



On Some Novel Results About Neutrosophic Square Complex Matrices

Ahmad Salama¹, Rasha Dalla², Malath Al Aswad³, Rozina Ali⁴

¹Department of Mathematics, port said, Egypt

²Departement of Mathematics, Albaath University, Homs, Syria

³PHD of Mathematics, GaziAntep, Turkey

⁴Department Of Mathematics, Cairo University, Egypt

Email: drsalama44@gmail.com; rasha.dallah20@gmail.com; Malaz.Aswad@yahoo.com; rozyyy123n@gmail.com

Abstract

The objective of this paper is to study algebraic properties of complex neutrosophic matrices, where a necessary and sufficient condition for the invertibility of a complex square neutrosophic matrix is presented by defining the complex neutrosophic determinant. On the other hand, this work introduces the concept of neutrosophic characteristic polynomial and neutrosophic Cayley-Hamilton theorem for the complex case.

Keywords: Neutrosophic complex matrix; neutrosophic real number; neutrosophic determinant; neutrosophic inverse

1. Introduction

2. Preliminaries

Definition 2.1 [28]: Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n .

Definition 2.2 [29]: Let $w_1 = a_1 + b_1I$, $w_2 = a_2 + b_2I$ Then we have:

$$\frac{w_1}{w_2} = \frac{a_1}{a_2} + \frac{a_1b_2 - a_2b_1}{a_2(a_2 + b_2)}$$

Definition 2.3 [10]: Let K be a field, the neutrosophic field generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 2.4 (Neutrosophic complex matrix) [16]. Let $M_{m \times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$ is the neutrosophic complex field. We call to be the neutrosophic complex matrix.

Definition 2.4 : Let $M_{m \times n}$ is a neutrosophic complex matrix. We call to be the neutrosophic square complex matrix if $m = n$.

Now a neutrosophic n square complex matrix is defined by form $M = A + BI$ where A and B are two n squares complex matrices.

3. Main discussion

Definition 3.1:

Let $M = A + BI$ be a neutrosophic n square complex matrix. The determinant of M is defined as

$$\det M = \det A + I[\det(A + B) - \det A].$$

Definition 3.2:

Let $M = A + BI$ a neutrosophic square $n \times n$ matrix, where A, B are two squares $n \times n$ complex matrices, then M is invertible if and only if A and $A + B$ are invertible matrices and

$$M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}].$$

Theorem 3.3:

M is invertible matrix if and only if $\det M$ is invertible.

Proof:

From **Definition 3.2** we find that M is invertible matrix if and only if $A + B, A$ are two invertible matrices, hence $\det[A + B] \neq 0, \det A \neq 0$ which means

$$\det M = \det A + I[\det(A + B) - \det A] \text{ is invertible.}$$

Example 3.4:

Consider the following neutrosophic complex matrix

$$M = A + BI = \begin{pmatrix} i & -1 \\ 0 & 1-i \end{pmatrix} + I \begin{pmatrix} 0 & i \\ -i & -1+i \end{pmatrix}.$$

(a) $\det A = 1 + i, A + B = \begin{pmatrix} i & -1+i \\ -i & 0 \end{pmatrix}, \det(A + B) = -1 - i, \det M = 1 + i + (-2 - 2i)I \neq 0$, hence M is invertible.

(b) We have:

$$A^{-1} = \begin{pmatrix} -i & \frac{1}{2} - \frac{i}{2} \\ 0 & \frac{1}{2} + \frac{i}{2} \end{pmatrix}, (A + B)^{-1} = \begin{pmatrix} 0 & i \\ -\frac{1}{2} - \frac{i}{2} & -\frac{1}{2} - \frac{i}{2} \end{pmatrix},$$

$$\text{thus } M^{-1} = (A^{-1}) + I[(A + B)^{-1} - A^{-1}] = \begin{pmatrix} -i & \frac{1}{2} - \frac{i}{2} \\ 0 & \frac{1}{2} + \frac{i}{2} \end{pmatrix} + I \begin{pmatrix} i & -\frac{1}{2} + \frac{3i}{2} \\ -\frac{1}{2} - \frac{i}{2} & -1 - i \end{pmatrix}.$$

(c) We can compute $MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = U_{2 \times 2}$.

Definition 3.5:

Let $M = A + BI$ be a neutrosophic n square complex matrix, where A and B are two n square complex matrices, then

$$M^T = A^T + I[(A + B)^T - A^T].$$

Definition 3.6:

Let $M = A + BI$ be a neutrosophic n square complex matrix, where A and B are two n square complex matrices, then

$$M^r = A^r + I[(A + B)^r - A^r].$$

Remark 3.7:

Let $M = A + BI$ and $N = C + DI$ be two neutrosophic n square complex matrices, then

$$(3.7.1) \det(M \cdot N) = \det M \cdot \det N.$$

$$(3.7.2) \det(M^{-1}) = (\det M)^{-1}.$$

$$(3.7.3) \det M = \det M^T.$$

Remark: The result in the section (c) can be generalized easily to the following fact:

$$\det M = \det A \text{ if and only if } \det A = \det(A + B).$$

Definition 3.8:

Let $M = A + BI$ be a neutrosophic n square complex matrix, where A and B are two n square complex matrices, And $Z = X + YI$. We define the neutrosophic characteristic polynomial by the form:

$$\varphi(Z) = \det[ZU_{n \times n} - M] = \det[ZU_{n \times n} - (A + BI)] = \det[(ZU_{n \times n} - A) + (-B)I]$$

$$\varphi(Z) = \det(ZU_{n \times n} - A) + I[\det(ZU_{n \times n} - (A + B)) - \det(ZU_{n \times n} - A)]$$

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)].$$

Where:

$$\alpha(Z) = \det(ZU_{n \times n} - A), \beta(Z) = \det(ZU_{n \times n} - (A + B))$$

Example 3.9:

Consider the following neutrosophic complex matrix

$$M = A + BI = \begin{pmatrix} i & -1 \\ 0 & 1-i \end{pmatrix} + I \begin{pmatrix} 0 & i \\ -i & -1+i \end{pmatrix}, A + B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}. \text{ Then.}$$

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)]$$

$$\alpha(Z) = \det(ZU_{2 \times 2} - A) = \begin{vmatrix} Z-i & -1 \\ 0 & Z-(1-i) \end{vmatrix}$$

$$\alpha(Z) = Z^2 - (1-i)Z - iZ + 1 + i = Z^2 - Z + (1+i)$$

$$\beta(Z) = Z^2 - iZ - i - 1$$

Then,

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)] = Z^2 - Z + (1+i) + I[(1-i)Z - 2 - 2i]$$

Theorem 3.10:

A neutrosophic characteristic polynomial of neutrosophic square complex matrix is equal a neutrosophic characteristic polynomial of its transpose.

Proof:

Let $M = A + BI$ be a neutrosophic n square complex matrix, where A and B are two n square complex matrices.

Let $\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)]$ a neutrosophic characteristic polynomial for M and M^T is transpose for M .

Let $\psi(Z)$ be the neutrosophic characteristic polynomial of M^T , then

$$\varphi(Z) = \det[ZU_{n \times n} - M]$$

$$\varphi(Z) = \det(ZU_{n \times n} - A) + I[\det(ZU_{n \times n} - (A + B)) - \det(ZU_{n \times n} - A)]$$

Now we have.

$$\psi(Z) = \det[ZU_{n \times n} - M]^T = \det[(ZU_{n \times n} - A) + (-B)I]^T$$

$$\psi(Z) = \det[(ZU_{n \times n} - A)^T + I[(ZU_{n \times n} - (A + B))^T - (ZU_{n \times n} - A)^T]]$$

$$\psi(Z) = \det(ZU_{n \times n} - A)^T + I \left[\det(ZU_{n \times n} - (A + B))^T - \det(ZU_{n \times n} - A)^T \right]$$

By **Remark 3.7** we have.

$$[\det(ZU_{n \times n} - A)]^T = \det(ZU_{n \times n} - A)$$

$$\det(ZU_{n \times n} - (A + B))^T = \det(ZU_{n \times n} - (A + B))$$

Then.

$$\psi(Z) = \det(ZU_{n \times n} - A) + I [\det(ZU_{n \times n} - (A + B)) - \det(ZU_{n \times n} - A)]$$

Then.

$$\phi(Z) = \psi(Z)$$

Example 3.11:

Consider the neutrosophic matrix defined in Example 3.9, we have:

$$\phi(Z) = Z^2 - Z + (1 + i) + I[(1 - i)Z - 2 - 2i]$$

Now.

$$A^T = \begin{pmatrix} i & 0 \\ -1 & 1 - i \end{pmatrix}, B^T = \begin{pmatrix} 0 & -i \\ i & -1 + i \end{pmatrix} \text{ Then.}$$

$$\psi(Z) = \alpha^*(Z) + I [\beta^*(Z) - \alpha^*(Z)]$$

$$\alpha^*(Z) = \det(ZU_{2 \times 2} - A^T) = \begin{vmatrix} Z - i & 0 \\ -1 & Z - (1 - i) \end{vmatrix}$$

$$\alpha^*(Z) = Z^2 - Z + (1 + i)$$

$$\beta^*(Z) = \det(ZU_{2 \times 2} - (A + B)^T) = \begin{vmatrix} Z & i \\ -i & Z - (-1 + i) \end{vmatrix}$$

$$\beta^*(Z) = Z^2 - iZ - i - 1$$

Then.

$$\phi(Z) = \psi(Z) = Z^2 - Z + (1 + i) + I[(1 - i)Z - 2 - 2i]$$

Theorem 3.12: (Neutrosophic complex Cayley-Hamilton): Any neutrosophic square complex matrix is a root of its a neutrosophic characteristic polynomial.

Example 3.13:

Consider the neutrosophic matrix defined in Example 3.9, we have:

$$\phi(Z) = Z^2 - Z + (1 + i) + I[(1 - i)Z - 2 - 2i]$$

Now we find $\phi(M)$.

$$\phi(M) = M^2 - M + (1 + i)U_{2 \times 2} + (1 - i)MI + (-2 - 2i)U_{2 \times 2}I$$

$$M^2 = A^2 + I[(A + B)^2 - A^2] = \begin{pmatrix} -1 & -1 \\ 0 & -2i \end{pmatrix} + I \begin{pmatrix} i + 1 & -i \\ 1 & 1 + 3i \end{pmatrix}$$

$$\begin{aligned} \phi(M) = & \begin{pmatrix} -1 & -1 \\ 0 & -2i \end{pmatrix} + I \begin{pmatrix} i + 1 & -i \\ 1 & 1 + 3i \end{pmatrix} + \begin{pmatrix} -i & 1 \\ 0 & -1 + i \end{pmatrix} + I \begin{pmatrix} 0 & -i \\ i & 1 - i \end{pmatrix} + \begin{pmatrix} i + 1 & 0 \\ 0 & i + 1 \end{pmatrix} \\ & + I \begin{pmatrix} 1 + i & 2i \\ -1 - i & 0 \end{pmatrix} + I \begin{pmatrix} -2 - 2i & 0 \\ 0 & -2 - 2i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + I \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$\phi(M) = 0$$

4. Refined neutrosophic matrix

Definition 4.1: The structure of refined neutrosophic numbers is taken as $a + bI_1 + cI_2$ instead of (a, bI_1, cI_2) .

Definition 4.2: $I_1^2 = I_1$, $I_2^2 = I_2$, $I_1 \cdot I_2 = I_2 \cdot I_1 = I_1$

Definition 4.3: (Refined neutrosophic complex matrix).

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$ be an $m \times n$ matrix: if $a_{ij} = a + bI_1 + cI_2 \in C_2(I)$, then it is called a refined neutrosophic complex matrix, where $C_2(I)$ is an refined neutrosophic complex field.

Example 4.4: Let $A = \begin{pmatrix} 1 + i + (-1 - i)I_1 & (1 + i)I_1 - iI_2 \\ 3 - iI_1 & (2 - i)I_2 \end{pmatrix}$ as a 2×2 refined neutrosophic complex matrix.

Theorem 4.5: Let $M = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic complex matrix; then it is invertible if only of A , $A + C$ and $A + B + C$ are invertible. The inverse of M is

$$M^{-1} = A^{-1} + ((A + B + C)^{-1} - (A + C)^{-1})I_1 + ((A + C)^{-1} - A^{-1})I_2$$

Proof: The proof holds as a special case of invertible elements in refined neutrosophic rings [30].

Definition 4.6:

Let $M = A + BI_1 + CI_2$ be a refined neutrosophic n square complex matrix, where A, B and C are n square complex matrices, then.

$$M^T = A^T + [(A + B + C)^T - (A + C)^T]I_1 + [(A + C)^T - A^T]I_2.$$

Definition 4.7:

Let $M = A + BI_1 + CI_2$ be a refined neutrosophic n square complex matrix, where A, B and C are n square complex matrices, then

$$\det M = \det(A + BI_1 + CI_2) = \det A + [\det(A + B + C) - \det(A + C)]I_1 + [\det(A + C) - \det A]I_2.$$

Remark 4.8:

(a). If A is an $m \times n$ matrix, then it can be represented as an element of the refined neutrosophic ring of matrices such as the following: $M = A + BI_1 + CI_2$, where A, B and C are complex matrices with elements from ring C and from size $m \times n$.

$$\text{For example, } M = \begin{pmatrix} (-1 - i) + I_1 + 3iI_2 & 1 - (1 - i)I_1 - I_2 \\ 3 + (1 + i)I_2 & 1 + (2 + i)I_1 \end{pmatrix} = \begin{pmatrix} -1 - i & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 + i \\ 0 & 2 + i \end{pmatrix} I_1 + \begin{pmatrix} 3i & -1 \\ 1 + i & 0 \end{pmatrix} I_2.$$

(b). Multiplication can be defined by using the same representation as a special case multiplication on refined neutrosophic rings as follows:

$$(A + BI_1 + CI_2)(X + YI_1 + ZI_2) = (AX) + (AY + BX + BY + BZ + CY)I_1 + (AZ + CZ + CX)I_2$$

Theorem 4.9:

Let $M = A + BI_1 + CI_2$ be a neutrosophic n square complex matrix, where A, B and C are two n square complex matrices, And $Z = X + YI_1 + TI_2$. We define the neutrosophic characteristic polynomial by form:

$$\varphi(z) = \det[ZU_{n \times n} - M] = \det[ZU_{n \times n} - (A + BI_1 + CI_2)] = \det[(ZU_{n \times n} - A) + (-B)I_1 + (-C)I_2]$$

$$\varphi(z) = \det(ZU_{n \times n} - A) + [\det(ZU_{n \times n} - (A + B + C)) - \det(ZU_{n \times n} - (A + C))]I_1 + [\det(ZU_{n \times n} - (A + C)) - \det(ZU_{n \times n} - A)]I_2$$

$$\varphi(z) = \alpha(Z) + [\beta(Z) - \gamma(Z)]I_1 + [\gamma(Z) - \alpha(Z)]I_2.$$

Where:

$$\alpha(Z) = \det(ZU_{n \times n} - A), \beta(Z) = \det(ZU_{n \times n} - (A + B + C)), \gamma(Z) = \det(ZU_{n \times n} - (A + C))$$

Example 4.10:

Consider the following neutrosophic matrix

$$M = A + BI_1 + CI_2. \text{ Where } A = \begin{pmatrix} 2+i & 1 \\ -i & -1+i \end{pmatrix}, B = \begin{pmatrix} 1 & -i \\ 0 & 2i \end{pmatrix}, C = \begin{pmatrix} 1-i & -1 \\ i & 0 \end{pmatrix}$$

$$A + B + C = \begin{pmatrix} 4 & -i \\ 0 & -1+3i \end{pmatrix}, A + C = \begin{pmatrix} 3 & 0 \\ 0 & -1+i \end{pmatrix}.$$

Then,

$$\varphi(Z) = \alpha(Z) + [\beta(Z) - \gamma(Z)]I_1 + [\gamma(Z) - \alpha(Z)]I_2$$

$$\alpha(Z) = \det(ZU_{n \times n} - A) = \begin{vmatrix} Z - (2+i) & -1 \\ i & Z + (1-3i) \end{vmatrix}$$

$$\alpha(Z) = Z^2 - (1+2i)Z - 3 + 2i$$

$$\beta(Z) = \det(ZU_{n \times n} - (A + B + C)) = \begin{vmatrix} Z - 4 & i \\ 0 & Z + (1-3i) \end{vmatrix}$$

$$\beta(Z) = Z^2 - (1+4i)Z - (5-5i)$$

$$\gamma(Z) = \det(ZU_{n \times n} - (A + C)) = \begin{vmatrix} Z - 3 & 0 \\ 0 & Z + (1-i) \end{vmatrix}$$

$$\gamma(Z) = Z^2 - (2+i)Z + (-3+3i)$$

Then,

$$\varphi(Z) = \alpha(Z) + [\beta(Z) - \gamma(Z)]I_1 + [\gamma(Z) - \alpha(Z)]I_2$$

$$\varphi(Z) = Z^2 - (1+2i)Z - 3 + 2i + [(1-3i)Z + (-2+2i)]I_1 + [(-1+i)Z + i]I_2$$

Conclusion

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