



## A Review Study On Neutrosophic AH-Algebraic Structures

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**Abstract:** Neutrosophic Algebraic structures are rich fields for researchers to get many interesting generalizations of classical and fuzzy structures. This Study is dedicated to give the interested reader some of special neutrosophic algebraic substructures of neutrosophic algebraic structures, especially AH-substructures in neutrosophic rings, spaces, modules, and their generalizations.

**Key words:** Neutrosophic set, neutrosophic number, neutrosophic integer, Neutrosophic Algebra.

### 1. Introduction

Neutrosophy is considered as a new generalization of fuzzy and intuitionistic fuzzy ideas proposed by Smarandache [1]. Neutrosophic sets were very applicable in many areas of pure and applied mathematic such as matrix theory, topology, number theory, and algebraic structures [2-12].

Neutrosophic algebra began with Smarandache et. al, where they have defined neutrosophic rings, groups, vector spaces, and modules [20-30, 60-75].

In the literature many researchers have studied and contributed to these structures and their generalizations, where we can find many generalizations such as refined neutrosophic structures [13-20, 36-40], and n-refined neutrosophic structures [46-54].

An interesting direction was opened by Abobala. Et. al, where they defined AH-substructures such as AH-subspaces, AH-homomorphisms, AH-ideals, and AH-submodules [6-10]. Recently, AH-substructures were used to

build neutrosophic system of Euclidean geometry, especially AH-isometries[66], and Turiyam symbolic rings [78-90]. The Turiyam set given a way to represent the data set beyond the Non-Euclidean and NeutroGeoemtry [91-92].

Through this work, we review the basic concepts of AH-substructures in neutrosophic rings, refined neutrosophic rings, neutrosophic vector spaces, n-refined neutrosophic modules, and many other neutrosophic algebraic structures.

### Main Discussion

#### Definition 1:

Let  $R(I)$  be a neutrosophic ring and  $P = P_0 + P_1I = \{a_0 + a_1I; a_0 \in P_0, a_1 \in P_1\}$ .

(a) We say that  $P$  is an AH-ideal if  $P_0$  and  $P_1$  are ideals in the ring  $R$ .

(b) We say that  $P$  is an AHS-ideal if  $P_0 = P_1$ .

#### Definition 2:

Let  $(R(I_1, I_2), +, \times)$  be a refined neutrosophic ring, and  $P_0, P_1, P_2$  be three ideals in the ring  $R$  then the set

$P = (P_0, P_1I_1, P_2I_2) = \{(a, bI_1, cI_2); a \in P_0, b \in P_1, c \in P_2\}$  is called a refined neutrosophic AH-ideal.

If  $P_0 = P_1 = P_2$  then  $P$  is called a refined neutrosophic AHS-ideal.

#### Definition 3:

Let  $V(I) = V + VI$  be a strong/weak neutrosophic vector space, the set

$S = P + QI = \{x + yI; x \in P, y \in Q\}$ , where  $P$  and  $Q$  are subspaces of  $V$  is called an AH-subspace of  $V(I)$ .

If  $P = Q$  then  $S$  is called an AHS-subspace of  $V(I)$ .

#### Example 4:

We have  $V = R^2$  is a vector space,  $P = \langle (0,1) \rangle, Q = \langle (1,0) \rangle$ , are two subspaces of  $V$ . The set

$S = P + QI = \{(0, a) + (b, 0)I; a, b \in R\}$  is an AH-subspace of  $V(I)$ .

The set  $L = P + PI = \{(0, a) + (0, b)I\}; a, b \in R$  is an AHS-subspace of  $V(I)$ .

#### Theorem 5:

Let  $V(I) = V + VI$  be a neutrosophic weak vector space, and let  $S = P + QI$  be an AH-subspace of  $V(I)$ , i.e  $P, Q$  are subspaces of  $V$ , then  $S$  is a subspace by the classical meaning.

Proof:

Suppose that  $x = a + bI, y = c + dI \in S; a, c \in P, b, d \in Q$ , we have

$x + y = (a + c) + (b + d)I \in S$ . For each scalar  $m \in K$  we obtain  $m \cdot x = m \cdot a + (m \cdot b)I \in S$ , since  $P$  and  $Q$  are subspaces; thus  $S = P + QI$  is a subspace of  $V(I)$  over the field  $K$ .

**Theorem 6:**

Let  $V(I)$  be a neutrosophic strong vector space over a neutrosophic field  $K(I)$ , let  $S = P + PI$  be an AHS-subspace.  $S$  is a subspace of  $V(I)$ .

Proof:

Suppose that  $x = a + bI, y = c + dI \in S; a, c, b,$  and  $c \in P$ , we have

$x + y = (a + c) + (b + d)I \in S$ . Let  $m = x + yI \in K(I)$  be a neutrosophic scalar, we find

$m \cdot x = (x \cdot a) + (y \cdot a + y \cdot b + x \cdot b)I \in S$ , since  $y \cdot a + y \cdot b + x \cdot b \in P$ , thus we get the desired result.

**Definition 7:**

(a) Let  $V$  and  $W$  be two vector spaces,  $L_V: V \rightarrow W$  be a linear transformation. The AHS-linear transformation can be defined as follows:

$$L: V(I) \rightarrow W(I); L(a + bI) = L_V(a) + L_V(b)I.$$

(b) If  $S = P + QI$  is an AH-subspace of  $V(I)$ ,  $L(S) = L_V(P) + L_V(Q)I$ .

(c) If  $S = P + QI$  is an AH-subspace of  $W(I)$ ,  $L^{-1}(S) = L_V^{-1}(P) + L_V^{-1}(Q)I$ .

(d)  $AH - Ker L = Ker L_V + Ker L_V I = \{x + yI; x, y \in Ker L_V\}$ .

**Theorem 8:**

Let  $W(I)$  and  $V(I)$  be two neutrosophic strong/weak vector spaces, and  $L: V(I) \rightarrow W(I)$  be an AHS-linear transformation, we have:

(a)  $AH - Ker L$  is an AHS-subspace of  $V(I)$ .

(b) If  $S = P + QI$  is an AH-subspace of  $V(I)$ ,  $L(S)$  is an AH-subspace of  $W(I)$ .

(c) If  $S = P + QI$  is an AH-subspace of  $W(I)$ ,  $L^{-1}(S)$  is an AH-subspace of  $V(I)$ .

Proof:

(a) Since  $Ker L_V$  is a subspace of  $V$ , we find that  $AH - Ker L = Ker L_V + Ker L_V I$  is an AHS-subspace of  $V(I)$ .

(b) We have  $L(S) = L_V(P) + L_V(Q)I$ ; thus  $L(S)$  is an AH-subspace of  $W(I)$ , since  $L_V(P), L_V(Q)$  are subspaces of  $W$ .

(c) By regarding  $L^{-1}(S) = L_V^{-1}(P) + L_V^{-1}(Q)I$ ,  $L_V^{-1}(P)$  and  $L_V^{-1}(Q)$  are subspaces of  $V$ , we obtain that  $L^{-1}(S)$  is an AH-subspace of  $V(I)$ .

**Theorem 9:**

Let  $W(I)$  and  $V(I)$  be two neutrosophic strong vector spaces over a neutrosophic field  $K(I)$ , and  $L: V(I) \rightarrow W(I)$  be an AHS-linear transformation, we have:

$$L(x + y) = L(x) + L(y), L(m \cdot x) = m \cdot L(x), \text{ for all } x, y \in V(I), m \in K(I).$$

Proof:

Suppose  $x = a + bI, y = c + dI; a, b, c, d \in V$ , and  $m = s + tI \in K(I)$ , we have

$$\begin{aligned} L(x + y) &= L([a + c] + [b + d]I) = L_V(a + c) + L_V(b + d)I = \\ &= [L_V(a) + L_V(b)I] + [L_V(c) + L_V(d)I] = L(x) + L(y). \end{aligned}$$

$$\begin{aligned} m \cdot x &= (s \cdot a) + (s \cdot b + t \cdot a + t \cdot b)I, L(m \cdot x) = L_V(s \cdot a) + L_V(s \cdot b + t \cdot a + t \cdot b)I \\ &= s \cdot L_V(a) + [s \cdot L_V(b) + t \cdot L_V(a) + t \cdot L_V(b)]I = (s + tI) \cdot (L_V(a) + L_V(b)I) = m \cdot L(x). \end{aligned}$$

**Theorem 10:**

Let  $S = P + QI$  be an AH-subspace of a neutrosophic weak vector space  $V(I)$  over a field  $K$ , suppose that  $X = \{x_i; 1 \leq i \leq n\}$  is a bases of  $P$  and  $Y = \{y_j; 1 \leq j \leq m\}$  is a bases of  $Q$  then  $X \cup YI$  is a bases of  $S$ .

Proof:

Let  $z = x + yI$  be an arbitrary element in  $S$ ;  $x \in P, y \in Q$ . Since  $P$  and  $Q$  are subspaces of  $V$  we can write  $x = a_1x_1 + a_2x_2 + \dots + a_nx_n$ ;  $a_i \in K$  and  $x_i \in X, y = b_1y_1 + b_2y_2 + \dots + b_my_m$ ;  $b_i \in K, y_i \in Y$ .

Now we obtain  $z = (a_1x_1 + \dots + a_nx_n) + (b_1y_1I + \dots + b_my_mI)$ ; thus  $X \cup YI$  generates the subspace  $S$ .

$X \cup YI$  is linearly independent set. Assume that  $(a_1x_1 + \dots + a_nx_n) + (b_1y_1I + \dots + b_my_mI) = 0$ , this implies

$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$  and  $(b_1y_1 + b_2y_2 + \dots + b_my_m)I = 0$ . Since  $X$  and  $Y$  are linearly independent sets over  $K$ , we get  $a_i = b_j = 0$  for all  $i, j$  and  $X \cup YI$  is linearly independent then it is a bases of  $S$ .

Definition 11:

Let  $V(I)$  be a neutrosophic strong/weak vector space,  $S = P + QI$  be an AH-subspace of  $V(I)$ , we define

AH-Quotient as:

$$V(I)/S = V/P + (V/Q)I = (x + P) + (y + Q)I; x, y \in V.$$

**Theorem 12:**

Let  $V(I)$  be a neutrosophic weak vector space over a field  $K$ , and  $S = P + QI$  be an AH-subspace of  $V(I)$ . The AH-Quotient  $V(I)/S$  is a vector space over the field  $K$  with respect to the following operations:

$$\text{Addition: } [(x + P) + (y + Q)I] + [(a + P) + (b + Q)I] = (x + a + P) + (y + b + Q)I; x, y, a, b \in V.$$

$$\text{Multiplication by a scalar: } (m) \cdot [(x + P) + (y + Q)I] = (m \cdot x + P) + (m \cdot y + Q)I;$$

$x, y \in V$  and  $m \in K$ .

**Definition 13:**

Let  $(V, +, \cdot)$  be a vector space over a field  $K$ ,  $V_n(I)$  be the corresponding weak  $n$ -refined neutrosophic vector space over  $K$ . Consider the set  $\{M_i; 0 \leq i \leq n\}$ , where  $M_i$  is a subspace of  $V$ . We say:

$M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n = \{m_0 + m_1I_1 + \dots + m_nI_n; m_i \in M_i\}$  is a weak  $n$ -refined AH-subspace of the weak  $n$ -refined vector space  $V_n(I)$ .

We say that  $M_n(I)$  is a weak  $n$ -refined AH-subspace if  $M_j = M_i$  for all  $i, j$ .

**Definition 14:**

Let  $(V, +, \cdot)$  be a vector space over a field  $K$ ,  $V_n(I)$  be the corresponding strong  $n$ -refined neutrosophic vector space over the  $n$ -refined neutrosophic field  $K_n(I)$ . Consider the set  $\{M_i; 0 \leq i \leq n\}$ , where  $M_i$  is a subspace of  $V$ . We say:

$M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n = \{m_0 + m_1I_1 + \dots + m_nI_n; m_i \in M_i\}$  is a strong  $n$ -refined AH-subspace of the strong  $n$ -refined vector space  $V_n(I)$ .

We say that  $M_n(I)$  is a strong  $n$ -refined AH-subspace if  $M_j = M_i$  for all  $i, j$ .

**Theorem 15:**

Let  $(V, +, \cdot)$  be a vector space over a field  $K$ ,  $V_n(I)$  be the corresponding weak  $n$ -refined neutrosophic vector space over  $K$ ,  $M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n$  be a weak  $n$ -refined AH-subspace. Then

- (a)  $M_n(I)$  is a vector subspace of  $V_n(I)$ .
- (b) If  $X_i$  is a bases of  $M_i$ ,  $X = \cup_{i=0}^n X_iI_i$  is a bases of  $M_n(I)$ .
- (c)  $\dim(M_n(I)) = \sum_{i=0}^n \dim(M_i)$ .

Proof:

(a) Let  $x = \sum_{i=0}^n a_iI_i, y = \sum_{i=0}^n b_iI_i; b_i, a_i \in M_i$  be two arbitrary elements in  $M_n(I)$ ,  $r$  be an arbitrary element in  $K$ , we have:

$x + y = \sum_{i=0}^n (a_i + b_i)I_i \in M_n(I)$ , since  $a_i + b_i \in M_i$  because  $M_i$  is a subspace of  $V$ .

$r \cdot x = \sum_{i=0}^n ra_iI_i \in M_n(I)$ , since  $ra_i \in M_i$  because  $M_i$  is a subspace of  $V$ . Thus  $M_n(I)$  is a vector subspace of  $V_n(I)$ .

(b) Suppose that  $X_0 = \{x_1^{(0)}, \dots, x_{s_0}^{(0)}\}, X_1 = \{x_1^{(1)}, \dots, x_{s_1}^{(1)}\}, \dots, X_n = \{x_1^{(n)}, \dots, x_{s_n}^{(n)}\}$ , let  $x = \sum_{i=0}^n a_iI_i$  be an arbitrary element of  $M_n(I)$ , since  $X_i$  is a basis of  $M_i$  for all  $i$ . We can write:

$a_i = \sum_{j=0}^{s_i} t_j^{(i)} x_j^{(i)}; t_j \in K$ , so  $x = \sum_{j=0}^{s_0} t_j^{(0)} x_j^{(0)} + \sum_{j=0}^{s_1} t_j^{(1)} x_j^{(1)}I_1 + \dots + \sum_{j=0}^{s_n} t_j^{(n)} x_j^{(n)}I_n$ . This implies that  $X$  is a generating set of  $M_n(I)$ .

Now we prove that  $X$  is linearly independent. For our purpose we assume

$\sum_{j=0}^{s_0} t_j^{(0)} x_j^{(0)} + \sum_{j=0}^{s_1} t_j^{(1)} x_j^{(1)} I_1 + \dots + \sum_{j=0}^{s_n} t_j^{(n)} x_j^{(n)} I_n = 0$ , by definition of n-refined vector space we find  $\sum_{j=0}^{s_i} t_j^{(i)} x_j^{(i)}$  for all i, hence  $t_j^{(i)} = 0$  for all i,j, since each  $X_i$  is linearly independent itself. Thus our proof is complete.

(c) It holds directly from (b).

**Theorem 16:**

Let  $(V, +, \cdot)$  be a vector space over a field  $K$ ,  $V_n(I)$  be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field  $K_n(I)$ ,  $M_n(I) = M + M_1 I_1 + \dots + M_n I_n$  be a strong n-refined AHS-subspace. Then:

(a)  $M_n(I)$  is a submodule of  $V_n(I)$ .

(b) If  $Y$  is a bases of  $M$ ,  $X = \cup_{i=0}^n Y I_i$  is a bases of  $M_n(I)$ .

(c)  $\dim(M_n(I)) = \sum_{i=0}^n \dim(M) = n \cdot \dim(M)$ .

**Remark 17:**

If  $V_n(I)$  is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field  $K_n(I)$ , and

$M_n(I) = M_0 + M_1 I_1 + \dots + M_n I_n$  is a strong n-refined AH-subspace, then it is not supposed to be a submodule.

**Example18 :**

Let  $V = R^2$  be a vector space over  $R$ ,  $V_2(I) = R_2^2(I) = \{(a, b) + (c, d)I_1 + (e, f)I_2; a, b, c, d, e, f \in R\}$  be the corresponding strong 2-refined neutrosophic vector space over the neutrosophic field  $R_2(I)$ .

$M = \langle (0,1) \rangle, N = \langle (1,0) \rangle$  are two subspaces of  $V$ ,  $T = M + N I_1 + N I_2$  is a strong AH-subspace of  $V_2(I)$ .

$x = (0,1) + (2,0)I_1 + (1,0)I_2 \in T, r = 1 + 1 \cdot I_1 + 1 \cdot I_2 \in R_2(I)$ ,

$r \cdot x = 1 \cdot (0,1) + 1 \cdot (0,1)I_1 + 1 \cdot (0,1)I_2 + 1 \cdot (2,0)I_1 I_1 + 1 \cdot (2,0)I_1 + 1 \cdot (1,0)I_1 I_2 + 1 \cdot (0,1)I_2 + 1 \cdot (2,0)I_1 I_2 + 1 \cdot (2,0)I_2 I_2 = (0,1) + [(0,1) + (2,0) + (1,0) + (2,0)]I_1 + [(0,1) + (0,1) + (2,0)]I_2 =$

$(0,1) + (5,1)I_1 + (2,2)I_2$ ,  $r \cdot x$  does not belong to  $T$ , thus  $T$  is not a submodule.

**Definition 19:**

Let  $V_n(I)$  be a weak/strong n-refined neutrosophic vector space,  $M_n(I) = M_0 + M_1 I_1 + \dots + M_n I_n$ ,

$W_n(I) = W_0 + W_1 I_1 + \dots + W_n I_n$  be two weak/strong AH-subspaces of  $V_n(I)$ , we define:

(a)  $M_n(I) \cap W_n(I) = (M_0 \cap W_0) + (M_1 \cap W_1)I_1 + \dots + (M_n \cap W_n)I_n$ .

(b)  $M_n(I) + W_n(I) = (M_0 + W_0) + (M_1 + W_1)I_1 + \dots + (M_n + W_n)I_n$ .

**Theorem 20:**

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Let  $V_n(I)$  be a weak  $n$ -refined neutrosophic vector space,  $M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n$ ,

$W_n(I) = W_0 + W_1I_1 + \dots + W_nI_n$  be two weak AH-subspaces of  $V_n(I)$ . Then:

$M_n(I) \cap W_n(I), M_n(I) + W_n(I)$  are two weak AH-subspaces of  $V_n(I)$ .

Proof:

Since  $M_i \cap W_i, M_i + W_i$  are subspaces of  $V$  for all  $i$ , we obtain the proof.

**Theorem 21:**

Let  $V_n(I)$  be a strong  $n$ -refined neutrosophic vector space,  $M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n$ ,

$W_n(I) = W_0 + W_1I_1 + \dots + W_nI_n$  be two strong AH-subspaces of  $V_n(I)$ . Then:

(a)  $M_n(I) \cap W_n(I)$  is a strong AH-subspaces of  $V_n(I)$ .

(b)  $M_n(I) + W_n(I)$  is not supposed to be a strong AH-subspace of  $V_n(I)$ .

**Definition 22:**

Let  $V, W$  be two vector spaces over the field  $K$ ,  $f_i: V \rightarrow W; 0 \leq i \leq n+1$  be  $n+1$  linear transformations,  $V_n(I), W_n(I)$  be the corresponding weak  $n$ -refined neutrosophic vector spaces over the field  $K$  respectively. We say:

(a)  $f: V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^n f_i(a_i)I_i$  is a weak AH-linear transformation.

(b) If  $f_i = f_j$  for all  $i, j$ , we call  $f$  a weak AHS-linear transformation.

**Example 23:**

(a) Let  $V = R^3, W = R^2$  be two vector spaces over the field  $R$ ,  $V_2(I) = R_2^3(I) = \{(x_0, y_0, z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2; x_i, y_i, z_i \in R\}$ ,

$W_2(I) = \{(x_0, y_0) + (x_1, y_1)I_1 + (x_2, y_2)I_2; x_i, y_i \in R\}$  be the corresponding weak 2-refined neutrosophic vector spaces. We have  $g: V \rightarrow W; g(a, b, c) = (b, c), h: V \rightarrow W; h(a, b, c) = (2a, 0)$

$s: V \rightarrow W; s(a, b, c) = (2b, 3c)$  are three linear transformations.

(b)  $f: V_2(I) \rightarrow W_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m, n, q \in V$  is a weak AH-linear transformation.

(c) We clarify  $f$  as follows:

$$x = (1, 2, 2) + (1, 0, 1)I_1 + (3, -1, 0)I_2 \in V_2(I),$$

$$f(x) = g(1, 2, 2) + [h(1, 0, 1)]I_1 + [s(3, -1, 0)]I_2 = (2, 2) + (2, 0)I_1 + (-2, 0)I_2.$$

(d)  $k: V_2(I) \rightarrow W_2(I); k(m + nI_1 + qI_2) = g(m) + g(n)I_1 + g(q)I_2; m, n, q \in V$  is a weak AHS-linear transformation.

**Definition 24:**

Let  $V, W$  be two vector spaces over the field  $K$ ,  $f_i: V \rightarrow W; 0 \leq i \leq n+1$  be  $n+1$  linear transformations,  $V_n(I), W_n(I)$  be the corresponding strong  $n$ -refined neutrosophic vector spaces over the  $n$ -refined neutrosophic field  $K_n(I)$  respectively. We say:

(a)  $f: V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^n f_i(a_i)I_i$  is a strong AH-linear transformation.

(b) If  $f_i = f_j$  for all  $i, j$ , we call  $f$  a strong AHS-linear transformation.

**Example 25:**

(a) Let  $V = R^3, W = R^2$  be two vector spaces over the field  $R$ ,  $V_2(I) = R_2^3(I) = \{(x_0, y_0, z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2; x_i, y_i, z_i \in R\}$ ,

$W_2(I) = \{(x_0, y_0) + (x_1, y_1)I_1 + (x_2, y_2)I_2; x_i, y_i \in R\}$  be the corresponding strong 2-refined neutrosophic vector spaces over the 2-refined neutrosophic field  $R_2(I)$ . We have  $g: V \rightarrow W; g(a, b, c) = (b, c), h: V \rightarrow W; h(a, b, c) = (2a, 0)$ ,

$s: V \rightarrow W; s(a, b, c) = (2b, 3c)$  are three linear transformations.

(b)  $f: V_2(I) \rightarrow W_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m, n, q \in V$  is a strong AH-linear transformation.

(c) We clarify  $f$  as follows:

$$x = (1, 2, 2) + (1, 0, 1)I_1 + (3, -1, 0)I_2 \in V_2(I),$$

$$f(x) = g(1, 2, 2) + [h(1, 0, 1)]I_1 + [s(3, -1, 0)]I_2 = (2, 2) + (2, 0)I_1 + (-2, 0)I_2.$$

(d)  $k: V_2(I) \rightarrow W_2(I); k(m + nI_1 + qI_2) = g(m) + g(n)I_1 + g(q)I_2; m, n, q \in V$  is a strong AHS-linear transformation.

**Definition 26:**

Let  $V_n(I), W_n(I)$  be two weak/strong  $n$ -refined neutrosophic vector spaces,

$f: V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^n f_i(a_i)I_i$  be a weak/strong AH-linear transformation. We define:

(a)  $AH - Ker(f) = Ker(f_0) + Ker(f_1)I_1 + \dots + Ker(f_n)I_n.$

(b)  $AH - Im(f) = Im(f_0) + Im(f_1)I_1 + \dots + Im(f_n)I_n.$

**Theorem 27:**

Let  $V_n(I), W_n(I)$  be two weak  $n$ -refined neutrosophic vector spaces,

$f: V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^n f_i(a_i)I_i$  be a weak AH-linear transformation. Then:

(a)  $AH - Ker(f)$  is a weak AH-subspace of  $V_n(I)$ .

(b)  $AH - Im(f)$  is a weak AH-subspace of  $W_n(I)$ .

(c) If  $M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n$  is a weak AH-subspace of  $V_n(I)$ ,  $f(M_n(I))$  is a weak AH-subspace of  $W_n(I)$ .

Proof:

(a) Since  $Ker(f_i)$  is a subspace of  $V$ , we find that

$AH - Ker(f) = Ker(f_0) + Ker(f_1)I_1 + \dots + Ker(f_n)I_n$  is a weak AH-subspace of  $V_n(I)$ .

(b) Since  $Im(f_i)$  is a subspace of  $W$ , we find that  $AH - Im(f) = Im(f_0) + Im(f_1)I_1 + \dots + Im(f_n)I_n$  is a weak AH-subspace of  $W_n(I)$ .

(c) It is known that  $f_i(M_i)$  is a subspace of  $W$ , hence

$f(M_n(I)) = f_0(M_0) + f_1(M_1)I_1 + \dots + f_n(M_n)I_n$  is a weak AH-subspace of  $W_n(I)$ .

**Theorem 28:**

Let  $V_n(I), W_n(I)$  be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field  $K_n(I)$ ,

$f: V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^n f_i(a_i)I_i$  be a strong AH-linear transformation. Then:

(a)  $AH - Ker(f)$  is a strong AH-subspace of  $V_n(I)$ .

(b)  $AH - Im(f)$  is a strong AH-subspace of  $W_n(I)$ .

(c) If  $M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n$  is a strong AH-subspace of  $V_n(I)$ ,  $f(M_n(I))$  is a strong AH-subspace of  $W_n(I)$ .

**Theorem 29 :**

Let  $V_n(I), W_n(I)$  be two weak n-refined neutrosophic vector spaces over the field  $K$ ,

$f: V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^n f_i(a_i)I_i$  be a weak AH-linear transformation. Then:

$f(x + y) = f(x) + f(y), f(r \cdot x) = r \cdot f(x)$  for all  $x, y \in V_n(I), r \in K$ .

Proof:

Let  $x = \sum_{i=0}^n a_i I_i, y = \sum_{i=0}^n b_i I_i$  be two arbitrary elements in  $V_n(I)$ ,  $r \in K$  be any element in the field  $K$ , we have:

$f(x + y) = f(\sum_{i=0}^n (a_i + b_i)I_i) = \sum_{i=0}^n f_i(a_i + b_i)I_i = \sum_{i=0}^n f_i(a_i)I_i + \sum_{i=0}^n f_i(b_i)I_i = f(x) + f(y)$ .

$f(r \cdot x) = f(\sum_{i=0}^n r a_i I_i) = \sum_{i=0}^n f_i(r a_i)I_i = r \cdot \sum_{i=0}^n f_i(a_i)I_i = r \cdot f(x)$ .

**Theorem 30:**

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Let  $V_n(I), W_n(I)$  be two strong  $n$ -refined neutrosophic vector spaces over the  $n$ -refined neutrosophic field  $K_n(I)$ ,

$f: V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^n f_i(a_i)I_i$  be a strong AH-linear transformation. Then:

$$f(x + y) = f(x) + f(y), f(r \cdot x) = r \cdot f(x) \text{ for all } x, y \in V_n(I), r \in K_n(I).$$

Proof:

Let  $x = \sum_{i=0}^n a_i I_i, y = \sum_{i=0}^n b_i I_i$  be two arbitrary elements in  $V_n(I), r = \sum_{i=0}^n r_i I_i \in K_n(I)$  be any element in the  $n$ -refined neutrosophic field  $K_n(I)$ , we have:

$$f(x + y) = f(\sum_{i=0}^n (a_i + b_i) I_i) = \sum_{i=0}^n f_i(a_i + b_i) I_i = \sum_{i=0}^n f_i(a_i) I_i + \sum_{i=0}^n f_i(b_i) I_i = f(x) + f(y).$$

For the proof of the second proposition we use induction on  $n$ . If  $n=0$ , the theorem is true clearly.

Suppose that it is true for  $n-1$ , we must prove it for  $n$ .

$$f(r \cdot x) = f(\sum_{i,j=0}^n r_i a_j I_i I_j) = f(\sum_{i,j=0}^{n-1} r_i a_j I_i I_j + (\sum_{i=0}^n r_i I_i) a_n I_n), \text{ we can write}$$

$$\sum_{i,j=0}^{n-1} r_i a_j I_i I_j = m_0 + m_1 I_1 + \dots + m_{n-1} I_{n-1},$$

$$(\sum_{i=0}^n r_i I_i) a_n I_n = r_1 a_n I_1 + r_2 a_n I_2 + \dots + (r_0 a_n + r_n a_n) I_n,$$

$$r \cdot x = \sum_{i,j=0}^{n-1} r_i a_j I_i I_j + (\sum_{i=0}^n r_i I_i) a_n I_n = m_0 + (m_1 + r_1 a_n) I_1 + (m_2 + r_2 a_n) I_2 + \dots + (r_0 a_n + r_n a_n) I_n,$$

$$f(r \cdot x) = f_0(m_0) + f_1(m_1 + r_1 a_n) I_1 + f_2(m_2 + r_2 a_n) I_2 + \dots + f_n(r_0 a_n + r_n a_n) I_n =$$

$$f_0(m_0) + [f_1(m_1) + r_1 f_1(a_n)] I_1 + \dots + [r_0 f_n(a_n) + r_n f_n(a_n)] I_n = r \cdot f(x).$$

**Theorem 31:**

Let  $V_n(I), W_n(I), U_n(I)$  be three weak  $n$ -refined neutrosophic vector spaces over the field  $K$ ,

$$f: W_n(I) \rightarrow U_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1) I_1 + \dots + f_n(a_n) I_n = \sum_{i=0}^n f_i(a_i) I_i,$$

$$g: V_n(I) \rightarrow W_n(I); g(\sum_{i=0}^n a_i I_i) = g_0(a_0) + g_1(a_1) I_1 + \dots + g_n(a_n) I_n = \sum_{i=0}^n g_i(a_i) I_i,$$

be two weak AH-linear transformations. Then:

$$(a) f \circ g = \sum_{i=0}^n (f_i \circ g_i).$$

$$(b) f \circ g \text{ is a weak AH-linear transformation between } V_n(I), U_n(I).$$

Proof:

$$(a) \text{ Let } x = \sum_{i=0}^n a_i I_i \in V_n(I), f \circ g(x) = f(\sum_{i=0}^n g_i(a_i) I_i) = f(g_0(a_0) + g_1(a_1) I_1 + \dots + g_n(a_n) I_n) =$$

$$f_0(g_0(a_0)) + f_1(g_1(a_1)) I_1 + \dots + f_n(g_n(a_n)) I_n = \sum_{i=0}^n (f_i \circ g_i)(a_i) I_i.$$

(b) Since  $f_i \circ g_i$  is a linear transformation for all  $i$ , then we get the proof.

**Theorem 32:**

Let  $V_n(I), W_n(I), U_n(I)$  be three strong  $n$ -refined neutrosophic vector spaces over the  $n$ -refined neutrosophic field  $K$ ,

$$f: W_n(I) \rightarrow U_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^n f_i(a_i)I_i,$$

$$g: V_n(I) \rightarrow W_n(I); g(\sum_{i=0}^n a_i I_i) = g_0(a_0) + g_1(a_1)I_1 + \dots + g_n(a_n)I_n = \sum_{i=0}^n g_i(a_i)I_i,$$

be two strong AH-linear transformations. Then:

$$(a) f \circ g = \sum_{i=0}^n (f_i \circ g_i).$$

(b)  $f \circ g$  is a strong AH-linear transformation between  $V_n(I), U_n(I)$ .

**Definition 33:**

Let  $M(I_1, I_2)$  be a strong refined neutrosophic module over the refined neutrosophic ring  $R(I_1, I_2)$ ,  $P, Q, S$  be three submodules of  $M$ . The set  $N = (P, QI_1, SI_2) = \{(a, bI_1, cI_2); a \in P, b \in Q, c \in S\}$  is called a strong AH-submodule of the strong refined neutrosophic module  $M(I_1, I_2)$ .

If  $P = Q = S$ , we call  $N$  a strong AHS-submodule.

**Theorem 34:**

Let  $M(I_1, I_2)$  be a strong refined neutrosophic module over the refined neutrosophic ring  $R(I_1, I_2)$ ,

$N = (P, PI_1, PI_2)$  be a strong AHS-submodule. Then  $N$  is a submodule by classical meaning.

**Definition 35:**

Let  $M, W$  be two modules over the ring  $R$ ,  $M(I_1, I_2)$  and  $W(I_1, I_2)$  be the corresponding strong refined neutrosophic modules over the refined neutrosophic ring  $R(I_1, I_2)$ . Let  $f, g, h: M \rightarrow W$  be three homomorphisms, then  $[f, g, h]: M(I_1, I_2) \rightarrow W(I_1, I_2); [f, g, h](a, bI_1, cI_2) = (f(a), g(b)I_1, h(c)I_2)$  is called a strong AH-homomorphism. If  $f = g = h$ , we get the strong AHS-homomorphism.

**Definition 36:**

Let  $M(I_1, I_2), W(I_1, I_2)$  be two strong refined neutrosophic modules over the refined neutrosophic ring  $R(I_1, I_2)$ ,  $[f, g, h]: M(I_1, I_2) \rightarrow W(I_1, I_2)$  be a strong AH-homomorphism, we define

$$(a) AH - Ker[f, g, h] = (Ker(f), Ker(g)I_1, Ker(h)I_2) = \{(a, bI_1, cI_2); a \in Ker(f), b \in Ker(g), c \in Ker(h)\}.$$

$$(b) AH - Im[f, g, h] = (Im(f), Im(g)I_1, Im(h)I_2).$$

**Theorem 37:**

Let  $M(I_1, I_2), W(I_1, I_2)$  be two strong refined neutrosophic modules over the refined neutrosophic ring  $R(I_1, I_2)$ ,  $[f, g, h]: M(I_1, I_2) \rightarrow W(I_1, I_2)$  be a strong AH-homomorphism.

(a) If  $N = (P, QI_1, SI_2)$  is a strong AH-submodule of  $M(I_1, I_2)$ , then  $[f, g, h](N)$  is a strong AH-submodule of  $W(I_1, I_2)$ .

- (b)  $[f, g, h]$  is a classical module homomorphism.
- (c)  $AH - Ker[f, g, h]$  is a strong AH-submodule of  $M(I_1, I_2)$ .
- (d)  $AH - Im[f, g, h]$  is a strong AH-submodule of  $W(I_1, I_2)$ .

Proof:

(a) Since  $f(P), g(Q), h(S)$  are submodules of  $N$ , we find that  $[f, g, h](N) = (f(P), g(Q)I_1, h(S)I_2)$  is a strong AH-submodule of  $W(I_1, I_2)$ .

(b) Let  $m = (x, yI_1, zI_2), n = (a, bI_1, cI_2)$  be two arbitrary elements in  $M(I_1, I_2), r = (t, uI_1, vI_2)$  be any element in  $R(I_1, I_2)$ ,

$$m + n = (x + a, [y + b]I_1, [z + c]I_2), r \cdot m = (tx, [xu + yt + yu + yv + zu]I_1, [xv + zt + zv]I_2),$$

$$[f, g, h](m + n) = (f(x + a), g([y + b]I_1), h([z + c]I_2)) = (f(x), g(y)I_1, h(z)I_2) + (f(a), g(b)I_1, h(c)I_2) = [f, g, h](m) + [f, g, h](n).$$

$$[f, g, h](r \cdot m) = (f(tx), g([xu + yt + yu + yv + zu]I_1), h([xv + zt + zv]I_2)) =$$

$$(t, uI_1, vI_2) \cdot (f(x), g(y)I_1, h(z)I_2) = r \cdot [f, g, h](m). \text{ Thus } [f, g, h] \text{ is a classical homomorphism.}$$

(c) Since  $Ker(f), Ker(g), Ker(h)$  are submodules of  $M$ , we get  $AH - Ker[f, g, h] = (Ker(f), Ker(g)I_1, Ker(h)I_2)$  as a strong AH-submodule of  $M(I_1, I_2)$ .

(d) Since  $Im(f), Im(g), Im(h)$  are submodules of  $W$ , we get  $AH - Im[f, g, h] = (Im(f), Im(g)I_1, Im(h)I_2)$  as a strong AH-submodule of  $W(I_1, I_2)$ .

**Example 38:**

(a) Let  $M = R^2, W = R$  be two modules over the ring  $R$ ,

$f: M \rightarrow W; f(x, y) = 2x, g: M \rightarrow W; g(x, y) = 3y, h: M \rightarrow W; h(x, y) = x + y$  are three homomorphisms.

(b)  $[f, g, h]: M(I_1, I_2) \rightarrow W(I_1, I_2); [f, g, h]((x, y), (z, t)I_1, (s, m)I_2) = (f(x, y), g(z, t)I_1, h(s, m)I_2) =$

$(2x, 3tI_1, [s + m]I_2)$  is a strong AH-homomorphism, where  $x, y, z, t, s, m \in R$ .

(c)  $P = \{(0, x); x \in R\}, Q = \{(x, 0); x \in R\}$  are two submodules of  $M$ ,

$N = (P, PI_1, QI_2) = \{(0, x), (0, y)I_1, (z, 0)I_2; x, y, z \in R\}$  is a strong AH-submodule of  $M(I_1, I_2)$ .

(d)  $f(P) = \{0\}, g(P) = \{3y; y \in R\} = R, h(Q) = \{z; z \in R\} = R,$

$[f, g, h](N) = (f(P), g(P)I_1, h(Q)I_2) = (0, RI_1, RI_2) = \{(0, xI_1, yI_2); x, y \in R\}$  is a strong AH-submodule of  $W(I_1, I_2)$ .

(e)  $Ker(f) = \{(0, x); x \in R\}, Ker(g) = \{(x, 0); x \in R\}, Ker(h) = \{(y, -y); y \in R\},$

$AH - Ker[f, g, h] = (Ker(f), Ker(g)I_1, Ker(h)I_2) = \{(0, x), (y, 0)I_1, (z, -z)I_2); x, y, z \in R\}.$

**Definition 39:**

Let  $M(I) = M + MI$  be a strong/weak neutrosophic module, the set

$S = P + QI = \{x + yI; x \in P, y \in Q\}$ , where  $P$  and  $Q$  are submodules of  $V$  is called an AH-submodule of  $M(I)$ .

If  $P = Q$ ,  $S$  is called an AHS-submodule of  $M(I)$ .

**Example 40:**

We have  $M = Z^2 = Z \times Z$  is a module over  $R$ ,  $P = \langle (0,1) \rangle, Q = \langle (1,0) \rangle$ , are two submodules of  $M$ . The set

$S = P + QI = \{(0, a) + (b, 0)I; a, b \in Z\}$  is an AH-submodule of  $M(I)$ .

The set  $L = P + PI = \{(0, a) + (0, b)I; a, b \in Z\}$  is an AHS-submodule of  $M(I)$ .

**Theorem 41:**

Let  $M(I) = M + MI$  be a neutrosophic weak module over the ring  $R$ , and let  $S = P + QI$  be an AH-submodule of  $M(I)$ , then  $S$  is a submodule.

Proof:

Suppose that  $x = a + bI, y = c + dI \in S; a, c \in P, b, d \in Q$ ,

$x + y = (a + c) + (b + d)I \in S$ . For each scalar  $m \in R$  we obtain  $m.x = m.a + (m.b)I \in S$ , since  $P$  and  $Q$  are submodules; thus  $S = P + QI$  is a submodule of  $M(I)$  over the ring  $R$ .

**Theorem 42:**

Let  $M(I)$  be a neutrosophic strong module over a neutrosophic ring  $R(I)$ , let  $S = P + PI$  be an AHS-submodule. Then  $S$  is a submodule of  $M(I)$ .

**Definition 43:**

(a) Let  $M$  and  $W$  be two modules,  $L_M: M \rightarrow W$  be a homomorphism. The AHS-homomorphism can be defined as follows:

$$L: M(I) \rightarrow W(I); L(a + bI) = L_M(a) + L_M(b)I.$$

(b) If  $S = P + QI$  is an AH-submodule of  $M(I)$ ,  $L(S) = L_M(P) + L_M(Q)I$ .

(c) If  $S = P + QI$  is an AH-submodule of  $W(I)$ ,  $L^{-1}(S) = L_M^{-1}(P) + L_M^{-1}(Q)I$ .

(d)  $AH - Ker(L) = Ker(L_M) + Ker(L_M)I = \{x + yI; x, y \in Ker(L_M)\}$ .

**Theorem 44:**

Let  $W(I)$  and  $M(I)$  be two neutrosophic strong/weak modules, and  $L: M(I) \rightarrow W(I)$  be an AHS-homomorphism:

(a)  $AH - Ker(L)$  is an AHS-submodule of  $M(I)$ .

(b) If  $S = P + QI$  is an AH-submodule of  $M(I)$ ,  $L(S)$  is an AH-submodule of  $W(I)$ .

(c) If  $S = P + QI$  is an AH-submodule of  $W(I)$ ,  $L^{-1}(S)$  is an AH-submodule of  $M(I)$ .

Proof:

(a) Since  $\text{Ker}(L_M)$  is a submodule of  $M$ , we find that  $AH - \text{Ker}(L) = \text{Ker}(L_M) + \text{Ker}(L_M)I$  is an AHS-submodule of  $M(I)$ .

(b) We have  $L(S) = L_M(P) + L_M(Q)I$ ; thus  $L(S)$  is an AH-submodule of  $W(I)$ , since  $L_M(P), L_M(Q)$  are submodules of  $W$ .

(c) By regarding that  $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$ ,  $L_W^{-1}(P)$  and  $L_W^{-1}(Q)$  are submodules of  $M$ , we obtain that  $L^{-1}(S)$  is an AH-subModule of  $M(I)$ .

#### Theorem 45:

Let  $W(I)$  and  $M(I)$  be two neutrosophic strong modules over a neutrosophic ring  $R(I)$ , and  $L: M(I) \rightarrow W(I)$  be an AHS-homomorphism. Then

$$L(x + y) = L(x) + L(y), L(m \cdot x) = m \cdot L(x), \text{ for all } x, y \in M(I), m \in R(I).$$

Proof:

Suppose  $x = a + bI, y = c + dI; a, b, c, d \in M$ , and  $m = s + tI \in K(I)$ , we have

$$L(x + y) = L([a + c] + [b + d]I) = L_M(a + c) + L_M(b + d)I = [L_M(a) + L_M(b)I] + [L_M(c) + L_M(d)I] = L(x) + L(y).$$

$$m \cdot x = (s \cdot a) + (s \cdot b + t \cdot a + t \cdot b)I, L(m \cdot x) = L_M(s \cdot a) + L_M(s \cdot b + t \cdot a + t \cdot b)I$$

$$= s \cdot L_M(a) + [s \cdot L_M(b) + t \cdot L_M(a) + t \cdot L_M(b)]I = (s + tI) \cdot (L_M(a) + L_M(b)I) = m \cdot L(x).$$

#### Theorem 46:

Let  $S = P + QI$  be an AH-submodule of a neutrosophic weak module  $M(I)$  over a ring  $R$ , suppose that

$X = \{x_i; 1 \leq i \leq n\}$  is a bases of  $P$  and  $Y = \{y_j; 1 \leq j \leq m\}$  is a bases of  $Q$  then  $X \cup YI$  is a bases of  $S$ .

Proof:

Let  $z = x + yI$  be an arbitrary element in  $S$ ;  $x \in P, y \in Q$ . Since  $P$  and  $Q$  are submodules of  $M$  we can write

$$x = a_1x_1 + a_2x_2 + \dots + a_nx_n; a_i \in R \text{ and } x_i \in X, y = b_1y_1 + b_2y_2 + \dots + b_my_m; b_i \in K, y_i \in Y.$$

Now we obtain  $z = (a_1x_1 + \dots + a_nx_n) + (b_1y_1I + \dots + b_my_mI)$ ; thus  $X \cup YI$  generates the subspace  $S$ .

$X \cup YI$  is linearly independent set. Assume that  $(a_1x_1 + \dots + a_nx_n) + (b_1y_1I + \dots + b_my_mI) = 0$ , this implies

$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$  and  $(b_1y_1 + b_2y_2 + \dots + b_my_m)I = 0$ . Since  $X$  and  $Y$  are linearly independent sets over  $R$ , we get  $a_i = b_j = 0$  for all  $i, j$  and  $X \cup YI$  is linearly independent then it is a basis of  $S$ .

**Result 47:**

Let  $S = P + QI$  be an AH-submodule of a neutrosophic weak module  $M(I)$  with finite dimension over a ring  $R$ , from Theorem 3.8 and the fact that  $X \cap YI = \emptyset$ , we find  $\dim(S) = \dim(P) + \dim(Q)$ .

**Example 48:**

Let  $M = Z^3 = Z \times Z \times Z$  is a module over the ring  $Z$ ,  $P = \langle (0,0,1) \rangle$ ,  $Q = \langle (0,1,0) \rangle$  be two submodules of  $M$ ,

(a)  $S = P + QI = \{(0,0,m) + (0,n,0)I; m, n \in Z\}$  is an AH-submodule of  $M(I)$ .

(b) The set  $\{(0,0,1), (0,1,0)I\}$  is a bases of  $S$ ,  $\dim(S) = \dim(P) + \dim(Q) = 1 + 1 = 2$ .

(c)  $L_M: M \rightarrow M$ ;  $L_M(x, y, z) = (x + y, y, z)$  for all  $x, y, z \in Z$  is a homomorphism, the corresponding AHS-homomorphism is

$$L: M(I) \rightarrow M(I); L[(x, y, z) + (a, b, c)I] = L_M(x, y, z) + L_M(a, b, c)I = (x + y, y, z) + (a + b, b, c)I.$$

(d)  $L(S) = L_M(P) + L_M(Q) = L_M\{(0,0,m)\} + L_M\{(0,n,0)I\} = \{(0,0,m) + (n,n,0)I\}; m, n \in Z$ , which is an AH-submodule of  $M(I)$ .

**Example 49:**

Let  $M = Z^2 = Z \times Z$ ,  $W = Z^3 = Z \times Z \times Z$  be two modules over the ring  $Z$ ,  $L_M: M \rightarrow W$ ;  $L_M(x, y) = (x + y, x + y, x + y)$  is a homomorphism. The corresponding AHS-homomorphism is

$$L: M(I) \rightarrow W(I); L[(x, y) + (a, b)I] = (x + y, x + y, x + y) + (a + b, a + b, a + b)I.$$

$$\text{Ker } L_M = \langle (1, -1) \rangle, \text{AH} - \text{Ker}(L) = \text{Ker}(L_M) + \text{Ker}(L_M)I = \langle (1, -1) \rangle + \langle (1, -1) \rangle I =$$

$$\{(a, -a) + (b, -b)I; a, b \in Z\} \text{ which is an AHS-submodule of } M(I).$$

We find  $\dim(\text{Ker}(L)) = 1 + 1 = 2$ .

**Definition 50:**

Let  $M(I)$  be a neutrosophic strong/weak module,  $S = P + QI$  be an AH-submodule of  $M(I)$ , we define

AH-Quotient module as:

$$M(I)/S = M/P + (M/Q)I = (x + P) + (y + Q)I; x, y \in M.$$

**Theorem 51:**

Let  $M(I)$  be a neutrosophic weak module over a ring  $R$ , and  $S = P + QI$  be an AH-submodule of  $M(I)$ . The AH-Quotient  $M(I)/S$  is a module with respect to the following operations:

$$\text{Addition: } [(x + P) + (y + Q)I] + [(a + P) + (b + Q)I] = (x + a + P) + (y + b + Q)I; x, y, a, b \in M.$$

$$\text{Multiplication by a scalar: } (m) \cdot [(x + P) + (y + Q)I] = (m \cdot x + P) + (m \cdot y + Q)I;$$

$x, y \in M$  and  $m \in R$ .

Proof:

It is easy to check that operations are well defined, and  $(M(I)/S, +)$  is abelian group.

Let  $z = [(x + P) + (y + Q)I] \in M(I)/S$ , we have  $1.z = z$ .

Assume that  $m, n \in R$ , we have  $m.(n.z) = m.[(n.x + P) + (n.y + Q)I] = (m.n.x + P) + (m.n.y + Q)I = (m.n).z$ .

$(m + n).z = [(m + n).x + P] + [(m + n).y + Q]I = m.z + n.z$ .

Let  $h = [(a + P) + (b + Q)I] \in M(I)/S$ ,  $z + h = (x + a + P) + (y + b + Q)I$ ,

$m.(z + h) = (m.x + m.a + P) + (m.y + m.b + Q)I = m.z + m.h$ .

## Conclusion

In this review, we have discussed many neutrosophic algebraic AH-structures, such as AH-subspaces, AH-submodules, AH-ideals and n-refined AH-ideals and spaces.

This work maybe very useful in the future studies in Turiyam algebraic AH-structures, especially spaces and modules, where they can be built as new generalizations of corresponding neutrosophic ones.

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