Conjectures for Invertible Diophantine Equations Of 3-Cyclic and 4-Cyclic Refined Integers

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Abstract

The objective of this paper is to suggest two new conjectures concerning the invertible elements in 3-cyclic, and 4-cyclic refined neutrosophic rings of integers, where the invertibility condition shows that the solution of some Diophantine equations may determine the classification of the group of units of these algebraic rings.

Keywords: n-cyclic refined neutrosophic ring; n-cyclic refined integer; group of units; Turiyam rings

Introduction

In the theory of rings, it was very useful to build a bigger ring contains another one, which it is called a ring extension [1-2, 11-15]. Recently, it is studied for dealing the four way data analysis using Turiyam ring [16] and its extensive properties [17-19] for solving various decision making problems. However basic proof of some conjecture and equations are required to understand the usability of mathematical algebra [20]. To achieve this goal, the current paper focused on invertibility conditions for some Diophantine equations and its extension to Turiyam rings.

In [3], Abobala has presented a novel way to build a ring extension by using a neutrosophic idea of splitting and indeterminate into some other elements. In our work, we will concentrate the algebraic structure of this extension not the logical one.

If R is a ring, the corresponding n-cyclic refined neutrosophic ring is considered as

$R_n(I) = \{a_0 + a_1 I_1 + \cdots + a_n I_n \,;\ a_i \in R\}$. It is clear that $R_n(I)$ contains R.

The n-cyclic refined ring is different from the direct product of the ring R with itself n+1 times, that is because the multiplication is defined to be distributive on addition with an additional property which is similar to structure of cyclic group of integers, it can be described as follows:

$I_i I_j = I_{(i+j \bmod n)}$.

In [4], Sadiq has studied the group of units structure of 2-cyclic refined neutrosophic rings, and provided some interesting open problems. This motivated us to study the invertibility conditions in 3-cyclic and 4-cyclic refined rings of integers.

For more information about neutrosophic algebra [5-10] or four way Turiyam ring author can refer to [16-19] for basic understanding of mathematical proof [20].

For definitions and properties of n-cyclic refined neutrosophic rings see [3].

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Main Discussion

Definition:

Let $Z_3(I) = \{a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3;\ a_i \in Z\}$ be the 3-cyclic refined neutrosophic ring of integers, then $A = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3$ is called invertible (unit) if and only if there exists $B = b_0 + b_1 I_1 + b_2 I_2 + b_3 I_3$ such that AB=1.

Theorem:

Let $Z_3(I) = \{a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3;\ a_i \in Z\}$ be the 3-cyclic refined neutrosophic ring of integers, then $A = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3$ is invertible if and only if one of the following two Diophantine equations are true.

$(a_0 + a_3)^3 + (a_1)^3 + (a_2)^3 - 3a_1 a_2 (a_0 + a_3) = 1\ or - 1$. With $a_0 \in \{1, -1\}$.

Proof:

The study of the invertibility of A means that we are forced to study the 3-cyclic refined equation

AX=1. Where $= x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3$.

Now, we compute the product AX as follows:

$AX = a_0 x_0 + I_1[x_1(a_0 + a_3) + x_2 a_2 + x_3 a_1 + x_0 a_1] + I_2[x_1 a_1 + x_2(a_0 + a_3) + x_3 a_2 + x_0 a_2] + I_3[x_1 a_2 + x_2 a_1 + x_3(a_0 + a_3) + x_0 a_3] = 1$. This implies that:

$a_0 x_0 = 1, thus\ x_0, a_0 \in \{1, -1\}$. Also,

$x_1(a_0 + a_3) + x_2 a_2 + x_3 a_1 = -x_0 a_1, x_1 a_1 + x_2(a_0 + a_3) + x_3 a_2 = -x_0 a_2, x_1 a_2 + x_2 a_1 + x_3(a_0 + a_3) = -x_0 a_3$.

So that, we got three linear equations with three variables x_1, x_2, x_3 . This system is solvable uniquely if and only if the determinant of the coefficients matrix is invertible, i.e. $\begin{vmatrix} a_0 + a_3 & a_2 & a_1 \\ a_1 & a_0 + a_3 & a_2 \\ a_2 & a_1 & a_0 + a_3 \end{vmatrix} = 1\ or - 1$.

By easy computing, we get the equations

$(a_0 + a_3)^3 + (a_1)^3 + (a_2)^3 - 3a_1 a_2 (a_0 + a_3) = 1\ or - 1$. With $a_0 \in \{1, -1\}$.

Conjecture 1:

The previous two equations has finite number of integer solutions (We mean that the order of the group of units of the 3-cyclic refined ring of integers is finite). Also, this number must be divisible by 6. (That is because the group of units of the integer ring Z has order 2, and the index of the extension is 3).

Definition:

Let $Z_4(I) = \{a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4;\ a_i \in Z\}$ be the 4-cyclic refined neutrosophic ring of integers, then $A = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4$ is called invertible (unit) if and only if there exists $B = b_0 + b_1 I_1 + b_2 I_2 + b_3 I_3 + b_4 I_4$ such that AB=1.

Theorem:

Let $Z_4(I) = \{a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4;\ a_i \in Z\}$ be the 4-cyclic refined neutrosophic ring of integers, then $A = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4$ is invertible if and only if one of the following two Diophantine equations are true.

$(a_0 + a_4)^4 - 4a_1 a_3 (a_0 + a_4)^2 + 4a_2 {a_3}^2 (a_0 + a_4) - 4a_1 a_3 {a_2}^2 - {a_1}^4 - {a_3}^4 + {a_2}^4 + 4a_2 {a_1}^2 (a_0 + a_4) + 2{a_1}^2 {a_3}^2 - 2{a_2}^2 (a_0 + a_4)^2 = 1\ or - 1, with\ a_0 \in \{1, -1\}$.

Proof:

The study of the invertibility of A means that we are forced to study the 4-cyclic refined equation

AX=1. Where $= x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3 + x_4 I_4$.

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Now, we compute the product AX as follows:

$AX = a_0x_0 + I_1[x_1(a_0 + a_4) + x_2a_3 + x_3a_2 + x_0a_1 + x_4a_1] + I_2[x_1a_1 + x_2(a_0 + a_4) + x_3a_3 + x_0a_2 + x_4a_2] + I_3[x_1a_2 + x_2a_1 + x_3(a_0 + a_4) + x_0a_3 + x_4a_3] + I_4[x_1a_3 + x_2a_2 + x_3a_1 + x_0a_4 + x_4(a_0 + a_4)] = 1.$
This implies that:

$a_0x_0 = 1$, $thus\ a_0, x_0 = 1\ or - 1$. On the other hand, we get:

$x_1(a_0 + a_4) + x_2a_3 + x_3a_2 + x_4a_1 = -x_0a_1, x_1a_1 + x_2(a_0 + a_4) + x_3a_3 + x_4a_2 = -x_0a_2, x_1a_2 + x_2a_1 + x_3(a_0 + a_4) + x_4a_3 = -x_0a_3, x_1a_3 + x_2a_2 + x_3a_1 + x_4(a_0 + a_4) = -x_0a_4.$

The previous system of 4 linear equations with 4 variables is solvable if and only if the following determinant is equal to 1 or -1.

$\begin{vmatrix} a_0 + a_4 & a_3 & a_2 & a_1 \\ a_1 & a_0 + a_4 & a_3 & a_2 \\ a_2 & a_1 & a_0 + a_4 & a_3 \\ a_3 & a_2 & a_1 & a_0 + a_4 \end{vmatrix}$ = 1 or -1. By computing the value of the determinant, we get the desired equations.

Conjecture 2:

The equations shown in the previous Theorem have finite number of solutions. Also, this number is divisible by 8.

A Discussion Of The Case Of Symbolic Turiyam Ring of Integers

Recently, symbolic Turiyam rings [16] its modules [17] and matrix [18] is studied for dealing the data with Turiyam set [19]. In this section, the invertibility of a symbolic Turiyam integer is discussed as follows:

Let $A = x_0 + x_1T + x_2F + x_3I + x_4Y,\ B = y_0 + y_1T + y_2F + y_3I + y_4Y$ be two symbolic Turiyam integers such that AB=1.

We get:

$x_0, y_0 = 1\ or - 1, y_1(x_0 + x_1) = -x_1y_0, y_1(x_2) + y_2(x_0 + x_1 + x_2) = -x_2y_0,$

$y_1(x_3) + y_2(x_3) + y_3(x_0 + x_1 + x_2 + x_3 + x_4) + y_4(x_3) = 0,$

$y_1(x_4) + y_2(x_4) + y_3(0) + y_4(x_0 + x_1 + x_2 + x_3 + x_4) = 0.$

The determinant of the coefficients matrix is:

$\begin{vmatrix} x_0 + x_1 & 0 & 0 & 0 \\ x_2 & x_0 + x_1 + x_2 & 0 & 0 \\ x_3 & x_3 & x_0 + x_1 + x_2 + x_3 + x_4 & x_3 \\ x_4 & x_4 & 0 & x_0 + x_1 + x_2 + x_3 + x_4 \end{vmatrix} =$

$(x_0 + x_1)(x_0 + x_1 + x_2)(x_0 + x_1 + x_2 + x_3 + x_4)(x_0 + x_1 + x_2 + x_3 + x_4) = 1\ or - 1.$

The previous Diophantine equation can be solved easily, that is because it is written as a product.

From the previous discussion, we get the following result:

A symbolic Turiyam integer x is invertible if and only if $(x_0 + x_1)(x_0 + x_1 + x_2)(x_0 + x_1 + x_2 + x_3 + x_4)(x_0 + x_1 + x_2 + x_3 + x_4) = 1\ or - 1.$

For example, the following symbolic Turiyam integer $x = 1 - 2I$ is invertible. In this way the proposed method can be helpful for dealing several decision making process as discussed in[20]. In near future the author will focus on solving some new problems of these conjectures to classify the group of units.

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Conclusion

This paper discussed the invertibility of a 3-cyclic\ 4-cyclic refined neutrosophic integer. Also, we have presented two conjectures concerning the order of the group of units of 3-cyclic\ 4-cyclic refined neutrosophic ring of integers. Same time condition for the invertibility of a symbolic Turiyam integer is also discussed.

In the future, we aim to look for the solutions of these conjectures and to classify the group of units of these rings.

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