On the Neutrosophic Characteristic Polynomials and Neutrosophic Cayley-Hamilton Theorem

Malath F. Alaswad

Faculty of science , Department of mathematices,AL- Baath University, Homs,Syria
Email: Malaz.aswad@yahoo.com

Abstract:

The objective of this paper is to study some novel algebraic properties of neutrosophic matrices. Also, this work introduces the concept of neutrosophic characteristic polynomial of a square neutrosophic matrix and the neutrosophic version of Cayley-Hamilton.On the other hand, we illustrate many examples about the validity of this work.

Keywords: Neutrosophic matrix; neutrosophic Real number; neutrosophic determinant; neutrosophic inverse

1.Introduction

The theory of neutrosophic algebraic structures was released many years ago. In the literature, we find a lot of structures defined by using indeterminacy ideas such as spaces, rings and homomorphisms.

The concept of neutrosophic matrix [1-5,6-12] and its generalizations played an important role in the study of neutrosophic linear algebra, where we can find conditions for invertibility, diagonalization, and algebraic equations systems in neutrosophic and refined neutrosophic terms [13-14].

In this work, we continuo the previous efforts in the study of neutrosophic matrix theory, where we present the neutrosophic form of famous Cayley-Hamilton theorem, and the neutrosophic characteristic polynomials. Also, we illustrate many examples to clarify the validity of our work, with respect to real matrices as well as complex matrices.

2. Main Discussion

Definition [14]: Let $M_{m\times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$,where $K\ (I)$is a neutrosophic field. We call to be the neutrosophic matrix.

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Definition [14]: Let $M_{m\times n}$ is a neutrosophic matrix. We call to be the neutrosophic square matrix if $m = n$.
Now a neutrosophic n square matrix is defined by form $M = A + BI$ where A and B are two n squares matrices.
Definition : [14]

Let $M = A + BI$ be a neutrosophic n square matrix. The determinant of M is defined as

$$detM = detA + I[det(A + B) - detA].$$

Definition :[14]

Let $M = A + BI$ a neutrosophic square $n \times n$ matrix, where , B are two squares $n \times n$ matrices, then M is invertible if and only if A and $A + B$ are invertible matrices and

$$M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}].$$

Theorem [14]

M is invertible matrix if and only if $detM\ is\ invertible$.

Definition:

Let $M = A + BI$ be a neutrosophic n square matrix, where A and B are two n square matrices, And $Z = X + YI$. We define the characterisitic neutrosophic Polynomial characteristic by the following:

$$\varphi(z) = det[ZU_{n\times n} - M] = det[ZU_{n\times n} - (A + BI)] = det[(ZU_{n\times n} - A) + (-B)I]$$

$$\varphi(z) = det(ZU_{n\times n} - A) + I\left[det\left(ZU_{n\times n} - (A + B)\right) - det(ZU_{n\times n} - A)\right]$$

$$\varphi(z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)].$$

Where:

$$\alpha(Z) = det(ZU_{n\times n} - A)\ , \beta(Z) = det\left(ZU_{n\times n} - (A + B)\right)$$

Example:

Consider the following neutrosophic matrix

$M = A + BI = \begin{pmatrix} 1 & 3I \\ 1 + I & 2 + 2I \end{pmatrix}$. Where $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

$A + B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. Then.

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)]$$

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$$\alpha(Z) = det(ZU_{2\times 2} - A) = \begin{vmatrix} X + YI - 1 & -1 \\ 0 & X + YI + 1 \end{vmatrix}$$

$$\alpha(Z) = (X + YI)^2 - 1 = Z^2 - 1$$

$$\beta(Z) = det\big(ZU_{2\times 2} - (A + B)\big) = \begin{vmatrix} X + YI - 1 & -2 \\ -1 & X + YI \end{vmatrix}$$

$$\beta(Z) = (X + YI)^2 - (X + YI) - 2 = Z^2 - Z - 2$$

Then.

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)] = Z^2 - 1 + I[-Z - 1]$$

Example:

Consider the following neutrosophic matrix

$M = A + BI = \begin{pmatrix} 1 & -1 + I \\ I & 2 + I \end{pmatrix}$. Where $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

$A + B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$. Then.

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)]$$

$$\alpha(Z) = det(ZU_{2\times 2} - A) = \begin{vmatrix} X + YI - 1 & 1 \\ 0 & X + YI - 2 \end{vmatrix}$$

$$\alpha(Z) = (X + YI - 1)(X + YI - 2) = (X + YI)^2 - 3(X + YI) + 2 = Z^2 - 3Z + 2$$

$$\beta(Z) = det\big(ZU_{2\times 2} - (A + B)\big) = \begin{vmatrix} X + YI - 1 & 0 \\ -1 & X + YI - 3 \end{vmatrix}$$

$$\beta(Z) = (X + YI)^2 - 4(X + YI) + 3 = Z^2 - 4Z + 3$$

Then.

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)] = (Z^2 - 3Z + 2) + I[-Z + 1]$$

Theorem :

A neutrosophic characterisitic Polynomial of a neutrosophic square matrix is equal to the neutrosophic characteristics Polynomial of its transposed matrix.

Proof:

Let $M = A + BI$ be a neutrosophic n square matrix, where A and B are two n square matrices.

Let$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)]$a neutrosophic Polynomial characterstic for M and $M^T$is transpose for M.

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Let$\psi(Z)$a neutrosophic Polynomial characterstic forM^T. Then.

$\varphi(Z) = det[ZU_{n\times n} - M]$

$\varphi(Z) = det(ZU_{n\times n} - A) + I\big[det\big(ZU_{n\times n} - (A + B)\big) - det(ZU_{n\times n} - A)\big]$

Now we have.

$\psi(Z) = det[ZU_{n\times n} - M]^T = det[(ZU_{n\times n} - A) + (-B)I]^T$

$\psi(Z) = \det\Big[(ZU_{n\times n} - A)^T + I\left[\big(ZU_{n\times n} - (A + B)\big)^T - (ZU_{n\times n} - A)^T\right]\Big]$

$\psi(Z) = \det(ZU_{n\times n} - A)^T + I\left[det\big(ZU_{n\times n} - (A + B)\big)^T - det(ZU_{n\times n} - A)^T\right]$, hence

$[det(ZU_{n\times n} - A)]^T = det(ZU_{n\times n} - A)$

$det\big(ZU_{n\times n} - (A + B)\big)^T = det\big(ZU_{n\times n} - (A + B)\big)$

Then.

$\psi(Z) = det(ZU_{n\times n} - A) + I\big[det\big(ZU_{n\times n} - (A + B)\big) - det(ZU_{n\times n} - A)\big]$

Then.

$$\varphi(Z) = \psi(Z)$$

Example:

Consider the previous neutrosophic matrix defined above, we have:

$\varphi(Z) = Z^2 - 1 + I[-Z - 1]$

Now.

$A^T = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, $B^T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ Then.

$\psi(Z) = \alpha^*(Z) + I[\beta^*(Z) - \alpha^*(Z)]$

$\alpha^*(Z) = det(ZU_{2\times 2} - A^T) = \begin{vmatrix} X + YI - 1 & 0 \\ -1 & X + YI + 1 \end{vmatrix}$

$\alpha^*(Z) = (X + YI)^2 - 1 = Z^2 - 1$

$\beta^*(Z) = det(ZU_{2\times 2} - (A + B)^T) = \begin{vmatrix} X + YI - 1 & -2 \\ -1 & X + YI \end{vmatrix}$

$\beta^*(Z) = (X + YI)^2 - (X + YI) - 2 = Z^2 - Z - 2$

Then,

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$\varphi(Z) = \psi(Z) = Z^2 - 1 + I[-Z - 1]$

Example:

Consider the neutrosophic matrix defined above, we have:

$\varphi(Z) = (Z^2 - 3Z + 2) + I[-Z + 1]$

Now.

$A^T = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, B^T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ Then.

$\psi(Z) = \alpha^*(Z) + I[\beta^*(Z) - \alpha^*(Z)]$

$\alpha^*(Z) = det(ZU_{2\times 2} - A^T) = \begin{vmatrix} X + YI - 1 & 0 \\ 1 & X + YI - 2 \end{vmatrix}$

$\alpha^*(Z) = (X + YI)^2 - 3(X + YI) + 2 = Z^2 - 3Z + 2$

$\beta^*(Z) = det(ZU_{2\times 2} - (A + B)^T) = \begin{vmatrix} X + YI - 1 & -1 \\ 0 & X + YI - 3 \end{vmatrix}$

$\beta^*(Z) = (X + YI)^2 - 4(X + YI) + 3 = Z^2 - 4Z + 3$

Then.

$\varphi(Z) = \psi(Z) = (Z^2 - 3Z + 2) + I[-Z + 1]$

Theorem :(Neutrosophic Cayely-Hamilton):

Any neutrosophic square matrix is a root of its polynomial.

The proof is similar to the classical case, by using the definition of the determinant directly.

Example:

Consider the neutrosophic matrix defined above, we have:

$\varphi(Z) = (Z^2 - 3Z + 2) + I[-Z + 1]$

Now we find $\varphi(M)$.

$\varphi(M) = (M^2 - 3M + 2) + I[-M + 1] = M^2 - 3M - IM + (2 + I)U_{2\times 2}$

$\varphi(M) = \begin{pmatrix} 1 & -3 + 3I \\ 4I & 4 + 5I \end{pmatrix} - \begin{pmatrix} 3 & -3 + 3I \\ 3I & 6 + 3I \end{pmatrix} - \begin{pmatrix} I & 0 \\ I & 3I \end{pmatrix} + \begin{pmatrix} 2 + I & 0 \\ 0 & 2 + I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

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$\varphi(M) = 0$

Example:

Consider the neutrosophic matrix defined previously

(a). Find M^{-1} by using Cayley-Hamilton theorem.

Solution.

(a) We have $\varphi(Z) = (Z^2 - 3Z + 2) + I[-Z + 1]$. Now we have by using previous approach, $\varphi(M) = 0 \Rightarrow M^2 - 3M - IM + (2 + I) = 0 \Rightarrow M^2 - 3M - IM = -(2 + I)$

$$\Rightarrow M^2 - (3 + I)M = -(2 + I) \Rightarrow \frac{-1}{(2 + I)}[M^2 - (3 + I)M] = U_{2\times 2}$$

$\Rightarrow \frac{-1}{(2+I)}[M^2 - (3 + I)M] = MM^{-1}$.

$detM = 2 + I \neq 0$ Then.

$$\frac{-1}{(2 + I)}[M - (3 + I)U_{2\times 2}] = M^{-1} \Rightarrow M^{-1} = \frac{-1}{(2 + I)}[M - (3 + I)U_{2\times 2}]$$

$$\Rightarrow M^{-1} = \frac{-1}{(2 + I)}\begin{pmatrix} -2 - I & -1 + I \\ I & -1 \end{pmatrix}$$

$$\Rightarrow M^{-1} = \begin{pmatrix} \frac{2 + I}{2 + I} & \frac{1 - I}{2 + I} \\ \frac{-I}{2 + I} & \frac{1}{2 + I} \end{pmatrix}$$

We have:

$$\frac{2 + I}{2 + I} = 1, \frac{1 - I}{2 + I} = \frac{1}{2} - \frac{1}{2}I, \frac{-I}{2 + I} = \frac{-1}{3}I, \frac{1}{2 + I} = \frac{1}{2} - \frac{1}{6}I$$

Then.

$$M^{-1} = \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2}I \\ \frac{-1}{3}I & \frac{1}{2} - \frac{1}{6}I \end{pmatrix}.$$

Theorem :

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A neutrosophic Polynomial of any neutrosophic square matrix M is exactly equal to the neutrosophic polynomial of any other neutrosophic matrix N with property $M = P^{-1}NP$. Where P is an invertible neutrosophic matrix.

Proof:

Let $M = A + BI$ be a neutrosophic n square matrix, where A and B are two n square matrices and similar a neutrosophic$n$ square matrix $N = C + DI$. Then $N = P^{-1}MP$ where $P = K + LI$.

Let $\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)]$ be the neutrosophic Polynomial of M

Let $\psi(Z)$ be the neutrosophic Polynomial of N, then

$$\psi(Z) = det[ZU_{n\times n} - N] = det[ZU_{n\times n} - P^{-1}MP] = det[ZP^{-1}P - P^{-1}MP]$$

$$\psi(Z) = det[P^{-1}(ZU_{n\times n} - M)P]$$

hence, we get

$$\psi(Z) = det(P^{-1})\,det(ZU_{n\times n} - M)det(P) = det(P^{-1})\,det(P)\,det(ZU_{n\times n} - M)$$

$$\psi(Z) = det(U_{n\times n})\,det(ZU_{n\times n} - M) = 1\,det(ZU_{n\times n} - M) = \varphi(Z).$$

Example:

Consider the following neutrosophic complex matrix

$$M = A + BI = \begin{pmatrix} i & -1 \\ 0 & 1-i \end{pmatrix} + I\begin{pmatrix} 0 & i \\ -i & -1+i \end{pmatrix}, A + B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}. \text{ Then.}$$

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)]$$

$$\alpha(Z) = det(ZU_{2\times 2} - A) = \begin{vmatrix} Z-i & -1 \\ 0 & Z-(1-i) \end{vmatrix}$$

$$\alpha(Z) = Z^2 - (1-i)Z - iZ + 1 + i = Z^2 - Z + (1+i)$$

$$\beta(Z) = Z^2 - iZ - i - 1$$

Then.

$$\varphi(Z) = \alpha(Z) + I[\beta(Z) - \alpha(Z)] = Z^2 - Z + (1+i) + I[(1-i)Z - 2 - 2i].$$

Example:

Consider the neutrosophic matrix defined above, we have:

$$\varphi(Z) = Z^2 - Z + (1+i) + I[(1-i)Z - 2 - 2i]$$

Now.

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$A^T=\begin{pmatrix} i & 0 \\ -1 & 1-i \end{pmatrix}, B^T=\begin{pmatrix} 0 & -i \\ i & -1+i \end{pmatrix}$ Then.

$$\psi(Z)=\alpha^*(Z)+I[\beta^*(Z)-\alpha^*(Z)]$$

$$\alpha^*(Z)=det(ZU_{2\times 2}-A^T)=\begin{vmatrix} Z-i & 0 \\ -1 & Z-(1-i) \end{vmatrix}$$

$$\alpha^*(Z)=Z^2-Z+(1+i)$$

$$\beta^*(Z)=det(ZU_{2\times 2}-(A+B)^T)=\begin{vmatrix} Z & i \\ -i & Z-(-1+i) \end{vmatrix}$$

$$\beta^*(Z)=Z^2-iZ-i-1$$

Then.

$$\varphi(Z)=\psi(Z)=Z^2-Z+(1+i)+I[(1-i)Z-2-2i].$$

Example:

Consider the neutrosophic matrix defined above, we have:

$$\varphi(Z)=Z^2-Z+(1+i)+I[(1-i)Z-2-2i]$$

Now we find $\varphi(M)$ as follows:

$$\varphi(M)=M^2-M+(1+i)U_{2\times 2}+(1-i)MI+(-2-2i)U_{2\times 2}I$$

$$M^2=A^2+I[(A+B)^2-A^2]=\begin{pmatrix} -1 & -1 \\ 0 & -2i \end{pmatrix}+I\begin{pmatrix} i+1 & -i \\ 1 & 1+3i \end{pmatrix}$$

$$\varphi(M)=\begin{pmatrix} -1 & -1 \\ 0 & -2i \end{pmatrix}+I\begin{pmatrix} i+1 & -i \\ 1 & 1+3i \end{pmatrix}+\begin{pmatrix} -i & 1 \\ 0 & -1+i \end{pmatrix}+\mathrm{I}\begin{pmatrix} 0 & -i \\ i & 1-i \end{pmatrix}+\begin{pmatrix} i+1 & 0 \\ 0 & i+1 \end{pmatrix}+ I\begin{pmatrix} 1+i & 2i \\ -1-i & 0 \end{pmatrix}+I\begin{pmatrix} -2-2i & 0 \\ 0 & -2-2i \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}+I\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\varphi(M)=0.$$

3. Refined Neutrosophic Matrices

Definition [5]: The structure of refined neutrosophic numbers is taken as $a+bI_1+cI_2$ instead of (a,bI_1,cI_2).

Definition [5]: $I_1{}^2=I_1\ ,I_2{}^2=I_2\ ,I_1.I_2=I_2.I_1=I_1$

Definition [5]: (**Refined neutrosophic matrix).**

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Let$A=\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$ be an $m\times n$ matrix: if $a_{ij}=a+bI_1+cI_2\in R_2(I)$, then it is called an refined neutrosophic matrix, where $R_2(I)$ is an refined neutrosophic field.

Theorem : [5] Let $M=A+BI_1+CI_2$ be a square $n\times n$ refined neutrosophic matrix, then it is invertible if only of $A, A+C$ and $A+B+C$ are invertible. The inverse of M is

$$M^{-1}=A^{-1}+((A+B+C)^{-1}-(A+C)^{-1})I_1+((A+C)^{-1}-A^{-1})I_2$$

Definition: [5]

Let $M=A+BI_1+CI_2$ be a refined neutrosophic n square matrix, where A, B and C are n square matrices, then,

$$detM=det(A+BI_1+CI_2)=detA+[det(A+B+C)-det(A+C)]I_1+[det(A+C)-detA]I_2.$$

Theorem :

Let $M=A+BI_1+CI_2$ be a neutrosophic n square matrix, where A, B and C are two n square matrices, And $Z=X+YI_1+TI_2$. We define the refined neutrosophic Polynomial as follows:

$$\begin{aligned}\varphi(z)=det[ZU_{n\times n}-M]&=det[ZU_{n\times n}-(A+BI_1+CI_2)]\\&=det[(ZU_{n\times n}-A)+(-B)I_1+(-C)I_2]\end{aligned}$$

$$\begin{aligned}\varphi(z)=det(ZU_{n\times n}-A)&+\left[det\left(ZU_{n\times n}-(A+B+C)\right)-det\left(ZU_{n\times n}-(A+C)\right)\right]I_1\\&+\left[det\left(ZU_{n\times n}-(A+C)\right)-det(ZU_{n\times n}-A)\right]I_2\end{aligned}$$

$$\varphi(z)=\alpha(Z)+[\beta(Z)-\gamma(Z)]I_1+[\gamma(Z)-\alpha(Z)]I_2.$$

Where:

$$\alpha(Z)=det(ZU_{n\times n}-A),\beta(Z)=det\left(ZU_{n\times n}-(A+B+C)\right),\gamma(Z)=det\left(ZU_{n\times n}-(A+C)\right)$$

Example:

Consider the following neutrosophic matrix

$M=A+BI_1+CI_2$. Where $=\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, B=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, C=\begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$

$A+B+C=\begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}, A+C=\begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}$.

Then.

$$\varphi(z)=\alpha(Z)+[\beta(Z)-\gamma(Z)]I_1+[\gamma(Z)-\alpha(Z)]I_2$$

$$\alpha(Z)=det(ZU_{n\times n}-A)=\begin{vmatrix} X+YI_1+TI_2-2 & -1 \\ -3 & X+YI_1+TI_2-1 \end{vmatrix}$$

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$\alpha(Z) = (X + YI_1 + TI_2)^2 - 3X + YI_1 + TI_2 + 2 - 3 = Z^2 - 3Z - 1$

$\beta(Z) = det\big(ZU_{n\times n} - (A + B + C)\big) = \begin{vmatrix} X + YI_1 + TI_2 - 6 & 1 \\ -7 & X + YI_1 + TI_2 - 2 \end{vmatrix}$

$\beta(Z) = (X + YI_1 + TI_2)^2 - 8(X + YI_1 + TI_2) + 19 = Z^2 - 8Z + 19$

$\gamma(Z) = det\big(ZU_{n\times n} - (A + C)\big) = \begin{vmatrix} X + YI_1 + TI_2 - 5 & 0 \\ -7 & X + YI_1 + TI_2 - 1 \end{vmatrix}$

$\gamma(Z) = (X + YI_1 + TI_2)^2 - 6(X + YI_1 + TI_2) + 5 = Z^2 - 6Z + 5$

Then.

$\varphi(Z) = \alpha(Z) + [\beta(Z) - \gamma(Z)]I_1 + [\gamma(Z) - \alpha(Z)]I_2$

$$\varphi(Z) = Z^2 - 3Z - 1 + [-2\mathrm{Z} + 14]I_1 + [-3Z + 6]I_2$$

Theorem: :(Refined Neutrosophic Cayely-Hamilton theorem):

Any refined neutrosophic square matrix is a root of its refined neutrosophic Polynomial.

Example:

Consider the neutrosophic matrix defined above, we have:

$\varphi(Z) = Z^2 - 3Z - 1 + [-2\mathrm{Z} + 14]I_1 + [-3Z + 6]I_2$

Now we find $\varphi(M)$.

$\varphi(M) = M^2 - 3M - 1 + [-2\mathrm{M} + 14]I_1 + [-3M + 6]I_2 = M^2 + (-3M) - U_{2\times 2} + (-2\mathrm{M}I_1) + 14I_1U_{2\times 2} + (-3MI_2) + 6I_2U_{2\times 2}$.

Now we have:

$M^2 = AA + (AB + BA + BB + BC + CB)I_1 + (AC + CC + CA)I_2$

$AA = \begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix}$, $AB = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $BA = \begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix}$, $BB = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, $BC = \begin{pmatrix} -1 & -1 \\ 4 & 0 \end{pmatrix}$, $CB = \begin{pmatrix} 3 & -4 \\ 4 & -4 \end{pmatrix}$, $AC = \begin{pmatrix} 10 & -2 \\ 13 & -3 \end{pmatrix}$, $CC = \begin{pmatrix} 5 & -3 \\ 12 & -4 \end{pmatrix}$, $CA = \begin{pmatrix} 3 & 2 \\ 8 & 4 \end{pmatrix}$. *Then.*

$M^2 = \begin{pmatrix} 7 + 4I_1 + 18I_2 & 3 - 8I_1 - 3I_2 \\ 9 + 14I_1 + 33I_2 & 4 - 4I_1 - 3I_1 \end{pmatrix}$

$-3M = \begin{pmatrix} -6 - 3I_1 - 9I_2 & -3 + 3I_1 + 3I_2 \\ -9 - 12I_2 & -3 - 3I_1 \end{pmatrix}$

$-2\mathrm{M}I_1 = \begin{pmatrix} -12I_1 & 2I_1 \\ -14I_1 & -4I_1 \end{pmatrix}$

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$$-3MI_2 = \begin{pmatrix} -3I_1 - 15I_2 & -3I_1 - 6I_2 \\ -21I_2 & -3I_1 - 3I_2 \end{pmatrix}$$

$$\varphi(M) = \begin{pmatrix} 7 + 4I_1 + 18I_2 & 3 - 8I_1 - 3I_2 \\ 9 + 14I_1 + 33I_2 & 4 - 4I_1 - 3I_1 \end{pmatrix} + \begin{pmatrix} -6 - 3I_1 - 9I_2 & -3 + 3I_1 + 3I_2 \\ -9 - 12I_2 & -3 - 3I_1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -12I_1 & 2I_1 \\ -14I_1 & -4I_1 \end{pmatrix} + \begin{pmatrix} 14I_1 & 0 \\ 0 & 14I_1 \end{pmatrix} + \begin{pmatrix} -3I_1 - 15I_2 & -3I_1 - 6I_2 \\ -21I_2 & -3I_1 - 3I_2 \end{pmatrix} + \begin{pmatrix} 6I_2 & 0 \\ 0 & 6I_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\varphi(M) = 0$$

Example :

Consider the following neutrosophic matrix

$M = A + BI_1 + CI_2$. Where $= \begin{pmatrix} 2+i & 1 \\ -i & -1+i \end{pmatrix}$, $B = \begin{pmatrix} 1 & -i \\ 0 & 2i \end{pmatrix}$, $C = \begin{pmatrix} 1-i & -1 \\ i & 0 \end{pmatrix}$

$A + B + C = \begin{pmatrix} 4 & -i \\ 0 & -1+3i \end{pmatrix}$, $A + C = \begin{pmatrix} 3 & 0 \\ 0 & -1+i \end{pmatrix}$**.**

Then,

$$\varphi(z) = \alpha(Z) + [\beta(Z) - \gamma(Z)]I_1 + [\gamma(Z) - \alpha(Z)]I_2$$

$$\alpha(Z) = det(ZU_{n\times n} - A) = \begin{vmatrix} Z - (2+i) & -1 \\ i & Z + (1-3i) \end{vmatrix}$$

$$\alpha(Z) = Z^2 - (1+2i)Z - 3 + 2i$$

$$\beta(Z) = det\big(ZU_{n\times n} - (A + B + C)\big) = \begin{vmatrix} Z - 4 & i \\ 0 & Z + (1-3i) \end{vmatrix}$$

$$\beta(Z) = Z^2 - (1+4i)Z - (5 - 5i)$$

$$\gamma(Z) = det\big(ZU_{n\times n} - (A + C)\big) = \begin{vmatrix} Z - 3 & 0 \\ 0 & Z + (1-i) \end{vmatrix}$$

$$\gamma(Z) = Z^2 - (2+i)Z + (-3 + 3i)$$

Then.

$$\varphi(Z) = \alpha(Z) + [\beta(Z) - \gamma(Z)]I_1 + [\gamma(Z) - \alpha(Z)]I_2$$

$$\varphi(Z) = Z^2 - (1+2i)Z - 3 + 2i + [(1-3i)Z + (-2+2i)]I_1 + [(-1+i)Z + i]I_2.$$

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4. Conclusion

In this paper, we have presented the concept pf the characteristic polynomial of a neutrosophic matrix. Also, we have proved the neutrosophic version of of Cayley-Hamilton theorem with many examples that clarify the validity of this work. In the future, we aim to study the Cayley-Hamilton theorem in the case of n-refined neutrosophic matrices and n-cyclic refined neutrosophic matrices too.

References

[1] Olgun, N., Hatip, A., Bal, M., and Abobala, M., A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices, International Journal of Neutrosophic Science, Vol. 16, pp. 72-79, 2021.

[2] M. Abobala. On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021

[3] Dhar, M., Broumi, S., and, Smarandache F., A Note on Square Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems", Vol. 3, pp. 37-41 2014.

[4] Smarandache, F., Introduction to Neutrosophic Statistics, USA: Sitech & Education Publishing, 2014.

[5] Smarandache, F., Neutrosophic Set a Generalization of the Intuitionistic Fuzzy Sets, Inter. J. Pure Appl. Math., pp. 287-297, 2005.

[6] Kandasamy V, Smarandache F., and Kandasamy I., Special Fuzzy Matrices for Social Scientists . Printed in the United States of America,2007, book, 99 pages.

[7] Khaled, H., and Younus, A., and Mohammad, A., The Rectangle Neutrosophic Fuzzy Matrices, Faculty of Education Journal Vol. 15, 2019. (Arabic version).

[8] Thomas, M. G., Convergence of Powers of a Fuzzy Matrix, J. Math. Annal. Appl. 57 , pp. 476-480, 1977.

[9] W. B. Vasantha Kandasamy Florentin Smarandache K. Ilanthenral, INTRODUCTION TO BIMATRICES, HEXIS, 2005.

[10]W. B. V. Kandasamy and F. Smarandache, Fuzzy Relational Maps and Neutrosophic Relational Maps, HEXIS Church Rock,p 302.

[11] Abobala, M., On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations, Mathematical Problems in Engineering, Hindawi, 2021

[12] Abobala, M., Bal, M., and Hatip, A., A Review On Recent Advantages In Algebraic Theory Of Neutrosophic Matrices**,** International Journal of Neutrosophic Science, Vol. 17, 2021.

[13] Abobala, M., On The Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations, Journal of Mathematics, Hindawi, 2021.

Doi : https://doi.org/10.54216/IJNS.180305
Received: January 21, 2022 Accepted: April 13, 2022

[14] Abobala, M., Hatip, A., Olgun, N., Broumi, S., Salama, A,A., and Khaled, E, H., The algebraic creativity In The Neutrosophic Square Matrices, Neutrosophic Sets and Systems, Vol. 40, pp. 1-11, 2021.

[15] F .Smarandache.; Introduction to Neutrosophic statistics, Sitech-Education Publisher, PP:34-44. (2014).

Doi : https://doi.org/10.54216/IJNS.180305
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