k \in W$

Take the following functions:

$$g: V \rightarrow W; g(V_i) = w_i$$

$$h_1: V \rightarrow W; h_1(V_j) = w_j$$

$$q_1: V \rightarrow W; q_1(V_k) = w_k$$

$$f(L_{i,j,k}) = g(V_i) + (h_1(V_j) - q_1(V_k))I_1 + (q_1(V_k) - g(V_i))I_2$$

By taking $h = h_1 - g - q_1, q = q_1 - g$

We get:

$$f(L_{i,j,k}) = g(V_i) + I_1[(g + h + q)(V_j) - (q + g)(V_k)] + I_2[(q + g)(V_k) - g(V_i)], \text{ thus } f = g + hI_1 + qI_2.$$

It is clear that g, h_1 , and q_1 are linear hence, h , and g are linear too.

So, f is a full AH-linear transformation.

From theorem 6, we get the following result.

Every linear transformation $f: V(I_1, I_2) \rightarrow W(I_1, I_2)$ can be represented by a unique refined neutrosophic matrix, which is exactly equal to the corresponding matrix of the corresponding full AH-refined linear transformation.

Example 7:

Now, we give an example to clarify the structure of the refined neutrosophic basis of a refined neutrosophic vector space based on Theorem 5.

It is well known that the R^2 is a vector space over the real field R . Consider the corresponding refined neutrosophic vector space

$$R^2(I_1, I_2) = \{a + bI_1 + cI_2; a, b, c \in R^2\} = \{(x, y) + I_1(z, t) + I_2(m, n); m, n, z, t, x, y \in R\}.$$

The basis of R^2 is the set $\{e_1 = (1,0), e_2 = (0,1)\}$. According to Theorem 5, the related refined neutrosophic basis of $R^2(I_1, I_2)$ is:

$$v_1 = e_1 + I_1(e_1 - e_1) + I_2(e_1 - e_1), v_2 = e_1 + I_1(e_2 - e_1) + I_2(e_1 - e_1)$$

$$v_3 = e_1 + I_1(e_2 - e_2) + I_2(e_2 - e_1), v_4 = e_1 + I_1(e_1 - e_2) + I_2(e_2 - e_1),$$

$$v_5 = e_2 + I_1(e_1 - e_1) + I_2(e_1 - e_2), v_6 = e_2 + I_1(e_1 - e_2) + I_2(e_2 - e_2),$$

$$v_7 = e_2 + I_1(e_2 - e_1) + I_2(e_1 - e_2), v_8 = e_2 + I_1(e_2 - e_2) + I_2(e_2 - e_2).$$

Example 8:

Now, we illustrate an example to clarify the representation of refined linear transformations by refined neutrosophic matrices.

$$\text{Let } f: R^2(I_1, I_2) \rightarrow R(I_1, I_2); f[(x, y) + I_1(z, t) + I_2(m, n)] = f[(x + zI_1 + mI_2, y + tI_1 + nI_2)] =$$

$$2(x + zI_1 + mI_2) - (y + tI_1 + nI_2) = (2x - y) + (2z - t)I_1 + (2m - n)I_2.$$

be a refined linear transformation, we have to write f as an AH-refined linear transformation.

By Theorem 6, we know that f must be equal to $g + hI_1 + qI_2$, where $g, h, q: R^2 \rightarrow R$.

Firstly, we have $g: R^2 \rightarrow R; g(x, y) = 2x - y$, and $(g + q)[(x, y) + (m, n)] - g(x, y) = 2m - n$,

This means that: $g(x + m, y + n) + q(x + m, y + n) - (2x - y) = 2m - n$, so that

$$2(x + m) - (y + n) + q(x + m, y + n) - (2x - y) = 2m - n, \text{ hence}$$

$$2m - n + q(x + m, y + n) = 2m - n, \text{ thus } q(x + m, y + n) = 0 \text{ for all } x, y, m, n.$$

This implies that $q(x, y) = 0$ is a zero map.

For the computing of h , we regard that: $(g + h + q)[(x, y) + (z, t) + (m, n)] - (g + q)[(x, y) + (m, n)] = 2z - t$.

So that: $(g + q + h)[(x + z + m, y + t + n)] - (g + q)[(x + m, y + n)] = 2z - t$. By remarking that

$g + q = g$, we get:

$$(g + h)[(x + z + m, y + t + n)] - g[(x + m, y + n)] = 2z - t, \text{ hence}$$

$$g(x + z + m, y + t + n) + h(x + z + m, y + t + n) - [2(x + m) - (y + n)] = 2z - t,$$

This means that: $2(x + z + m) - (y + t + n) + h(x + z + m, y + t + n) - 2x - 2m + y + n = 2z - t$,

Thus $h(x + z + m, y + t + n) = 0$ for all x, y, z, m, n, t , i.e. h is a zero map.

By the previous discussion, we get that $f = g + 0.I_1 + 0.I_2$.

The matrix of g is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, and the matrix of q and h is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, thus the refined neutrosophic matrix of f is

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} + I_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + I_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = M.$$

Now, we check that M is refined neutrosophic matrix of f by the following computing:

$$M. [(x, y) + I_1(z, t) + I_2(m, n)] = M. [(x + zI_1 + mI_2, y + tI_1 + nI_2)] = 2(x + zI_1 + mI_2) - (y + tI_1 + nI_2) = f(x + zI_1 + mI_2, y + tI_1 + nI_2).$$

Example 9:

In example 8, we showed how a refined neutrosophic linear transformation can be turned into a refined neutrosophic matrix, now we do the converse. We illustrate an example to show how a refined neutrosophic matrix can be turned into a refined neutrosophic linear transformation.

Let $M = \begin{pmatrix} 1 & I_2 \\ 3 - I_1 + 2I_2 & 2I_1 \end{pmatrix}$ be a refined neutrosophic matrix. M can be written in the following form:

$$M = A + BI_1 + CI_2 = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix} I_1 + \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} I_2.$$

M can be represented by a unique refined neutrosophic linear transformation

$$f: R^2(I_1, I_2) \rightarrow R^2(I_1, I_2); f[(x + zI_1 + mI_2, y + tI_1 + nI_2)] = M. [(x + zI_1 + mI_2, y + tI_1 + nI_2)] = (x + I_1(z + t) + I_2(m + y + n), 3x + (4z - x + 2y + 2t + 2n - m)I_1 + (2x + 5m)I_2).$$

4. Conclusion

In this paper, we have presented a novel solution to the problem of representing refined neutrosophic matrices with refined neutrosophic linear transformations, where we proved that every refined neutrosophic matrix can be represented by a unique refined linear transformation between two refined neutrosophic vector spaces.

Also, we have introduced an algorithm to obtain a basis for any refined neutrosophic vector space by depending on any basis of its corresponding classical vector space V . Also, we have illustrated many interesting examples to clarify the validity of this work.

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